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Polarization interactions in multi-component defocusing media

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Abstract
We study dark–bright soliton interactions in multi-component media such as nonlinear optical media in the defocusing regime and repulsive Bose–Einstein condensates. This is achieved using the recently developed formalism of the inverse scattering transform for the defocusing multi-component nonlinear Schrödinger equation with non-zero boundary conditions. We show that, generically, these interactions result in a non-trivial polarization shift for the bright components. We compute such polarization shift analytically and compare it to that in focusing two-component nonlinear Schrödinger systems.

Keywords: nonlinear Schrödinger systems, solitons, integrable systems, inverse scattering

(Some figures may appear in colour only in the online journal)

1. Introduction

Since their first experimental realization [1, 2], Bose–Einstein condensates (BECs) have attracted considerable attention, and they continue to be the object of intense research. In particular, experiments in multi-component BECs have demonstrated a variety of dark–dark and dark–bright solitons [3–5]. The same kinds of solutions also appear in nonlinear optical media with defocusing dispersion [6–9]. The purpose of this work is to study dark–

bright soliton interactions in multi-component media of this kind. We show that such interactions result in non-trivial polarization shifts, i.e., energy and phase exchanges between the bright components of the interacting solitons, similar to those in focusing two-component nonlinear Schrödinger systems [10]. To the best of our knowledge, this is the first time that non-trivial soliton polarization interactions have been reported in a defocusing system.

Repulsive, cigar-shaped single-component BECs can be modeled by the defocusing nonlinear Schrödinger (NLS) equation [14]. Similarly, multi-component BECs are modeled by a vector NLS (VNLS) equation. In particular, the two-component case is referred to as the Manakov system [10]. To model dark–bright soliton interactions, one must consider non-zero boundary conditions (NZBC). Thus, here we study the three-component defocusing VNLS equation

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} - 2\left(\| \mathbf{q} \|^2 - q_o^2\right)\mathbf{q} = \mathbf{0}, \tag{1}$$

where $\mathbf{q}(x, t) = (q_1, q_2, q_3)^T$, and with the NZBC

$$\lim_{x\to\pm\infty} \mathbf{q}(x, t) = \mathbf{q}_\pm, \tag{2}$$

with $\| \mathbf{q}_\pm \| = q_o > 0$. The term proportional to q_o in equation (1) makes $\mathbf{q}_\pm$ independent of time but can be removed by a simple gauge transformation. Importantly, multicomponent VNLS equations such as equation (1) also arise in nonlinear optical media [11–13]. Therefore, the results of this work are applicable to both physical contexts.

Equation (1) is a completely integrable system, so its initial value problem can be solved by means of an appropriate inverse scattering transform (IST) [15]. The IST for the case of zero boundary conditions (ZBCs) was presented in [10] for the two-component case and is easily extended to the multi-component case [16]. The IST for NZBC is much more challenging, however. The IST for the scalar defocusing case with NZBC was done in [14], but the defocusing Manakov system with NZBC remained open for a long time and was recently done in [17] (see also [18, 19]). A general formulation of the IST for the multi-component defocusing case was then developed in [20, 21]. Here we employ the machinery of [20, 21] to study the resulting soliton interactions.

The structure of this work is the following: in section 2 we briefly recall the essential elements of the IST formalism developed in [20, 21] (referring the reader to those works for all details). In section 3 we discuss the symmetries and discrete spectrum of the scattering problem and use them to write exact expressions for general multi-soliton solutions. In section 4 we then use these expressions to study the soliton interactions, including the interaction-induced polarization shift. Section 5 concludes the work with some final remarks.

2. Direct and inverse scattering for the multi-component VNLS equation with NZBC

The VNLS equation (1) admits the following 4×4 Lax pair:

$$\phi_x = X\,\phi, \qquad \phi_t = T\,\phi, \tag{3}$$

where

$$X(x, t, k) = -\mathrm{i}kJ + Q, \tag{4a}$$

$$T(x, t, k) = 2\mathrm{i}k^2J - \mathrm{i}J\left(Q_x - Q^2 + q_o^2\right) - 2kQ, \tag{4b}$$

and with

$$J = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -I \end{pmatrix}, \qquad Q(x, t) = \begin{pmatrix} 0 & \mathbf{q}^\dagger \\ \mathbf{q} & O \end{pmatrix}, \tag{4c}$$

I and O denoting identity and zero matrices of appropriate size. That is, the VNLS equation (1) is the zero curvature condition

$$X_t - T_x + [X, T] = 0, \tag{5}$$

where $[A, B] = AB - BA$ is the matrix commutator. In turn, equation (5) is the compatibility condition $\phi_{tx} = \phi_{xt}$ of the overdetermined linear system (3).

For simplicity, we consider the case in which the asymptotic vectors $\mathbf{q}_\pm$ at $x \to \pm\infty$ are parallel. In this case, thanks to the $U(N)$ invariance of equation (1), without loss of generality they can be chosen in the form $\mathbf{q}_\pm = (0,0, q_\pm)^T$, with $q_\pm = q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm}$ and $\theta_\pm$ arbitrary real constants.

