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Novel systems of resonant wave interactions

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Abstract

A matrix Riemann–Hilbert problem (RHP) is constructed using the dressing method starting from two uncoupled, one-directional linear wave equations; the RHP thus obtained is then used to derive a novel integrable matrix nonlocal system of equations describing resonant wave interactions, together with its Lax pair. This system is shown to be a matrix generalization of the equations for resonant three-wave interactions and stimulated Raman scattering. Several compatible reductions admitted by this system are also discussed.

Keywords: Integrable systems, Lax pair, dressing method, Riemann–Hilbert problems, resonant interactions

1. Introduction

Infinite-dimensional, completely integrable systems continue to be the subject of considerable research. In particular, several approaches exist for obtaining nonlinear integrable partial differential equations (PDEs) starting from their linear counterpart. The method of Ablowitz, Kaup, Newell and Segur (AKNS) [2, 3] starts from the linear dispersion relation. More recently, it was shown in [7] how the dressing method, originally introduced in [18, 19], can be used to derive nonlinear systems starting from linear ones by appropriately modifying (i.e., 'dressing') a suitable Riemann–Hilbert problem (RHP) (see also [10] and [15] for further details). This was done using the results of [8], where the associated Lax pair was derived and used to solve the Cauchy problem on the infinite line for a large class of first-order linear evolution PDEs using a simpler version of the inverse scattering transform (IST). Recall that the IST was developed in the 1960s and 1970s to solve the initial value problem (IVP) for the Korteweg–deVries equation [11], the nonlinear Schrödinger equation [17] and other integrable nonlinear PDEs [3, 14].

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More precisely, the approach is the following: (i) Write the Lax pair for a given linear PDE, which for a large class of equations can be done algorithmically. (ii) Perform spectral analysis on said Lax pair to obtain an associated RHP. Note that this last step can be inverted: that is, one can recover both the Lax pair and the PDE itself starting from just the RHP, by looking at the asymptotic behavior of the solutions of the RHP. (iii) 'Dress' the RHP and perform similar steps as above on the new RHP to obtain a nonlinear PDE and its associated Lax pair. This use of the dressing method was demonstrated in [7, 15] by 'deriving' the nonlinear Schrödinger equation starting from its linear counterpart. On the other hand, the aim of this work is to show how this approach can also be used to derive *novel* integrable systems. Specifically, in this work we derive novel integrable coupled systems of nonlinear PDEs.

This paper is organized as follows. In section 2 we present the master system of equations and discuss its reductions. In section 3 we elucidate their relation to similar systems already known in the literature. Importantly, the coupled systems derived in this work are physically significant, as they describe special cases of the resonant interaction of three waves as well as stimulated Raman scattering. The physical meaning of these novel systems of equations is also discussed in section 3. A full derivation of these nonlinear systems is presented in section 4.

2. The master system of equations and its reductions

2.1. The master system

As discussed above, in section 4 we will use the dressing method to derive the following nonlocal system of equations:

$$Q_t - H_x + \left[Q, H\right] = 0, \tag{2.1a}$$

where $Q(x, t)$ is a 4×4 complex-valued matrix, subscripts x and t denote partial differentiation, $H(x, t)$ is given by

$$H_x = -\frac{1}{2}\sigma\left[J, Q_x - Q^2\right], \tag{2.1b}$$

and $[A, B] = AB - BA$ is the matrix commutator. Above, and throughout this work, we use the notation

$$\sigma = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad J = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{pmatrix}, \tag{2.2}$$

where $\mathbf{I}$ and $\mathbf{0}$ are the 2×2 identity and zero matrices, respectively, $\mathbf{C} = \mathrm{diag}(c_1, c_2)$ and c_1 and c_2 are arbitrary constants. We will also show that (2.1a) and (2.1b) are the compatibility condition of the following matrix Lax pair:

$$M_x - \mathrm{i}k\left[\sigma, M\right] = QM, \tag{2.3a}$$

$$M_t + \mathrm{i}k\left[J, M\right] = HM, \tag{2.3b}$$

with $M = M(x, t, k)$. In particular, we will show in section 4 that

$$Q(x, t) = -\mathrm{i}\lim_{k\to\infty} k\left[\sigma, M\right] = \begin{pmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{R} & \mathbf{0} \end{pmatrix}, \tag{2.4a}$$

$$H(x,\,t) = \mathrm{i}\lim_{k\to\infty} k\,[J,\,M] = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4 \end{pmatrix}, \tag{2.4b}$$

where $\mathbf{Q}(x,\,t)$, $\mathbf{R}(x,\,t)$ and $\mathbf{H}_j(x,\,t)$ for $j = 1,\ldots,4$ are all 2×2 matrices, and where the space and time dependence in the right-hand side (rhs) was omitted for brevity. For brevity, in the following we will occasionally refer to (2.1) as the '4×4 master system' or simply as the 'master system'.

