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Whitham modulation theory for the Kadomtsev-Petviashvili equation: Supplementary material

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The KdV-Whitham system

Here we briefly review the derivation of the Riemann invariant variables of the Whitham system for the KdV equation. We start from the three modulation equations, which can be obtained from the the first three of (2.25) in the main text by taking $\lambda = 0$ and removing the dependence with respect to y. The result are three quasi-linear (1+1)-dimensional PDEs for the independent variables V, $\beta = b/m$ and m. Next, we show that one can introduce the Riemann invariant variables to diagonalize the above system. Introducing the notation

$$\mathbf{w}(x,t) = (w_1\,,\, w_2\,,\, w_3)^T := (V\,,\, b/m\,,\, m)^T, \tag{S.1}$$

for brevity, we rewrite the system of modulation equations as a vector system of equations

$$R\,\mathbf{w}_t + S\,\mathbf{w}_x = 0\,, \tag{S.2}$$

or equivalently,

$$\mathbf{w}_t + A\,\mathbf{w}_x = 0\,, \tag{S.3}$$

where $A = R^{-1}S$. (Explicitly, the entries of A are obtained from the first three rows and columns of the corresponding matrix for the KP equation by setting $\lambda = q = 0$.) To diagonalize the system (S.3), one must diagonalize the matrix A. Straightforward calculations show that

$$A = P^{-1}DP\,, \tag{S.4a}$$

where

$$P = \frac{1}{3w_2}\begin{pmatrix} w_3 - 1 & 1 - w_3^2 & (1-w_3)w_2 \\ 1 & 2w_3 - 1 & 2w_2 \\ -w_3 & w_3(w_3-2) & w_3 w_2 \end{pmatrix} \tag{S.4b}$$

and $D = \mathrm{diag}(V_1, V_2, V_3)$, with V_1, V_2, V_3 given by (2.33a) in the main text. Using (S.4a), we can then write (S.3) in the main text as

$$P\,\mathbf{w}_t + DP\,\mathbf{w}_x = 0\,. \tag{S.5}$$

The key to find the Riemann invariants is to find dependent variables r_1, r_2 and r_3 such that

$$P\,\mathbf{w}_x = \mathbf{r}_x\,, \qquad P\,\mathbf{w}_t = \mathbf{r}_t\,, \tag{S.6}$$

with $\mathbf{r} = (r_1, r_2, r_3)^T$. After canceling out common factors in each row of the LHS of both parts of (S.6), the three rows in the LHS of (S.6) can be written as

$$\begin{aligned} w_{1,x} - (1+w_3)w_{2,x} - w_2\,w_{3,x} &= (w_1 - w_2 \ - w_2w_3)_x\,, \\ w_{1,x} + (2w_3 - 1)w_{2,x} + 2w_2\,w_{3,x} &= (w_1 - w_2 \ + 2w_2w_3)_x\,, \\ w_{1,x} + (2 - w_3)w_{2,x} - w_2\,w_{3,x} &= (w_1 + 2w_2 \ - w_2w_3)_x\,, \end{aligned}$$

plus identical expressions for the temporal derivatives. Finally, taking

$$r_1 = (w_1 - w_2 \ - w_2w_3)/6\,, \quad r_2 = (w_1 - w_2 \ + 2w_2w_3)/6\,, \quad r_3 = (w_1 + 2w_2 \ - w_2w_3)/6\,, \tag{S.7}$$

and solving for w_1, w_2, w_3 we obtain (2.26). That is, the change of variables (2.26) transforms the system of equations (S.2) into the diagonal system (3.1).

Stability analysis of periodic solutions via direct numerical simulations

In this section we briefly discuss the calculations of the stability properties of the cnoidal wave solutions for the KP equation by direct numerical simulations.

Let $u(x,y,t)=u_0(\xi)$ be a traveling wave (a.k.a. elliptic, periodic, cnoidal-wave or genus-1) solution of the KP equation (1.2) with $\xi=x-ct$. (Note that without loss of generality we can always align any traveling wave solution along the x-axis thanks to the pseudo-rotation invariance of the KP equation.) Next, consider a perturbed solution of the KP equation in the form $u(x,y,t)=u_0(\xi)+v(\xi,y,t)$ with $|v(\xi,y,t)|\ll 1$. Substituting into the KP equation (1.2) and dropping higher-order terms, we have

$$\Big(v_t - cv_\xi + 6(u_0 v)_\xi + \epsilon^2 v_{\xi\xi\xi}\Big)_\xi + \lambda v_{yy} = 0\,.$$

Using the Galilean invariance of the KP equation, we can always perform the transformation $u_0 \mapsto c/6+\tilde{u}_0$, which yields

$$\Big(v_t + 6(\tilde{u}_0 v)_\xi + \epsilon^2 v_{\xi\xi\xi}\Big)_\xi + \lambda v_{yy} = 0\,. \tag{S.8}$$