As in the scalar case, the scattering problem (i.e., the first equation in the Lax pair (3)) is self-adjoint, but the continuous spectrum $\mathbb{R}\backslash(-q_o, q_o)$ exhibits a gap, and the Jost solutions are expressed in terms of $\lambda = (k^2 - q_o^2)^{1/2}$ [14]. To deal effectively with the branching of λ, one introduces a Riemann surface by gluing two copies of the complex k-plane in which λ takes on either value of the square root. One then defines a uniformization variable $z = k + \lambda$ such that the first/second copy of the complex k-plane are mapped into the upper/lower half of the z-plane [22]. The original variables are $k = \frac{1}{2}(z + q_o^2/z)$ and $\lambda = \frac{1}{2}(z - q_o^2/z)$. Expressing all functional dependence on k and λ in the IST in terms of z then eliminates the branching.

For all $z \in \mathbb{R}$ one defines the Jost solutions $\phi_\pm(x, t, z)$ as the simultaneous solutions of both parts of the Lax pair with the free-particle asymptotic behavior

$$\phi_\pm(x, t, z) = E_\pm \mathrm{e}^{\mathrm{i}\Theta} + o(1), \qquad x \to \pm\infty, \tag{6}$$

with $\Theta(x, t, z) = \mathrm{diag}(\theta_1, \ldots, \theta_4) = \Lambda x - \Omega t$, where

$$\mathrm{i}\Lambda(z) = \mathrm{i}\,\mathrm{diag}(-\lambda, k, k, \lambda),$$
$$-\mathrm{i}\Omega(z) = -\mathrm{i}\,\mathrm{diag}\left(-2k\lambda, k^2 + \lambda^2, k^2 + \lambda^2, 2k\lambda\right)$$

are respectively the eigenvalue matrices of $X_\pm = \lim_{x\to\pm\infty} X$ and $T_\pm = \lim_{x\to\pm\infty} T$, and

$$E_\pm(z) = I + J\,Q_\pm/(\mathrm{i}z) \tag{7}$$

is the corresponding eigenvector matrix, with $Q_\pm = \lim_{x\to\pm\infty} Q$.

Note $\det \phi_\pm = \gamma\,\mathrm{e}^{2\mathrm{i}[kx-(k^2+\lambda^2)t]}$, where $\gamma = \det E_\pm = 1 - q_o^2/z^2$, so for $z \in \mathbb{R}\backslash \pm q_o\}$ one can introduce the scattering matrix $A(z) = (a_{i,j})$ via

$$\phi_-(x, t, z) = \phi_+(x, t, z)A(z), \tag{8}$$

with $\det A(z) = 1$. Also, since the Jost solutions solve the t-part of the Lax pair as well, all entries of the scattering matrix $A(z)$, which will enter in the definition of the scattering data, are time-independent.

Unlike the scalar case [14, 22], only two of the columns of each of $\phi_\pm$ admit analytic continuation onto the complex z-plane, specifically the first and the last columns $\phi_{\pm,1}$ and $\phi_{\pm,4}$. This is major obstruction in the development of the IST, since the solution of the inverse problem requires complete sets of analytical (or, more generally, meromorphic) eigenfunctions. This problem was circumvented in [20], where a fundamental set of meromorphic eigenfunctions $\Xi^\pm(x, t, z)$ in each half plane was constructed using an appropriate extension of the scattering problem to higher-dimensional tensors. (Hereafter, subscripts $\pm$ denote normalization as $x \to \pm\infty$, whereas superscripts $\pm$ denote analyticity or meromorphicity in the upper/lower half of the complex z-plane. Also, a subscript j in a matrix will be used to refer to the jth column of the matrix.)

For $z \in \mathbb{R}$, the meromorphic eigenfunctions can be written in terms of the Jost eigenfunctions as follows:

$$\Xi^+(x, t, z) = \phi_-(x, t, z)\alpha(z) = \phi_+(x, t, z)\beta(z), \tag{9a}$$

$$\Xi^-(x, t, z) = \phi_-(x, t, z)\tilde{\beta}(z) = \phi_+(x, t, z)\tilde{\alpha}(z), \tag{9b}$$

where α and $\tilde{\alpha}$ are upper triangular matrices, while β and $\tilde{\beta}$ are lower triangular ones. (In particular, the diagonal entries of α and $\tilde{\beta}$ are all unity.) Then equation (8) yields triangular decompositions of the scattering matrix $A(z) = \beta(z)\alpha^{-1}(z) = \tilde{\alpha}(z)\tilde{\beta}^{-1}(z)$, similarly to the N-wave interactions [23]. In turn, these decompositions allow one to express the entries of $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ in terms of the minors of A, denoted as

$$A\begin{pmatrix} i_1, ..., i_p \\ k_1, ..., k_p \end{pmatrix} = \det\begin{pmatrix} a_{i_1k_1} & \cdots & a_{i_1k_p} \\ \vdots & \ddots & \vdots \\ a_{i_pk_1} & \cdots & a_{i_pk_p} \end{pmatrix},$$

where $1 \leqslant i_1 < \cdots < i_p \leqslant 4$ and similarly for $k_1, ..., k_p$. In particular, the upper and lower principal minors of A are, respectively, determinants of the form

$$A_{[1,...,p]} = A\begin{pmatrix} 1, ..., p \\ 1, ..., p \end{pmatrix}, \qquad A_{[p,...,N]} = A\begin{pmatrix} p, ..., 4 \\ p, ..., 4 \end{pmatrix},$$

for $1 \leqslant p \leqslant 4$. Importantly, equation (9) allow one to obtain the analyticity properties of the minors of A. The upper principal minors $A_{[1]}, A_{[1,2]}, A_{[1,2,3]}$ are analytic in the upper half plane (UHP), while the lower principal minors $A_{[2,3,4]}, A_{[3,4]}, A_{[4]}$ are analytic in the lower half plane (LHP). In addition, the following minors are also analytic:

$$A\begin{pmatrix} 1,2 \\ 1,3 \end{pmatrix}, \ A\begin{pmatrix} 1,3 \\ 1,2 \end{pmatrix}: \ \text{Im}\, z > 0;$$
$$A\begin{pmatrix} 2,4 \\ 3,4 \end{pmatrix}, \ A\begin{pmatrix} 3,4 \\ 2,4 \end{pmatrix}: \ \text{Im}\, z < 0.$$