The system (2.1) is non-local since one can eliminate $H(x,\,t)$ by integrating (2.1b) to obtain

$$H(x,\,t) = -\frac{1}{2}\sigma\left[J,\,Q - \int Q^2\mathrm{d}x\right]. \tag{2.5}$$

Importantly, note that $H(x,\,t)$ itself is not block off-diagonal, but the block diagonal part of H_x cancels exactly that of $[Q,\,H]$ in (2.1a). So (2.1a) can be written in a 2×2 matrix form:

$$\mathbf{Q}_t - \mathbf{H}_{2,x} + \mathbf{Q}\mathbf{H}_4 - \mathbf{H}_1\mathbf{Q} = 0, \tag{2.6a}$$

$$\mathbf{R}_t - \mathbf{H}_{3,x} + \mathbf{R}\mathbf{H}_1 - \mathbf{H}_4\mathbf{R} = 0, \tag{2.6b}$$

while the auxiliary equation (2.1b) become

$$\mathbf{H}_{1,x} = -\frac{1}{2}(\mathbf{QRC} - \mathbf{CQR}), \tag{2.7a}$$

$$\mathbf{H}_{4,x} = -\frac{1}{2}(\mathbf{RQC} - \mathbf{CRQ}), \tag{2.7b}$$

$$\mathbf{H}_{2,x} = -\frac{1}{2}\left(\mathbf{Q}_x\mathbf{C} + \mathbf{C}\mathbf{Q}_x\right), \tag{2.7c}$$

$$\mathbf{H}_{3,x} = -\frac{1}{2}\left(\mathbf{R}_x\mathbf{C} + \mathbf{C}\mathbf{R}_x\right). \tag{2.7d}$$

Correspondingly, we will refer to (2.6) and (2.7) as the '2×2 master system'. Importantly, we note that it is only necessary to keep $\mathbf{H}_1$ and $\mathbf{H}_4$ out of the four auxiliary fields $\mathbf{H}_1,\ldots,\mathbf{H}_4$, since $\mathbf{H}_{2,x}$ and $\mathbf{H}_{3,x}$ can be eliminated by direct substitution into (2.6).

2.2. Reductions

As mentioned earlier, the system (2.1) (or, equivalently, (2.6) with (2.7)) admits several consistent reductions. The first such reduction is the constraint $\mathbf{R} = \mathbf{Q}^\dagger$ (where $\mathbf{Q}^\dagger = (\mathbf{Q}^*)^\mathsf{T}$ denotes the adjoint of $\mathbf{Q}$ and the asterisk $*$ and superscript T denote respectively complex conjugation and matrix transpose), which yields

$$\mathbf{Q}_t + \frac{1}{2}\left(\mathbf{Q}_x\mathbf{C} + \mathbf{C}\mathbf{Q}_x\right) + \mathbf{Q}\mathbf{H}_4 - \mathbf{H}_1\mathbf{Q} = 0, \tag{2.8}$$

with

$$\mathbf{H}_{1,x} = -\frac{1}{2}\left(\mathbf{Q}\mathbf{Q}^\dagger\mathbf{C} - \mathbf{C}\mathbf{Q}\mathbf{Q}^\dagger\right), \qquad \mathbf{H}_{4,x} = -\frac{1}{2}\left(\mathbf{Q}^\dagger\mathbf{Q}\mathbf{C} - \mathbf{C}\mathbf{Q}^\dagger\mathbf{Q}\right), \tag{2.9}$$

where from now on c_1 and c_2 are taken to be real. More in general, one can also go back to the 2×2 master system of (2.6) and (2.7), and write it in component form. Denoting the entries of $\mathbf{Q}$ and $\mathbf{R}$ as

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} r_1 & r_3 \\ r_2 & r_4 \end{pmatrix}, \tag{2.10}$$

we obtain a system of eight scalar equations in component form:

$$q_{1,t} + c_1 q_{1,x} = \frac{1}{2}(c_1 - c_2)(m_4 q_2 + m_1 q_3), \tag{2.11a}$$

$$r_{1,t} + c_1 r_{1,x} = \frac{1}{2}(c_1 - c_2)(m_3 r_2 + m_2 r_3), \tag{2.11b}$$

$$q_{2,t} + \frac{1}{2}(c_1 + c_2) q_{2,x} = -\frac{1}{2}(c_1 - c_2)(m_3 q_1 - m_1 q_4), \tag{2.11c}$$

$$r_{2,t} + \frac{1}{2}(c_1 + c_2) r_{2,x} = -\frac{1}{2}(c_1 - c_2)(m_4 r_1 - m_2 r_4), \tag{2.11d}$$

$$q_{3,t} + \frac{1}{2}(c_1 + c_2) q_{3,x} = -\frac{1}{2}(c_1 - c_2)(m_2 q_1 - m_4 q_4), \tag{2.11e}$$

$$r_{3,t} + \frac{1}{2}(c_1 + c_2) r_{3,x} = -\frac{1}{2}(c_1 - c_2)(m_1 r_1 - m_3 r_4), \tag{2.11f}$$

$$q_{4,t} + c_2 q_{4,x} = -\frac{1}{2}(c_1 - c_2)(m_2 q_2 + m_3 q_3), \tag{2.11g}$$

