Supplemental Document

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Experimental observation of the spatio-temporal dynamics of breather gases in a recirculating fiber loop: supplement

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In this Supplemental Material we concisely review some background material regarding the nonlinear Schrödinger equation (NLSE), some key aspects of the theoretical findings from Refs. [S1–S5] and we provide additional additional comments to further clarify the results of the present Letter. All equations, figures, reference numbers within this document are prepended with "S" to distinguish them from corresponding numbers in the main manuscript.

As mentioned in the main text, the NLSE, namely

$$i\frac{\partial A}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} - \gamma|A|^2 A, \tag{S1}$$

admits a broad class of solutions known as breathers, or solitons on finite background. This category includes the celebrated Peregrine, Akhmediev, Kuznetsov-Ma and Tajiri-Watanabe breathers. From a theoretical point of view, the difference between all these excitations lies in their spectral content, as we briefly review next.

Recall that the NLSE (S1) can be written as the compatibility condition of the matrix Lax pair (e.g., see [S6])

$$\phi_t = U\,\phi\,, \tag{S2a}$$
$$\phi_z = V\,\phi\,, \tag{S2b}$$

where

$$U(t,z,\zeta) = \begin{pmatrix} -i\zeta & A \\ -A^* & i\zeta \end{pmatrix}, \tag{S3}$$

i is the imaginary unit, the asterisk denotes complex conjugation, and where the explicit form of V is omitted since it is not relevant for our purposes. Equation (S2a), which is the celebrated Zakharov-Shabat problem, is referred to as the scattering problem, ζ as the scattering (or spectral) parameter, and $A(z,t)$ as the scattering potential. The spectrum of the inverse scattering transform (IST) is the set of values of $\zeta \in \mathbb{C}$ for which the scattering problem admits bounded solutions over all $t \in \mathbb{R}$, and is composed of a continuous spectrum plus discrete eigenvalues. For potentials with zero background, the continuous spectrum is simply the real ζ-axis, whereas for potentials on a non-zero background, the continuous spectrum also includes the vertical segment $[-iA_o, iA_o]$ of the complex ζ-plane (which is a branch cut in the IST with non-zero boundary conditions [S6]), where $A_o > 0$ denotes the background amplitude.

Also recall that each soliton or breather is associated with a discrete eigenvalue ζ_o of the scattering problem. Mathematically, the difference among the various breathers is simply the position of the eigenvalue: Kuznetsov-Ma breathers are obtained when ζ_o is located on the imaginary axis above iA_o, Akhmediev breathers when $\zeta_o \in (0, iA_o)$, the Peregrine breather when $\zeta_o = iA_o$ and Tajiri-Watanabe breathers when ζ_o is in a generic location in the complex plane. In turn, the different location results in different properties from a physical point of view: Kuznetsov-Ma breathers are localized in t and periodic in z, Akhmediev breathers are periodic in t and localized in z, and the Peregrine breather is localized in both t and z, while Tajiri-Watanabe breathers are traveling solutions with non-zero velocity.

The key quantity in the theory of soliton gases (SGs) and breather gases (BGs) is played by a spectral density of states (DOS) $f(\zeta,t,z)$ (e.g., see [S2]), defined so that $f(\zeta,t,z)\,d\Re(\zeta)d\Im(\zeta)dt$ is proportional to the number of solitons or breathers with spectral parameter in $[\Re(\zeta), \Re(\zeta) + d\Re(\zeta)] \times [\Im(\zeta), \Im(\zeta) + d\Im(\zeta)]$ at time $[t, t+dt]$ at distance z along the fiber. One of the central results in the kinetic theory of SGs/BGs is the kinetic equation satisfied by the DOS, and which comprises a continuity equation plus an equation of state that describes the cumulative effect of the gas on an individual soliton.