Using these results, one can write a fundamental set of analytic eigenfunctions in either half plane as

$$\chi^\pm(x, t, z) = \Xi^\pm(x, t, z)D^\pm(z), \tag{10}$$

where

$$D^+ = \text{diag}\left(1, A_{[1]}, A_{[1,2]}, A_{[1,2,3]}\right),$$
$$D^- = \text{diag}\left(A_{[2,3,4]}, A_{[3,4]}, A_{[4]}, 1\right).$$

Note $\phi_{-,1} = \chi_1^+$ and $\phi_{+,4} = \chi_4^+$ are analytic in the UHP, while $\phi_{+,1} = \chi_1^-$ and $\phi_{-,4} = \chi_4^-$ are in the LHP.

The inverse problem is formulated in terms of a Riemann–Hilbert problem (RHP) for the sectionally meromorphic matrix $M(x, t, z) = M^\pm$ for $\text{Im}\, z \gtrless 0$, with

$$M^+ = \left(\frac{\phi_{-,1}}{A_{[1]}},\ \frac{\chi_2^+}{A_{[1,2]}} - \frac{A\binom{1,3}{1,2}\chi_3^+}{A_{[1,2]}A_{[1,2,3]}},\ \frac{\chi_3^+}{A_{[1,2,3]}},\ \phi_{+,4} \right) \mathrm{e}^{-\mathrm{i}\Theta},$$

$$M^- = \left(\phi_{+,1},\ \frac{\chi_2^-}{A_{[2,3,4]}},\ \frac{\chi_3^-}{A_{[3,4]}} - \frac{A\binom{2,4}{3,4}\chi_2^-}{A_{[3,4]}A_{[2,3,4]}},\ \frac{\phi_{-,4}}{A_{[4]}} \right) \mathrm{e}^{-\mathrm{i}\Theta}.$$

Indeed, manipulating the scattering relation (8), one obtains the jump condition

$$M^+ = M^-\left(I - \mathrm{e}^{-\mathrm{i}K\Theta} L \mathrm{e}^{\mathrm{i}K\Theta}\right), \qquad z \in \mathbb{R}, \tag{11}$$

where $K = \mathrm{diag}(-1,\ 1,\ 1,\ -1)$ and $L(z)$ is explicitly determined in terms of the reflection coefficients of the problem: $\rho_1(z) = a_{21}/a_{11}$, $\rho_2(z) = a_{31}/a_{11}$ and $\rho_3(z) = a_{41}/a_{11}$.

Since $M^\pm(x,\ t,\ z) \to I$ as $z \to \infty$, one can use Cauchy projectors to reduce the RHP to a system of linear integral equations. In addition, if a nontrivial discrete spectrum is present, as usual one must supplement the system with appropriate algebraic equations, obtained by computing the residues of $M^\pm(x,\ t,\ z)$ at the discrete eigenvalues. Finally, computing the asymptotic behavior of the solution of the RHP as $z \to \infty$ and comparing with the asymptotic behavior obtained from the direct problem allows one to write down a reconstruction formula for the solution of the VNLS equation (1):

$$q_j(x,\ t) = -\mathrm{i} \lim_{z\to\infty} z M_{j+1,1}(x,\ t,\ z), \qquad j = 1,\ 2,\ 3. \tag{12}$$

As usual, in the reflectionless case $[\rho_j(z) \equiv 0]$ the RHP reduces to a linear algebraic system and one obtains the pure soliton solutions.

3. Symmetries, discrete spectrum, and reflectionless potentials

The richness of the three-component VNLS equation compared to the Manakov system comes from the discrete spectrum and symmetries. In turn, these features result in a larger variety of soliton solutions, as we discuss next.

Symmetries. Similarly to the scalar case [14, 22], the Lax pair admits two involutions: $z \mapsto z^*$, mapping the UHP into the LHP and viceversa, and $z \mapsto q_o^2/z$ mapping the exterior of the circle $C_o : |z| = q_o$ into the interior and viceversa. The behavior of the analytic eigenfunctions under these symmetries is different, however, and is obtained by first noting that, for $z \in \mathbb{R}$,

$$\phi_\pm(x,\ t,\ z) = J\left[\phi_\pm^\dagger(x,\ t,\ z)\right]^{-1} C = \phi_\pm\left(x,\ t,\ q_o^2/z\right)\Pi_\pm, \tag{13}$$

with $\Pi_\pm(z) = \mathrm{diag}(0,\ 1,\ 1,\ 0) - \mathrm{i}JQ_\pm^*/z$ and $C(z) = \mathrm{diag}(\gamma,\ -1,\ -1,\ -\gamma)$. One then expresses the non-analytic Jost eigenfunctions in equations (13) in terms of the columns of $\chi^\pm$ via equations (9) and (10) and uses the Schwarz reflection principle to lift the resulting relations off the real axis. In particular, wherever the eigenfunctions are analytic