$$r_{4,t} + c_2 r_{4,x} = -\frac{1}{2}(c_1 - c_2)(m_1 r_2 + m_4 r_3), \tag{2.11h}$$

together with the corresponding equations for the four auxiliary scalar fields:

$$m_{1,x} = q_1 r_3 + q_2 r_4, \tag{2.12a}$$

$$m_{2,x} = q_3 r_1 + q_4 r_2, \tag{2.12b}$$

$$m_{3,x} = q_2 r_1 + q_4 r_3, \tag{2.12c}$$

$$m_{4,x} = q_1 r_2 + q_3 r_4. \tag{2.12d}$$

Note that the terms $\frac{1}{2}(c_1 - c_2)m_1$ and $-\frac{1}{2}(c_1 - c_2)m_2$ appearing in (2.11) are the off-diagonal components of $\mathbf{H}_1$, whereas the terms $\frac{1}{2}(c_1 - c_2)m_3$ and $-\frac{1}{2}(c_1 - c_2)m_4$ are the off-diagonal components of $\mathbf{H}_4$. The off-diagonal components of $\mathbf{H}_2$ and $\mathbf{H}_3$ were explicitly taken into account in (2.11), which reduced the number of auxiliary fields from eight to four.

The reduction $\mathbf{R} = \mathbf{Q}^\dagger$ then yields $r_j = q_j^*$ for $j = 1,...,4$ as well as $m_2 = m_1^*$ and $m_4 = m_3^*$, resulting in the system

$$q_{1,t} + c_1 q_{1,x} = \frac{1}{2}(c_1 - c_2)(m_3^* q_2 + m_1 q_3), \tag{2.13a}$$

$$q_{2,t} + \frac{1}{2}(c_1 + c_2) q_{2,x} = -\frac{1}{2}(c_1 - c_2)(m_3 q_1 - m_1 q_4), \tag{2.13b}$$

$$q_{3,t} + \frac{1}{2}(c_1 + c_2) q_{3,x} = -\frac{1}{2}(c_1 - c_2)(m_1^* q_1 - m_3^* q_4), \tag{2.13c}$$

$$q_{4,t} + c_2 q_{4,x} = -\frac{1}{2}(c_1 - c_2)(m_1^* q_2 + m_3 q_3), \tag{2.13d}$$

together with the auxiliary equations

$$m_{1,x} = q_1 q_3^* + q_2 q_4^*, \qquad m_{3,x} = q_2 q_1^* + q_4 q_3^*. \tag{2.13e}$$

A further reduction can be obtained by taking $q_3 = q_4 = 0$, which simplifies the system (2.13) to

$$q_{1,t} + c_1 q_{1,x} = \frac{1}{2}(c_1 - c_2)\, m\, q_2, \tag{2.14a}$$

$$q_{2,t} + \frac{1}{2}(c_1 + c_2) q_{2,x} = -\frac{1}{2}(c_1 - c_2) m^*\, q_1, \tag{2.14b}$$

$$m_x = q_1 q_2^* \tag{2.14c}$$

(with $m = m_3^*$). An equivalent reduction is obtained by taking $q_1 = q_2 = 0$. Apparently, no other nontrivial reductions of (2.13) are possible.

One can also redefine the parameters by letting $\tilde{c}_1 = c_1$ and $\tilde{c}_2 = \frac{1}{2}(c_1 + c_2)$. Dropping tildes, the system (2.14) then becomes

$$q_{1,t} + c_1 q_{1,x} = (c_1 - c_2) m\, q_2, \tag{2.15a}$$

$$q_{2,t} + c_2 q_{2,x} = -(c_1 - c_2) m^*\, q_1, \tag{2.15b}$$

$$m_x = q_1 q_2^*. \tag{2.15c}$$

Furthermore, without loss of generality one can take $c_1 = -c_2 = c$ (e.g., by performing a Galilean transformation, i.e., by writing it in a moving coordinate frame). With the simple rescaling $u = \sqrt{2c}\, q_1$, $v = \sqrt{2c}\, q_2^*$ and $\tilde{m} = 2c\, m$, the system (2.15) then becomes (again dropping tildes)

$$u_t + c u_x = m\, v^*, \tag{2.16a}$$

$$v_t - c v_x = -m\, u^*, \tag{2.16b}$$

$$m_x = u\, v. \tag{2.16c}$$

Equations (2.16) are the simplest coupled system of equations that are a reduction of the master system (2.1). Note that, since all three fields u, v and m are complex, one does not need to assume c to be positive in obtaining (2.16) from (2.15). (Alternatively, one could also derive (2.16) when $c < 0$ via the modified change of variable $u = \sqrt{2|c|}\, q_2^*$, $v = \sqrt{2|c|}\, q_1$ and $\tilde{m} = 2|c|\, m$, the only change being that c is replaced by $|c|$ in (2.16).)