Now we look for plane wave solution of the above equation in the form

$$v(\xi,y,t)=w(\xi)\mathrm{e}^{i\zeta y+\mu t}\,. \tag{S.9}$$

Substituting (S.9) into (S.8) yields

$$\Big(\mu w + 6(\tilde{u}_0 w)_\xi + \epsilon^2 w_{\xi\xi\xi}\Big)_\xi - \lambda\zeta^2 w = 0\,,$$

or equivalently,

$$\mu w + 6(\tilde{u}_0 w)_\xi + \epsilon^2 w_{\xi\xi\xi} - \lambda\zeta^2\partial_\xi^{-1} w = 0\,, \tag{S.10}$$

where $\partial_\xi^{-1} w = \mathcal{F}^{-1}\Big[(1/ik)\,\mathcal{F}[w]\Big]$ with $\mathcal{F}$ denoting the Fourier transform. Note that in order for (S.10) to admit periodic solutions, one needs $\int_{-\infty}^{\infty} w(\xi)d\xi = 0$. That is, w should have zero mean (i.e., the Fourier transform of w should vanish at the zero wave number). Then we can finally write (S.10) as an eigenvalue problem for a differential operator

$$-6(\tilde{u}_0 w)_\xi - \epsilon^2 w_{\xi\xi\xi} + \lambda\zeta^2\partial_\xi^{-1} w = \mu w\,, \tag{S.11}$$

and solve it in the Fourier domain. This operation is well-defined precisely because w has no mean term. In order to compare the results of this calculation with those obtained from the Whitham approach discussed in section 4, note that when $r_1=q=0$, $r_2=m$ and $r_3=1$, the cnoidal wave solution (2.27) becomes $\tilde{u}_0 = 1-m+2m\,\mathrm{cn}^2\left(x-(2+2m)t,m\right)$. The eigenvalue problem was solved numerically with $0\le m\le 1$. The results, together with a numerical comparison with the Whitham approach, are given in Fig. 1.

Coefficient matrices for the un-regularized KP-Whitham system

The entries of the coefficient matrix B of the original Whitham system (2.29) are given by:

$$B_{11} = \lambda q\,\frac{-2b\big(E_m(1+m)-K_m(1+3m)\big)+(E_m-K_m)mV}{6bK_m m}\,,$$

$$B_{12} = \lambda q\,\frac{\big(E_m-K_m(1-m)\big)\big(2b\big(-K_m(1-m)^2+E_m(1+m)\big)-\big(E_m-K_m(1-m)\big)mV\big)}{6bK_m(E_m-K_m)(1-m)m}\,,$$

$$B_{13} = \lambda q\,\frac{E_m\big(2b\big(-2K_m(1-m)+E_m(1+m)\big)-E_m mV\big)}{6b(K_m-E_m)K_m(1-m)}\,,$$

$$B_{14} = \lambda\,\frac{2b\big(E_m(1+m)-K_m\big)+(K_m-E_m)mV\big)}{6(K_m-E_m)m}\,,\quad B_{15}=B_{25}=B_{35}=\lambda\,,$$

$$B_{21} = \lambda q\,\frac{(E_m-K_m)\big(2b\big(K_m-E_m(1-2m)\big)+(E_m-K_m)mV\big)}{6bK_m\big(E_m-K_m(1-m)\big)m},$$

$$B_{22} = \lambda q\,\frac{2b\big(K_m(1-5m+4m^2)-E_m(1-2m)\big)+\big(E_m-K_m(1-m)\big)mV}{6bK_m(-1+m)m},$$

$$B_{23} = \lambda q\,\frac{E_m\big(-2b\big(E_m(1-2m)-2K_m(1-m)\big)+E_mmV\big)}{6bK_m\big(E_m-K_m(1-m)\big)(1-m)},$$

$$B_{24} = \lambda\,\frac{2b\big(K_m(1-m)^2-E_m(1-2m)\big)+\big(E_m-K_m(1-m)\big)mV}{6\big(E_m-K_m(1-m)\big)m},$$

$$B_{31} = \lambda q\,\frac{(K_m-E_m)\big(2b\big(K_m(2-3m)-E_m(2-m)\big)+(K_m-E_m)mV\big)}{6bE_mK_mm},$$

$$B_{32} = \lambda q\,\frac{\big(E_m-K_m(1-m)\big)\big(2b\big(E_m(-2+m)-K_m(-2+m+m^2)\big)-\big(E_m-K_m(1-m)\big)mV\big)}{6bE_mK_m(1-m)m},$$