$$\phi_{+,1}^*\left(x,\ t,\ z^*\right) = -\frac{\mathrm{e}^{-2\mathrm{i}\theta_2}}{A_{[1,2]}A_{[1,2,3]}} JL\left[\chi_2^+,\ \chi_3^+,\ \phi_{+,4}\right], \tag{14a}$$

$$\phi_{\pm,1}\left(x,\ t,\ q_o^2/z\right) = \left(\mathrm{i}z/q_\pm\right)\phi_{\pm,4}, \tag{14b}$$

$$\left[\chi_2^+(x,t,z^*)\right]^* = -\frac{e^{-2i\theta_2}}{A_{[4]}\gamma} JL\left[\phi_{+,1}, \chi_3^-, \phi_{-,4}\right], \tag{14c}$$

$$\chi_2^+\left(x,t,q_o^2/z\right) = \frac{e^{i\Delta\theta}}{A_{[3,4]}}\left(A_{[4]}\chi_2^- + A_{\binom{3,4}{2,4}}\chi_3^-\right), \tag{14d}$$

and similarly for the other columns of $\chi^\pm$. Here

$$L[\mathbf{u},\mathbf{v},\mathbf{w}] = \det\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{pmatrix}$$

is the multilinear and totally antisymmetric operator which generalizes the familiar cross-product to four dimensions, and $\Delta\theta = \theta_+ - \theta_-$. Corresponding symmetries exist for the scattering data. In particular, wherever the minors are analytic,

$$A_{[1]}(z) = A^*_{[2,3,4]}(z^*) = e^{i\Delta\theta}A_{[4]}\left(q_o^2/z\right), \tag{15a}$$

$$A_{[1,2]}(z) = A^*_{[3,4]}(z^*), \quad A_{[1,2,3]}(z) = A^*_{[4]}(z^*), \tag{15b}$$

$$A_{\binom{1,2}{1,3}}(z) = -A^*_{\binom{3,4}{2,4}}(z^*) = e^{i\Delta\theta}A_{\binom{2,4}{3,4}}\left(q_o^2/z\right), \tag{15c}$$

$$A_{\binom{1,3}{1,2}}(z) = -A^*_{\binom{2,4}{3,4}}(z^*). \tag{15d}$$

Moreover

$$e^{i\Delta\theta}A_{[1,2]}(z)A^*_{[1,2]}\left(q_o^2/z^*\right) = A_{[1]}(z)A_{[1,2,3]}(z) + A_{\binom{1,2}{1,3}}(z)A_{\binom{1,3}{1,2}}(z). \tag{15e}$$

These symmetries play a crucial role in the characterization of the discrete spectrum. Indeed, it is the presence of $L[\cdot]$ and the non-principal analytic minors in equations (14), (15*e*) and $M^\pm(x,t,z)$ that make the discrete spectrum and the corresponding soliton solutions of the three-component case much richer and more complex than those of the Manakov system.

Discrete spectrum and reflectionless solutions. The discrete spectrum is comprised of the values of $z \in \mathbb{C}$ for which the columns of $\chi^\pm$ are linearly dependent. Equations (9)–(10) yield

$$\det \chi^+ = A_{[1]}A_{[1,2]}A_{[1,2,3]}\gamma e^{2i\theta_2}, \tag{16a}$$

$$\det \chi^- = A_{[4]}A_{[3,4]}A_{[2,3,4]}\gamma e^{2i\theta_2}. \tag{16b}$$

We therefore see from equation (16) that the zeros of the upper/lower principal minors of the scattering matrix $A_{[1]}, A_{[1,2]}, A_{[1,2,3]}$ and $A_{[2,3,4]}, A_{[3,4]}, A_{[4]}$ play the role of discrete eigenvalues of the scattering problem. The scattering problem is self-adjoint, so bound states can only occur for $k \in \mathbb{R}$ [i.e., $|z| = q_o$]. As in the scalar case [14], these give rise to dark solitons. On the other hand, the analytic principal minors in equation (16) can have zeros for $|z| \neq q_o$. As in the Manakov system [17], such zeros yield dark–bright solitons [6–9, 24]. This is not a contradiction, as such solutions do not lead to bound states for the eigenfunctions [17, 18].

The symmetries of the scattering problem imply that, as in the scalar and two-component case [17, 22], the discrete eigenvalues appear in symmetric quartets $Z_n = \{z_n, z_n^*, \hat{z}_n, \hat{z}_n^*\}$, where $\hat{z} = q_o^2/z^*$. When the non-principal analytic minors are identically zero, each quartet yields a dark–bright soliton in which the bright component is aligned exclusively with either the first or the second component of $\mathbf{q}(x, t)$, while the dark part is along the third component of $\mathbf{q}(x, t)$ [21]. Here we discuss the novel case in which some of the extra analytic minors are non-zero. Specifically, for each quartet Z_n we consider the following configuration of simple zeros for the analytic minors: $A_{[1]}(z_n) = 0$ and $A_{[1,2]}(z_n)A_{[1,2,3]}(z_n) \neq 0$, as well as

$$A_{\binom{1,2}{1,3}}(z_n) = 0, \quad A_{\binom{1,3}{1,2}}(z_n) \neq 0.$$

The eigenfunctions can then be shown to be related as follows:

$$\phi_{-,1}(x, t, z_n) = b_n \chi_2^+(x, t, z_n), \tag{17a}$$

$$\chi_2^-\left(x, t, z_n^*\right) = d_n \phi_{+,1}\left(x, t, z_n^*\right), \tag{17b}$$