Of course the Lax pair (2.3) simplifies accordingly with each of the above reductions. In particular, the Lax pair for the system (2.16) is given by (2.3) with

$$Q = \frac{1}{\sqrt{2c}} \begin{pmatrix} 0 & 0 & u & v^* \\ 0 & 0 & 0 & 0 \\ u^* & 0 & 0 & 0 \\ v & 0 & 0 & 0 \end{pmatrix}, \quad H = \sqrt{\frac{c}{2}} \begin{pmatrix} 0 & 0 & -u & v^* \\ 0 & 0 & 0 & 0 \\ -u^* & 0 & 0 & 0 \\ v & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m^* \\ 0 & 0 & -m & 0 \end{pmatrix},$$

and $J = \mathrm{diag}(c, -3\,c, -c, 3c)$. Note that the second row and the second column of both matrices Q and H are zero. Therefore, the corresponding 4×4 Lax pair can be reduced to a

3×3 one by simply deleting the second row and the second column. Explicitly, a 3×3 Lax pair for the system (2.16) is still given by (2.3), but where now $M(x, t, k)$ is a 3×3 matrix, and

$$Q = \frac{1}{\sqrt{2c}}\begin{pmatrix} 0 & u & v^* \\ u^* & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \; H = \sqrt{\frac{c}{2}}\begin{pmatrix} 0 & -u & v^* \\ -u^* & 0 & 0 \\ v & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m^* \\ 0 & -m & 0 \end{pmatrix}, \tag{2.17}$$

together with

$$\sigma = \mathrm{diag}(1, -1, -1), \quad J = \mathrm{diag}(c, -c, 3c). \tag{2.18}$$

3. Physical significance

It should be clear that the system (2.16) describes the resonant interaction between three waves, since, when $c = 0$ (2.16) is a special case of the classical equations for sum-frequency generation [4, 13] of continuous waves. When $c \neq 0$, the system (2.16) generalizes those equations to take into account the transverse profiles of the interacting fields. Indeed, when $c \neq 0$, the system (2.16) is a special case of the three-wave interaction equations, and is closely related to the equations for simulated Raman scattering, both of which also describe resonant interactions, and both of which have been extensively studied for many years. We next briefly elaborate on these relations.

3.1. Relation to the three-wave interaction equations

In [16], Zakharov and Manakov first showed that the three-wave interaction equations could be written in Lax form, and that they were associated with a third order scattering problem. In [12], Kaup also wrote a Lax pair in AKNS form. Specifically, Kaup considered the following system of equations:

$$Q_{1,t} + c_1 Q_{1,x} = \mathrm{i}\gamma_1 Q_2^* Q_3^*, \tag{3.1a}$$

$$Q_{2,t} + c_2 Q_{2,x} = \mathrm{i}\gamma_2 Q_1^* Q_3^*, \tag{3.1b}$$

$$Q_{3,t} + c_3 Q_{3,x} = \mathrm{i}\gamma_3 Q_1^* Q_2^*, \tag{3.1c}$$

which is the one-dimensional form of the three-wave resonant interaction equations. The Lax pair associated with (3.1) is

$$v_x = -\mathrm{i}\big(V - A\zeta\big)v, \tag{3.2a}$$

$$v_t = -\mathrm{i}\big(B_0 + B_1\zeta\big)v, \tag{3.2b}$$

where ζ is the scattering parameter $A = -\mathrm{diag}(c_1, c_2, c_3)$, and the potentials are given by

$$V_{23} = Q_1\Big/\big(\beta_{12}\beta_{13}\big)^{1/2}, \quad V_{31} = Q_2\Big/\big(\beta_{12}\beta_{23}\big)^{1/2}, \quad V_{12} = Q_3\Big/\big(\beta_{13}\beta_{23}\big)^{1/2},$$

with $\beta_{ij} = c_j - c_i (i, j = 1, 2, 3)$, and other components of V_{ij} are given by the relation

$$V_{mm} = 0, \quad V_{mn} = \epsilon_m \epsilon_n V_{nm}^*,$$

where $(\epsilon_1, \epsilon_2, \epsilon_3) = (\gamma_1, -\gamma_2, \gamma_3)$. Also B_0 and B_1 are given by

$$\left(B_0\right)_{ij} = -\frac{c_1 c_2 c_3}{c_i c_j} V_{ij}, \quad \left(B_1\right)_{ij} = \delta_j^i \frac{c_1 c_2 c_3}{c_j}. \tag{3.3}$$

Note that in order for the entries of V to be real and well-defined, the velocities c_1, c_2 and c_3 must be non-degenerate and sorted in increasing order. Note also that the constraint $\gamma_1\, \gamma_2\, \gamma_3 = -1$ is needed to obtain (3.1) from (3.2).

The system (3.1) is closely related to our system (2.16). Indeed, if one and chooses the group velocities and the constants to be $c_3 = -c_1 = c$ as well as $c_2 = 0$ and $\gamma_1 = \gamma_2 = \gamma_3 = -1$, (3.1) becomes

$$Q_{1,t} - cQ_{1,x} = -\mathrm{i}Q_2^* Q_3^*, \qquad Q_{2,t} = -\mathrm{i}Q_1^* Q_3^*, \qquad Q_{3,t} + cQ_{3,x} = -\mathrm{i}Q_1^* Q_2^*.$$

Further defining $Q_1 = i\sqrt{c}\,v$, $Q_2 = im^*$ and $Q_3 = i\sqrt{c}\,u$ and performing the change of independent variables $x = c\tilde{t}$ and $t = \tilde{x}/c$, the above system then (dropping tildes) reduces to (2.16).