$$B_{33} = \lambda q\,\frac{2b\big(E_m(2-m)+2K_m(1-m)\big)-E_mmV}{6bK_m(1-m)},$$

$$B_{34} = \lambda\,\frac{-2b\big(K_m(1-m)+E_m(-2+m)\big)+E_mmV}{6E_mm},$$

$$B_{41} = 2+(V+\lambda q^2)\frac{E_m-K_m}{bK_m},\quad B_{42} = 2-(V+\lambda q^2)\frac{E_m-K_m(1-m)}{bK_m(1-m)},$$

$$B_{43} = 2+(V+\lambda q^2)\frac{E_mm}{bK_m(1-m)},\quad B_{44} = 2\lambda q,\quad B_{45} = B_{54} = B_{55} = 0,$$

$$B_{51} = -\frac{6(E_m-K_m)^2}{K_m^2m},\quad B_{52} = -\frac{6(E_m-K_m(1-m))^2}{K_m^2(-1+m)m},\quad B_{53} = -\frac{6E_m^2}{K_m^2(1-m)}.$$

Instead of listing all the entries of the coefficient matrix A, we point out the important fact that, even though both A and B are full matrices with complicated entries, the following combination of them takes on a particularly simple form:

$$A+qB = \begin{pmatrix} V_1+\lambda q^2 & 0 & 0 & \gamma_1 & -\lambda q\,\frac{K_m\left(2b(1+m)+mV\right)}{6b(E_m-K_m)} \\ 0 & V_2+\lambda q^2 & 0 & \gamma_2 & -\lambda q\,\frac{K_m\left(2b(1-2m)+mV\right)}{6b\left(E_m-K_m(1-m)\right)} \\ 0 & 0 & V_3+\lambda q^2 & \gamma_3 & -\lambda q\,\frac{K_m\left(2b(-2+m)+mV\right)}{6bE_m} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V+\frac{2b\left(3E_m-K_m(2-m)\right)}{K_mm} & 6 \end{pmatrix}$$

with V_1, V_2, V_3 given by (2.33*a*) as before and

$$\gamma_1 = \lambda q\,\frac{4b^2\big(3E_m(1+m)-K_m(4-7m+m^2)\big)+2bm(-3E_m+K_m-2K_mm)V-K_mm^2V^2}{36b(E_m-K_m)m},$$

$$\gamma_2 = \lambda q\,\frac{4b^2\big(3E_m(1-2m)+K_m(-4+m+2m^2)\big)+2bm(-3E_m+K_m+K_mm)V-K_mm^2V^2}{36b\big(E_m-K_m(1-m)\big)m},$$

$$\gamma_3 = \lambda q\,\frac{4b^2\big(K_m(8-8m-m^2)-3E_m(2-m)\big)-2b\big(3E_m+2K_m(-2+m)\big)mV-K_mm^2V^2}{36bE_mm}.$$

Finally, to remove singularities from the system (2.29) one must first subtract the product of the compatibility equation (2.32) times the diagonal matrix $\operatorname{diag}(c_1,c_2,c_3,c_4,c_5)$, with

$$c_1 = \frac{\lambda q\sqrt{\beta}}{12b}\,\frac{-2b\big(K_m(1-m)^2+E_m(1+m)\big)+\big(E_m-K_m(1-m)\big)mV}{E_m-K_m},$$

$$c_2 = \frac{\lambda q\sqrt{\beta}}{12b}\,\frac{-2b\big(E_m(1-2m)+2K_m(-1+m)\big)+E_m mV}{E_m-K_m(1-m)},$$

$$c_3 = \frac{\lambda q\sqrt{\beta}}{12b}\,\frac{2b\big(E_m(2-m)+K_m(-2+m+m^2)\big)+\big(E_m-K_m(1-m)\big)mV}{E_m}$$

$$-\lambda q\,\frac{K_m(1-m)\sqrt{\beta}\big(2b(1-2m)+mV\big)}{12b\big(E_m-K_m(1-m)\big)},$$

$$c_4 = \frac{V+\lambda q^2}{2\sqrt{\beta}} - m\sqrt{\beta}\,\frac{K_m(1-m)}{E_m-K_m(1-m)}, \quad c_5 = -3\sqrt{\beta}\,\frac{E_m-K_m(1-m)}{K_m},$$

and $\beta = b/m$. Then one need to subtract the product of the constraint (1.4*c*) times the diagonal matrix $\mathrm{diag}(d_1, d_2, d_3, 0, 0)$, with

$$d_1 = -\lambda q\,\frac{K_m\big(2b(1+m)+mV\big)}{36b(E_m-K_m)}, \qquad d_2 = -\lambda q\,\frac{K_m q\big(2b(1-2m)+mV\big)}{36b\big(E_m-K_m(1-m)\big)},$$

$$d_3 = -\lambda q\,\frac{K_m\big(2b(-2+m)+mV\big)}{36bE_m}.$$

Doing so yields the singularity-free system (1.4).