$$\chi_3^+\left(x, t, \hat{z}_n\right) = e_n \phi_{+,4}\left(x, t, \hat{z}_n\right), \tag{17c}$$

$$\phi_{-,4}\left(x, t, \hat{z}_n^*\right) = g_n \chi_2^-\left(x, t, \hat{z}_n^*\right) + h_n \tilde{\chi}_3^-\left(x, t, \hat{z}_n^*\right), \tag{17d}$$

where d_n, e_n and g_n are determined in terms of b_n via the symmetries, $\tilde{\chi}_3^-(x, t, z) = \chi_3^-(x, t, z)/A_{[3,4]}(z)$ is finite at $\hat{z}_o^*$, and h_n is proportional to the non-zero analytic non-principal minor $A_{\binom{1,3}{1,2}}(z_n)$. The residues of $M^\pm(x, t, z)$ at each point of Z_n can then be computed via equation (17) and expressed in terms of a single, arbitrary complex vector norming constant $\mathbf{c}_n = (c_{1,n}, c_{2,n})^T$. Finally, the solution of the VNLS equation in the reflectionless case is recovered from equation (12) as:

$$q_j(x, t) = q_{+,j} - \mathrm{i}\sum_{n=1}^{N}\left[c_{1,n}M_{2,j+1}^+(x, t, z_n) + c_{2,n}M_{3,j+1}^+(x, t, z_n)\right]\mathrm{e}^{\mathrm{i}\left(\theta_2(z_n) - \theta_1(z_n)\right)}, \tag{18}$$

for $j = 1, 2, 3$, where N is the total number of eigenvalue quartets.

4. General dark–bright soliton solutions and interaction-induced polarization shift

We now use the results presented in the previous sections to study the dark–bright soliton solutions of equation (1) and their interactions.

One-soliton solutions and dark–bright solitons. When only one quartet of discrete eigenvalues is present, letting $z_o = v_o + \mathrm{i}\eta_o = |z_o|\,\mathrm{e}^{\mathrm{i}\alpha_o}$, with $|z_o| < q_o$ and $\eta_o > 0$, one obtains a dark–bright soliton solution of the VNLS equation:

$$q_j(x, t) = -\mathrm{i}p_{j,o}\, w_o \sin\alpha_o\, \mathrm{e}^{\mathrm{i}\Phi_o}\,\mathrm{sech}\, S_o, \quad j = 1, 2, \tag{19a}$$

$$q_3(x, t) = q_+\mathrm{e}^{\mathrm{i}\alpha_o}\left(\cos\alpha_o - \mathrm{i}\sin\alpha_o \tanh S_o\right), \tag{19b}$$

where

$$w_n = \sqrt{q_o^2 - |z_n|^2}, \qquad \Phi_n = v_n x - \left(v_n^2 - \eta_n^2\right)t, \tag{20a}$$

(in this case with $n = 0$), with

$$S_o = \eta_o\left(x - 2v_o t - x_o\right), \qquad e^{\eta_o x_o} = \frac{|z_o| \| \mathbf{c}_o \|}{2\eta_o w_o}, \tag{21a}$$

and the polarization vector for the bright components is

$$\mathbf{p}_o = (p_{1,o}, p_{2,o})^T = \mathbf{c}_o / \| \mathbf{c}_o \|. \tag{21b}$$

For the dark component, the special case $\alpha_o = 0$, in which the intensity of the dark part dips all the way to zero at $x = x_o$, is referred to as a black soliton, while the generic case $\alpha_o \neq 0$ is referred to as a gray soliton.

The complex, unit-magnitude two-component vector $\mathbf{p}_o$ uniquely determines the state of polarization (SOP) of the bright part; that is, the relative amplitude and phase of the first two components of the soliton. Of course the SOP could be equivalently described by the real, unit-length three-component Stokes vector $\hat{s} = (s_1, s_2, s_3)^T$ with $s_j = \mathbf{p}_o^\dagger \sigma_j \mathbf{p}_o$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices [25]. The SOP is then identified by a point on the so-called Poincaré sphere $s_1^2 + s_2^2 + s_3^2 = 1$. Indeed, the real representation is the one most commonly used in nonlinear optics and telecommunications [26]. However, it is straightforward to go back and forth between the two representations of the SOP, and the complex representation given above will be sufficient for our purposes.

Two-soliton solutions. When two quartets Z_1 and Z_2 of discrete eigenvalues are considered, one obtains a two-soliton solution of the VNLS equation. We next discuss such kinds of solutions.

Denote the real and imaginary parts of the discrete eigenvalues z_n as $z_n = v_n + i\eta_n$ for $n = 1, 2$, with $\eta_n > 0$ and $|z_n| < q_o$. Let the two complex vector norming constants for soliton 1 and 2 be given by $\mathbf{c}_n = (c_{1n}, c_{2n})^T$ for $n = 1, 2$, respectively. Also define the coefficients

$$g = \frac{z_1 z_2^*}{\left(z_2^* - z_1\right)\left(q_o^2 - z_1 z_2^*\right)},$$
$$f_n = \frac{\left|z_n\right|^2}{2\eta_n\left(q_o^2 - \left|z_n\right|^2\right)}, \qquad n = 1, 2,$$

as well as the short-hand notations

$$\Gamma = \frac{\left|z_1 - z_2\right|^2}{4\eta_1\eta_2\left|z_1 - z_2^*\right|^2}, \qquad \Delta = \left|\mathbf{c}_1^\dagger \mathbf{c}_2\right|^2 |g|^2 - \| \mathbf{c}_1 \|^2 \| \mathbf{c}_2 \|^2 f_1 f_2 .$$