3.2. Relation to the equations for simulated Raman scattering

The phenomenon of stimulated Raman scattering is governed by the following system of equations (e.g., see [9]):

$$\frac{1}{c}\frac{\partial E_1}{\partial t} + \frac{\partial E_1}{\partial x} = -\mathrm{i}\frac{k_1}{k_2}\kappa_2 E_2 Q, \tag{3.4a}$$

$$\frac{1}{c}\frac{\partial E_2}{\partial t} + \frac{\partial E_2}{\partial x} = -\mathrm{i}\kappa_2 E_1 Q^*, \tag{3.4b}$$

$$\frac{\partial Q}{\partial t} + \frac{1}{T_2} Q = -\mathrm{i}\kappa_1 E_1 E_2^*. \tag{3.4c}$$

This systems is not integrable in general to the best of our knowledge. On the other hand, letting

$$\xi = \kappa_2 x, \quad \tau = \kappa_2 t - \frac{\kappa_2}{c} x, \quad A_1 = \sqrt{\frac{k_2 \kappa_1}{k_1 \kappa_2}} E_1,$$

$$A_2 = \sqrt{\frac{\kappa_1}{\kappa_2}} E_2, \quad X = \mathrm{i}\sqrt{\frac{k_1}{k_2}} Q,$$

and taking the limit $T_2 \to \infty$ (with κ_2 finite), (3.4) become the equations for the transient stimulated Raman scattering:

$$\frac{\partial A_1}{\partial \xi} = -A_2 X, \qquad \frac{\partial A_2}{\partial \xi} = A_1 X^*, \qquad \frac{\partial X}{\partial \tau} = A_1 A_2^*. \tag{3.5}$$

Furthermore, by introducing the complex-valued function $Y(\xi, \tau)$ and the real-valued function $b(\xi, \tau)$ to be

$$b = \left|A_1\right|^2 - \left|A_2\right|^2, \qquad Y = 2\mathrm{i}A_1 A_2^*, \tag{3.6}$$

the system (3.5) reduces to the following system of equations:

$$\frac{\partial b}{\partial \xi} = \mathrm{i}(X^*Y - XY^*), \qquad \frac{\partial Y}{\partial \xi} = 2\mathrm{i}bX, \qquad \frac{\partial X}{\partial \tau} = -\frac{i}{2}Y. \tag{3.7}$$

The above system admits the following Lax pair:

$$\frac{\partial \psi}{\partial \xi} = \left(-\mathrm{i}k\sigma_3 + \mathbf{X}\right)\psi, \qquad \frac{\partial \psi}{\partial \tau} = \frac{1}{4k}\left(\mathrm{i}b\sigma_3 - \mathbf{Y}\right)\psi, \tag{3.8}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix}.$$

Alternatively, if we define $u = \mathrm{i}\sqrt{c\kappa_1\kappa_2}E_1$, $v^* = \mathrm{i}\sqrt{c\kappa_1\kappa_2}E_2$ and $m = \mathrm{i}c\kappa_2 Q$ in (3.4), and take the limit $T_2 \to \infty$, the system (3.4) then becomes when $k_1 = k_2$,

$$u_t + cu_x = -mv^*, \tag{3.9a}$$

$$v_t + cv_x = mu^*, \tag{3.9b}$$

$$m_t = uv. \tag{3.9c}$$

This system is also closely related—but not equivalent—to our system of equations (2.16). In this case one can also interchange the role of x and t (to transform the temporal derivative in the third of (3.9) into a spatial derivative), but the group velocities in the first and second of (3.9) are the same, unlike in (2.16). Since the temporal and spatial derivatives have the same sign in both of the first two equations in (3.9), one can not perform a transformation to change only one of them into a different sign. Therefore, the systems (2.16) and (3.9) are not equivalent.

4. Derivation of the system and its Lax pair

We now present the detailed derivation of the master system (2.1). The derivation will proceed as follows: in section 4.1 we start from the RHP for a system of two uncoupled wave equations. By appropriately 'dressing' this RHP, we then obtain a 4×4 RHP for a matrix $M(x, t, k)$. Then, in section 4.2 we derive the 4×4 scattering problem starting from this RHP, and in section 4.3 we derive the corresponding 4×4 time evolution equation. In the process we relate the fields $Q(x, t)$ and $H(x, t)$ appearing in the resulting nonlinear system of PDEs to the asymptotic behavior of the solution $M(x, t, k)$ of the RHP as $k \to \infty$. The nonlinear system of equations obtained from this Lax pair is explicitly discussed in section 4.4, together with the more traditional representation of the Lax pair in AKNS form.