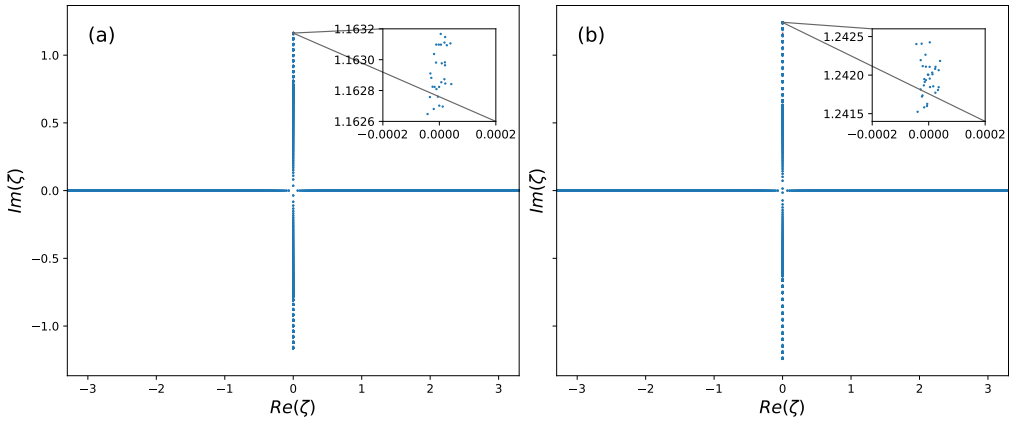

Fig. S1. (a) Numerically computed spectrum of the scattering problem Eq. S3 for the BG shown in Fig. 2(d)(e) of the Letter. (b) Same as (a) but for the BG shown in Fig. 3(d)(e) of the Letter.

As one can clearly see from Figs. 2 and 3, all the nonlinear excitations in the BGs realized in this work have zero or small velocity. This is because the spectral content of the input data consists only of eigenvalues located in the imaginary axis of the spectral parameter, which therefore corresponds to excitations with zero velocity. This is consistent with the asymptotic analysis of [S5], and is illustrated in Fig. S1.

We should clarify that, while Ref. [S1] dealt with an elliptic dn potential, it did so only for convenience (because in that case one can obtain an analytical representation for the spectrum of the inverse scattering problem associated with the NLSE). On the other hand, the theory formulated in Refs. [S3–S5] applies to all real-valued single lobe periodic potentials. This is indeed what makes it possible to generate these BGs experimentally, and this realization is precisely what motivated the present study in the first place. In other words, the dn potential does not play any special in the theory of BGs.

We also point out that the fission process in a BG is still caused by modulational instability, and is not different from that in a SG. The reason why a BG is qualitatively different from a SG is that a BG is a SG on a non-zero background. This has the consequence that, while in a typical SG the density of states is given by a single mathematical expression, in a BG the spectral density of states has different representations for different ranges of values for the spectral parameter: one for the region corresponding to the background [namely, the region $\zeta \in (0, iA_o)$, which breaks up into individual excitations due to modulational instability] and one for the region corresponding to the discrete eigenvalues [namely, the region $\zeta \in (iA_o, i\infty)$]. In other words, as was explained in Ref. [S1], a BG is equivalent to a "composite SG".

In this respect, we should briefly comment on two limiting cases, namely, the values $m = 0$ and $m = 1$ of the modulation depth parameter m in Eq. (2). When $m = 0$, the input pulse is constant. Conversely, when $m = 1$, the minimum value of input pulse dips all the way to zero. In both cases, the parameter A_o in the above discussion becomes zero, and one obtains a pure SG. Importantly, however, the theoretical predictions of [S3, S4] regarding the doubling of the kurtosis apply equally well to both SGs and BGs, and these predictions are indeed confirmed by the experimental results reported in the present Letter.

Finally, in Figure S1 we present the numerically computed IST spectrum for the scattering problem associated with the potential (i.e., the initial condition) used in the simulations reported in the Letter. Following Ref. [S1], the spectrum is obtained with the Fourier-Hill method. Figure S1(a) (computed for the input pulse in Fig. 2 of the Letter) shows the vertical band of the spectrum along the imaginary ζ axis, symmetrically located around the real axis, together with $N = 11$ narrow bands clustered along the imaginary axis above the continuous spectrum of the IST. Figure S1(b) (computed for the input pulse in Fig. 3 of the Letter) displays $N = 13$ bands. Recall, however, that a BG is equivalent a composite SG. Accordingly, both the discrete eigenvalues and the continuous spectrum in $[-iA_o, iA_0]$ contribute to the gas. To see why, consider the case in which there is no modulation (i.e., $m = 0$ in Eq. (2)). In this case, the input pulse is constant, and the IST spectrum along the imaginary axis is just the branch cut (continuous spectrum), with no additional discrete eigenvalues related to breathers. In this case, however, one would still obtain a noise-induced

destabilization of the plane wave, and would observe the formation in space and time of coherent structures very similar to what one obtains in the space-time diagrams of the BGs, namely, Figs. 2 and 3 (e.g., see Ref. [S7].) In the context of the noise-induced destabilization of the plane wave, it is the continuous part of the spectrum (in the IST with non-zero boundary conditions) that is at play and that drives the space-time dynamics. In other words, for the BG, the structures that are observed in the space-time plots results from the existence of *both* the discrete eigenvalues (the breathers) and also the continuous spectrum in $[-iA_o, iA_o]$.

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