Note that $\Gamma \geqslant 0$ and $\Delta \leqslant 0$, due to the inequalities

$$\| \mathbf{c}_1 \|^2 \| \mathbf{c}_2 \|^2 \geqslant \left|\mathbf{c}_1^\dagger \mathbf{c}_2\right|^2, \qquad \left|z_1 - z_2^*\right|^2 \geqslant 4\eta_1\eta_2,$$
$$\left|q_o^2 - z_1^* z_2\right|^2 \geqslant \left(q_o^2 - \left|z_1\right|^2\right)\left(q_o^2 - \left|z_2\right|^2\right).$$

The reconstruction formula for the two-soliton solution of the three-component defocusing VNLS equation (1) then yields

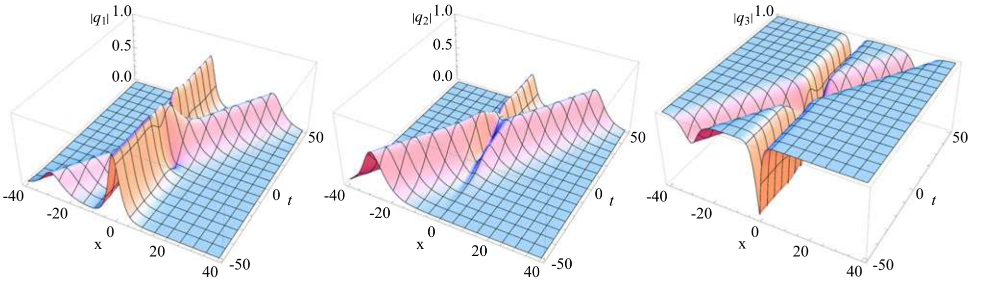

Figure 1. A two-soliton solution of the defocusing three-component VNLS equation exhibiting a polarization shift, obtained for $q_o = 1$ with $z_1 = \mathrm{i}/2$ (stationary soliton), $z_2 = (1+\mathrm{i})/4$ (moving soliton) and norming vectors $\mathbf{c}_1 = (1{,}0)^T$ and $\mathbf{c}_2 = (1{,}1+\mathrm{i}/2)^T$. Note how the bright component of soliton 1 is initially aligned exclusively with q_1, but acquires a component along q_2 as a result of the interaction.

$$\begin{aligned} q_j(x,\,t) &= c_{j,1}\frac{\mathrm{e}^{-S_1+\mathrm{i}\Phi_1}}{D}\Bigg[1 - \mathrm{i}\mathbf{c}_1^\dagger\mathbf{c}_2\frac{(z_2-z_1)g^*}{2\eta_1\left(z_2-z_1^*\right)}\,\mathrm{e}^{-S_1-S_2-i(\Phi_1-\Phi_2)} \\ &\quad + \|\,\mathbf{c}_2\,\|^2\,\frac{(z_2-z_1)f_2}{2\eta_2\left(z_2^*-z_1\right)}\,\mathrm{e}^{-2S_2}\Bigg] \\ &\quad + c_{j,2}\frac{\mathrm{e}^{-S_2+\mathrm{i}\Phi_2}}{D}\Bigg[1 + \mathrm{i}\mathbf{c}_2^\dagger\mathbf{c}_1\frac{(z_1-z_2)g}{2\eta_2\left(z_1-z_2^*\right)}\,\mathrm{e}^{-S_1-S_2+\mathrm{i}(\Phi_1-\Phi_2)} \\ &\quad + \|\,\mathbf{c}_1\,\|^2\,\frac{(z_1-z_2)f_1}{2\eta_1\left(z_1^*-z_2\right)}\,\mathrm{e}^{-2S_1}\Bigg], \end{aligned} \tag{22a}$$

for $j = 1,\,2$, together with

$$\begin{aligned} q_3(x,\,t) = q_+\Bigg(1 + \frac{1}{D}\Bigg[&\mathrm{i}\frac{\|\,\mathbf{c}_1\,\|^2}{z_1^*}f_1\,\mathrm{e}^{-2S_1} + \mathrm{i}\frac{\|\,\mathbf{c}_2\,\|^2}{z_2^*}f_2\,\mathrm{e}^{-2S_2} \\ &+ \frac{\mathbf{c}_2^\dagger\mathbf{c}_1}{z_2^*}g\,\mathrm{e}^{-S_1-S_2+\mathrm{i}(\Phi_1-\Phi_2)} - \frac{\mathbf{c}_1^\dagger\mathbf{c}_2}{z_1^*}g^*\,\mathrm{e}^{-S_1-S_2-\mathrm{i}(\Phi_1-\Phi_2)} \\ &- \frac{\mathrm{Re}(z_1z_2)\left|z_1-z_2\right|^2}{2\eta_1\eta_2z_1^*z_2^*\left|z_1-z_2^*\right|^2}\Delta\,\mathrm{e}^{-2S_1-2S_2}\Bigg]\Bigg), \end{aligned} \tag{22b}$$