4.1. Derivation of the matrix RHP

The starting point of our analysis is the following pair of uncoupled one-directional linear wave equations:

$$q_t + c_j q_x = 0, \qquad j = 1, 2, \tag{4.1}$$

each of which is associated with the linear dispersion relation $\omega(k) = c_j k$. As shown in [7], each of these equations admits the corresponding Lax pair

$$\mu_x - \mathrm{i}k\mu = q, \tag{4.2a}$$

$$\mu_t + \mathrm{i}c_j k\mu = -c_j q, \tag{4.2b}$$

for $j = 1,2$. (That is, each of (4.1) is the compatibility condition $\mu_{xt} = \mu_{tx}$ of the corresponding overdetermined linear system (4.2).) It was shown in [8] that the spectral analysis of (4.2) leads to a scalar jump condition:

$$\mu_j^+(x, t, k) - \mu_j^-(x, t, k) = \mathrm{e}^{\mathrm{i}\theta_j(x,t,k)} f_j(k), \quad j = 1, 2, \qquad k \in \mathbb{R}, \tag{4.3}$$

for the sectionally analytic function

$$\mu_j(x, t, k) = \begin{cases} \mu_j^+(x, t, k), & k \in \mathbb{C}^+, \\ \mu_j^-(x, t, k), & k \in \mathbb{C}^-, \end{cases} \tag{4.4}$$

with the phase function

$$\theta_j(x, t, k) = kx - c_j kt, \tag{4.5}$$

for $j = 1, 2$. (Here and below, the superscripts $\pm$ denote non-tangential projection to the real k-axis from the upper/lower half of the complex k-plane.) Note also that that $\mu(x, t, k)$ satisfies the asymptotic behavior

$$\mu_j(x, t, k) = O(1/k), \qquad k \to \infty \qquad j = 1, 2. \tag{4.6}$$

The problem of reconstructing the two sectionally analytic functions $\mu_1(x, t, k)$ and $\mu_2(x, t, k)$ given the jump conditions (4.3) and the normalizations (4.6) defines two uncoupled scalar RHPs. These RHPs can be trivially combined and converted into an equivalent 4×4 matrix RHP by introducing

$$M(x, t, k) = \begin{pmatrix} \mathbf{I} & \boldsymbol{\mu} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{4.7}$$

with $\boldsymbol{\mu}(x, t, k) = \mathrm{diag}(\mu_1, \mu_2)$. The unknown $M(x, t, k)$ then satisfies the matrix jump condition

$$M^+(x, t, k) = M^-(x, t, k)V(x, t, k) \qquad k \in \mathbb{R}, \tag{4.8}$$

with jump matrix

$$V(x, t, k) = \begin{pmatrix} \mathbf{I} & \mathbf{U} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{4.9}$$

and

$$\mathbf{U}(x, t, k) = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1(x,t,k)} f_1(k), \mathrm{e}^{\mathrm{i}\theta_2(x,t,k)} f_2(k)\right).$$

The corresponding normalization condition is

$$M(x, t, k) = I_4 + O(1/k) \qquad k \to \infty, \tag{4.10}$$

where I_4 is the 4×4 identity matrix.

The key to 'nonlinearize' the system (i.e., to obtain nonlinear evolution equations starting from the above linear PDEs) is to modify the jump matrix in (4.8). In particular, following [8], we replace $V(x, t, k)$ in (4.8) with:

$$R(x, t, k) = V^\dagger(x, t, 2k)V(x, t, 2k) = \mathrm{e}^{\mathrm{i}kx\sigma - \mathrm{i}ktJ} S(2k) \mathrm{e}^{-\mathrm{i}kx\sigma + \mathrm{i}ktJ} \tag{4.11}$$

with σ and J defined by (2.2), and where

$$S(k) = \begin{pmatrix} \mathbf{I} & \mathbf{F}(k) \\ \mathbf{F}^\dagger(k) & \mathbf{I} + \mathbf{F}^\dagger(k)\mathbf{F}(k) \end{pmatrix},$$

with $\mathbf{F}(k) = \mathrm{diag}(f_1(k), f_2(k))$. That is, we consider a modified matrix RHP for the sectionally analytic function

$$M(x, t, k) = \begin{cases} M^+(x, t, k) & k \in \mathbb{C}^+, \\ M^-(x, t, k) & k \in \mathbb{C}^-, \end{cases} \tag{4.12}$$

with the modified jump condition

$$M^+(x, t, k) = M^-(x, t, k)R(x, t, k), \qquad k \in \mathbb{R}, \tag{4.13}$$

and the normalization condition

$$M\big(x, t, k\big) = I_4 + O\big(1/k\big) \qquad k \to \infty. \tag{4.14}$$

In sections 4.2 and 4.3 we will show that $M(x, t, k)$ satisfies the matrix Lax pair (2.3), with $Q(x, t)$ given by (2.4*a*) and $H(x, t)$ given by (2.4*b*). Moreover, in section 4.4 we will show that $Q(x, t)$ and $H(x, t)$ satisfy the matrix system of nonlinear PDEs (2.1).

It should be noted that the jump contour and normalization at infinity for the matrix $M(x, t, k)$ and the jump matrix $R(x, t, k)$ depend in general on the particular boundary conditions (e.g., vanishing or non-vanishing) of the solutions sought. For example, while jump conditions over $k \in \mathbb{R}$ correspond to vanishing boundary conditions for the nonlinear Schrödinger equation, solutions which tend to non-zero values at infinity are associated with jump over $\mathbb{R}\backslash(-q_\infty, q_\infty)$ or $\mathbb{R} \cup \mathrm{i}(-q_\infty, q_\infty)$ in the defocusing and focusing case, respectively, where $q_\infty = \lim_{x\to\pm\infty}|q(x, t)|$.

4.2. Derivation of the scattering problem

The main idea of the present application of the dressing method, discussed in [7], is to construct two linear differential operators L and N such that (i) LM and NM satisfy the same jump condition as M, and (ii) LM and NM are of $O(1/k)$ as $k \to \infty$. Then, by applying the vanishing lemma (see below as well as [1] for further details), we will conclude that LM and NM vanish identically. As a result, these two operators yield the Lax pair associated with the RHP.

In this section we construct the operator L, which will yield the scattering problem (2.3*a*). Then, in section 4.4 we will construct N, which yields the time evolution equation (2.3*b*). Accordingly, we will look for L to be a differential operator in x and for N to be a differential operator in t. The simplest such choice is to take $LM = M_x + L_o$, where L_o is a multiplicative linear operator (and similarly for N). Next, one must select L_o such that $LM = O(1/k)$ as $k \to \infty$. Note that the jump condition (4.13) implies $M_x = \mathrm{i}k[\sigma, M]$ as $k \to \infty$. Thus, one needs to subtract M_x as well as the $O(1)$ term from it. We call the latter $Q(x, t)M$. That is, one needs to ensure $L_o = -\mathrm{i}k[\sigma, M] - QM + O(1/k)$ as $k \to \infty$.

Based on these considerations, we therefore define

$$LM = M_x - \mathrm{i}k\,\big[\sigma, M\big] - QM. \tag{4.15}$$

We then claim that LM satisfies the same jump condition as M, namely:

$$LM^+ = \big(LM^-\big)R, \qquad k \in \mathbb{R}. \tag{4.16}$$

To check that (4.16) holds, note that the left-hand side of (4.16) equals

$$LM^+ = L\left(M^-R\right) = M_x^-R + M^-R_x - \mathrm{i}k\sigma M^-R + \mathrm{i}kM^-R\sigma - QM^-R,$$

whereas the rhs of (4.16) is

$$\left(LM^-\right)R = \left(M_x^- - \mathrm{i}k\left[\sigma, M^-\right] - QM^-\right)R = M_x^-R - \mathrm{i}k\sigma M^-R + \mathrm{i}kM^-\sigma R - QM^-R.$$

The two sides are then equal since (4.11) implies

$$R_x = \mathrm{i}k\left[\sigma, R\right] = \mathrm{i}k\sigma R - \mathrm{i}kR\sigma.$$

We next check that $LM = O(1/k)$ as $k \to \infty$. To do so, using the normalization condition (4.14) we write an asymptotic expansion for $M(x, t, k)$ as $k \to \infty$ as:

$$M(x, t, k) = I_4 + M_1(x, t)\big/k + M_2(x, t)\big/k^2 + O\left(1\big/k^3\right) \qquad k \to \infty. \tag{4.17}$$

Substituting (4.17) into (4.15) leads to

$$LM = \{-\mathrm{i}\left[\sigma, M_1\right] - Q\} + \{M_{1,x} - \mathrm{i}\left[\sigma, M_2\right] - QM_1\}\big/k + O\left(1\big/k^2\right) \qquad k \to \infty. \tag{4.18}$$

In order for LM to be $O(1/k)$, one needs the $O(1)$ term in (4.18) to be identically zero, which implies

$$Q(x, t) = -\mathrm{i}\left[\sigma, M_1\right] = -\mathrm{i}\lim_{k\to\infty} k[\sigma, M], \tag{4.19}$$

consistently with (2.4*a*). This is the condition that determines $Q(x, t)$ from the solution of the RHP.

Assuming the RHP with jump condition (4.16) and normalization condition $(LM)(\infty) = 0$ as in (4.14) has a unique solution, we then conclude that $LM = 0$ for all $k \in \mathbb{C}$, which leads to the first half of (2.3).

The uniqueness of the solutions of the RHP follows from the so-called vanishing lemma, which establishes that a RHP with zero normalization at infinity only has the trivial solution [1]. Note that the 4×4 jump condition (4.13) is essentially a tensor product of two jumps for the focusing nonlinear Schrödinger equations. Thus, the vanishing lemma for the RHP with jump condition (4.13) can be proved using the techniques of [20] under mild conditions on the functions appearing in the jump condition, which are satisfied when these functions are the spectral data arising from the direct scattering problem. A detailed analysis of this issue, however, is outside the scope of this work. Indeed, in previous applications of the dressing method the relevant RHPs were simply assumed to admit a unique solution [7, 15].