with $|q_+| = q_o$ and where

$$\begin{aligned} D = 1 &+ \frac{\|\,\mathbf{c}_1\,\|^2\left|z_1\right|^2}{4\eta_1^2\left(q_o^2-\left|z_1\right|^2\right)}\,\mathrm{e}^{-2S_1} + \frac{\|\,\mathbf{c}_2\,\|^2\left|z_2\right|^2}{4\eta_2^2\left(q_o^2-\left|z_2\right|^2\right)}\,\mathrm{e}^{-2S_2} \\ &- 2\mathrm{e}^{-(S_1+S_2)}\mathrm{Re}\left(\frac{\mathbf{c}_2^\dagger\mathbf{c}_1\,z_1z_2^*}{\left(z_1-z_2^*\right)^2\left(q_o^2-z_1z_2^*\right)}\,\mathrm{e}^{\mathrm{i}(\Phi_1-\Phi_2)}\right) \\ &- \Delta\Gamma\,\mathrm{e}^{-2(S_1+S_2)}, \end{aligned} \tag{23}$$

where Φ_n and w_n are given by equation (20) for $n = 1, 2$, but now

$$S_n = \eta_n(x - 2v_n t). \tag{24}$$

Soliton interactions and polarization shift. The solution of VNLS solution in equation (22) describes a nonlinear superposition of two dark–bright solitons. As an example, figure 1 shows an interaction between a stationary dark–bright soliton (with a dark part comprised of a black soliton) and a moving dark–bright soliton (with a dark part comprised of a gray soliton).

The behavior of the solution and the effects of the interaction can be studied using a long-time asymptotic analysis similar to the one described in [28]. This analysis shows that, along the direction of S_n, as $t \to \pm\infty$, the solution takes the form of the one-soliton solution (19) but with $\mathbf{p}_n$, x_n and φ_n replaced by $\mathbf{p}_n^{\pm}$, $x_n^{\pm}$ and $\varphi_n^{\pm}$, with all these quantities expressed in terms of the discrete eigenvalues and norming constants. Explicitly

$$\mathbf{p}_1^- = \mathbf{c}_1/\| \mathbf{c}_1 \|, \qquad \mathbf{p}_2^+ = \mathbf{c}_2/\| \mathbf{c}_2 \|, \tag{25a}$$

$$\mathrm{e}^{\eta_1 x_1^-} = \frac{\| \mathbf{c}_1 \| |z_1|}{2\eta_1 w_1}, \qquad \mathrm{e}^{\eta_2 x_2^+} = \frac{\| \mathbf{c}_2 \| |z_2|}{2\eta_2 w_2}, \tag{25b}$$

and $\mathbf{p}_1^+$ and $\mathbf{p}_2^-$ expressed in terms of $\mathbf{p}_1^-$ and $\mathbf{p}_2^+$ as

$$\mathbf{p}_1^+ = \frac{1}{\chi}\left[\mathbf{p}_1^- + \frac{z_1(z_2 - z_2^*)(q_o^2 - |z_2|^2)}{z_2(z_2^* - z_1)(q_o^2 - z_1 z_2^*)}\langle \mathbf{p}_2^+, \mathbf{p}_1^- \rangle \mathbf{p}_2^+\right], \tag{25c}$$

$$\mathbf{p}_2^- = \frac{1}{\chi}\left[\mathbf{p}_2^+ + \frac{z_2(z_1 - z_1^*)(q_o^2 - |z_1|^2)}{z_1(z_1^* - z_2)(q_o^2 - z_1^* z_2)}\langle \mathbf{p}_1^-, \mathbf{p}_2^+ \rangle \mathbf{p}_1^-\right], \tag{25d}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\dagger \mathbf{b}$, with

$$\chi = 1\Big/\left[1 + 4\eta_1\eta_2\, |r|^2 w_1^2 w_2^2\, \left|\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle\right|^2\right]^{1/2}, \tag{26a}$$

$$r = 1\Big/\left[(z_1 - z_2)(q_o^2 - z_1 z_2)\right]. \tag{26b}$$

A non-trivial polarization shift for the bright components is evident in figure 1. In particular, the bright part of the stationary soliton, which is completely aligned along the first component before the interaction, acquires a nontrivial projection along the second component as a result of the interaction. A shift in the relative intensities of the projections of the moving soliton along the first two components is also clearly visible. We next quantify these effects.

Recall that the angle $\delta_{\mathbf{a},\mathbf{b}}$ between two complex two-component polarization vectors $\mathbf{a}$ and $\mathbf{b}$ is $\cos\delta_{\mathbf{a},\mathbf{b}} = |\langle \mathbf{a}, \mathbf{b} \rangle|$ [25]. Thus, the quadratic expressions $\left|\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle\right|$, $\left|\langle \mathbf{p}_1^+, \mathbf{p}_2^+ \rangle\right|$ $\left|\langle \mathbf{p}_1^+, \mathbf{p}_1^- \rangle\right|$ and $\left|\langle \mathbf{p}_2^+, \mathbf{p}_2^- \rangle\right|$ quantify respectively the degree of input copolarization between the two solitons, the degree of output copolarization between the two solitons, and the polarization shift of each of the solitons.

Using the expressions (25) obtained from the long-time aymptotics of the exact two-soliton solution (22), it is straightforward to write the output SOPs $\mathbf{p}_j^+$ in terms in terms of the input ones $\mathbf{p}_j^-$. Explicitly

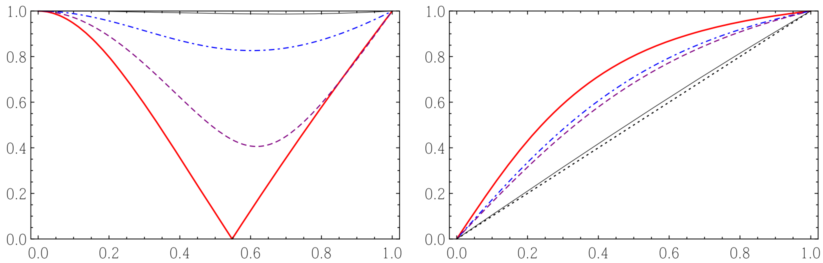

Figure 2. The polarization shift $|\langle \mathbf{p}_1^+, \mathbf{p}_1^- \rangle|$ (left) and output copolarization $|\langle \mathbf{p}_1^+, \mathbf{p}_2^+ \rangle|$ (right) as a function of the input copolarization $|\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle|$ for different combinations of soliton parameters. Red curves: $z_1 = \mathrm{i}/2$, $z_2 = \mathrm{i}/4$; blue curves: $z_1 = \mathrm{i}/2$, $z_2 = (1 + \mathrm{i})/4$; purple curves: $z_1 = (1 + \mathrm{i})/2$, $z_2 = (1 + \mathrm{i})/4$; black curves: $z_1 = (-1 + \mathrm{i})/2$, $z_2 = (1 + \mathrm{i})/2$. The input copolarization is also shown as a dotted black line in the figure to the right.

$$\mathbf{p}_1^+ = \chi\Big[\mathbf{p}_1^- - 2\mathrm{i}r^*\big(z_1^*/z_2\big)\eta_2 w_2^2 \left\langle \mathbf{p}_1^-, \mathbf{p}_2^- \right\rangle^* \mathbf{p}_2^-\Big], \tag{27a}$$

$$\mathbf{p}_2^+ = \chi\Big[\mathbf{p}_2^- - 2\mathrm{i}r\big(z_2/z_1^*\big)\eta_1 w_1^2 \left\langle \mathbf{p}_1^-, \mathbf{p}_2^- \right\rangle \mathbf{p}_1^-\Big]. \tag{27b}$$

From equations (27) we see that the SOP of each soliton always changes as a result of the interaction, unless the polarization vectors are either parallel or orthogonal to each other. The form of equations (27) is similar to that of the equations for the polarization shifts for the focusing Manakov system with ZBC [10, 27], but the dependence on the soliton parameters is of course different.

The input and output degrees of copolarization and the polarization shift can now be easily obtained from the exact expressions (27). These quantities are shown in figure 2 as a function of the input copolarization $\left|\left\langle \mathbf{p}_1^-, \mathbf{p}_2^- \right\rangle\right|$. Note how in some cases the output polarization is orthogonal to the input one. Also, note how the output copolarization is always larger than the input one, even though the interactions occur in a repulsive medium.

The long-time asymptotics also yields the position shift $x_n^+ - x_n^-$ of the solitons, which is found to be

$$\mathrm{e}^{\eta_1\left(x_1^+ - x_1^-\right)} = \mathrm{e}^{\eta_2\left(x_2^- - x_2^+\right)} = \chi \left|z_1 - z_2\right| / \left|z_1 - z_2^*\right|. \tag{28}$$

For the separation between the asymptotic soliton centers one then has

$$\Delta x = x_2^+ - x_2^- - x_1^+ + x_1^- = \frac{\eta_1 + \eta_2}{\eta_1 \eta_2}\Big[\log\Big(\left|z_1 - z_2^*\right| / \left|z_1 - z_2\right| - \log \chi\Big)\Big]. \tag{29}$$

When the polarizations of the interacting solitons are parallel or perpendicular, this expression reduces to the one found in [28] for the defocusing Manakov system with NZBC. But in general the position shift depends on the polarization vectors of the interacting solitons.

5. Discussion

The interaction-induced polarization shifts between bright solitons in focusing media is a well-known effect [10, 27] which has been experimentally observed [29]. To the best of our knowledge, however, the interaction-induced polarization shift between dark–bright solitons in defocusing media is a novel physical effect that had not previously been reported in the literature.

We emphasize that, in spite of the interaction-induced redistribution of energy between the bright components along q_1 and q_2, the total energy of each soliton and that of its bright part are both conserved, and a polarization shift is still consistent with elastic interactions, just like a position shift.

The formulae (27) for the polarization shift in the defocusing three-component VNLS equation with NZBC are similar to those for the equivalent effect in the focusing two-component VNLS equation with ZBC [10]. Also, the defocusing two-component VNLS equation with NZBC admits solutions arising from double zeros of the analytic scattering coefficients, leading to logarithmic interactions between dark–bright solitons [18]. Such solutions are not allowed in the scalar defocusing NLS equation [14], and their behavior is similar to that of double-pole solutions of the scalar focusing NLS equation with ZBC [30]. Thus, in both instances, the defocusing case of the VNLS equation with NZBC allows similar degrees of freedom as the focusing case with ZBC and one less component. It is an interesting question whether this analogy persists as the number of components increases further.

The multi-soliton solutions and the corresponding polarization interactions presented here are stable, as was demonstrated by performing careful direct numerical simulations of equation (1). Moreover, one can expect that the phenomenon will also be present for non-integrable versions of the coupled NLS equations, similarly to what happens for the interaction-induced polarization shifts in focusing media. Since scalar and VNLS equations arise in a wide variety of physical contexts, ranging from BEC to nonlinear optics, the polarization shift should therefore be a robust phenomenon that can in principle be verified experimentally.

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