4.3. Derivation of the time dependence equation

The derivation of the time dependence equation of the Lax pair follows similar steps as for the scattering problem. Namely, we define the operator N as

$$NM = M_t + \mathrm{i}k\left[J, M\right] - HM, \tag{4.20}$$

with $H(x, t)$ to be determined. We then have

$$NM^+ = \left(NM^-\right)R, \qquad k \in \mathbb{R}. \tag{4.21}$$

Equation (4.21) can be verified in a similar way as for the scattering problem. Similarly, substituting (4.17) into (4.20) yields

$$NM = \left\{ \mathrm{i}\left[J, M_1\right] - H \right\} + \left\{ M_{1,t} + \mathrm{i}\left[J, M_2\right] - HM_1 \right\} \Big/ k + O\left(1/k^2\right) \qquad k \to \infty. \tag{4.22}$$

Since we want NM to be $O(1/k)$ as well, we again need the $O(1)$ term to vanish identically, which implies

$$H(x, t) = \mathrm{i}\left[J, M_1\right] = \mathrm{i} \lim_{k\to\infty} k\,[J, M], \tag{4.23}$$

consistently with (2.4*b*). Then, by the vanishing lemma, we conclude $NM = 0$ for all $k \in \mathbb{C}$, which leads to the second half of (2.3).

To complete the process, however, we need to relate the fields $Q(x, t)$ and $H(x, t)$ appearing in the two halves of the Lax pair. To this end, we split $M_1(x, t)$ into its block-diagonal and block-off-diagonal parts M_1^d and M_1^o , respectively. From (4.19) we have immediately:

$$M_1^o = \frac{i}{2}\sigma Q.$$

Also, recalling $LM = 0$, the $O(1/k)$ term of (4.18) yields $M_{1,x} - \mathrm{i}\,[\sigma, M_2] - QM_1 = 0$, whose block-diagonal part is

$$M_{1,x}^d = Q\left(\frac{i}{2}\sigma Q\right) = -\frac{i}{2}\sigma Q^2.$$

Combining these results we then have

$$M_1(x, t) = M_1^o + M_1^d = \frac{i}{2}\sigma Q(x, t) - \frac{i}{2}\sigma \int Q^2(x, t)\mathrm{d}x. \tag{4.24}$$

Finally, inserting (4.24) into (4.23) yields (2.5).

4.4. From the Lax pair to the nonlinear coupled PDEs

Once the Lax pair (2.3) has been explicitly derived, it is trivial to obtain the resulting nonlinear system of equations. One way to do so is to note that an equivalent (and perhaps more traditional) Lax pair can be written by letting

$$\phi(x, t, k) = M(x, t, k)\exp[\mathrm{i}k(\sigma x - Jt)], \tag{4.25}$$

which yields the overdetermined linear system

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{4.26}$$

with

$$X = \mathrm{i}k\sigma + Q, \qquad T = -\mathrm{i}kJ + H. \tag{4.27}$$

The compatibility condition $\phi_{xt} = \phi_{tx}$ of the 4×4 matrix Lax pair (4.26) is

$$X_t - T_x + \left[X, T\right] = 0. \tag{4.28}$$

It is then trivial to check that, after some straightforward calculations, (4.28) yields (2.1*a*) and (2.1*b*), as announced earlier. Obviously this equivalent 4×4 Lax pair (4.26) for the eigenfunctions $\phi(x, t, k)$ can be simplified into a 3×3 Lax pair for the system (2.16) with Q, H defined by (2.17) and σ, J defined by (2.18).

5. Concluding remarks

The results of this work open up several interesting questions regarding the nonlocal system of equations (2.1), both from a physical and a mathematical point of view.

In particular, from a mathematical point of view, the following natural questions arise: (i) Can one formulate an IST for the matrix system (2.1) to solve the IVP? (ii) Does this system admit soliton solutions? (iii) Does this system have a bilinear form? (iv) What are the associated conservation laws? (v) What is the infinite hierarchy associated with these systems? (vi) Can the unified transform method of Fokas [7] be used to solve initial-boundary value problems? (vii) Does the system admit linearizable boundary conditions? If so, what are they? (viii) Can the nonlinear steepest descent method of Deift–Venakides–Zhou [5, 6] be applied to study the long-time asymptotics?

Further interesting questions concern the relation between the reduced system (2.16), the master system (2.1) and other related integrable systems. For example: (ix) Can the equations of second-harmonic generation, the full equations for three-wave interactions and those for stimulated Raman scattering also be 'derived' using the dressing method? If so, what needs to be different to obtain those equations instead of the ones presented here? In this respect, it is interesting to note that, even though the starting point for our derivation was the linear wave equation for *two* uncoupled fields, the arrival point is a system of coupled equations for *three* interacting fields. Also, unlike the three-wave interaction equations, there seems to be no consistent reduction that allows one to reduce the number of interacting fields to two not even as a singular limit [12]. (x) Is there a more general system of resonant wave interaction that includes (2.16), (3.1) and (3.6) as special cases or reduces to them in appropriate limits? (xi) Is there a discrete analogue for all of these coupled integrable systems?

Finally, several interesting questions also arise from a physical point of view. In particular: (a) What kind of physical behavior is described by the system of equations (2.1) and its solutions? (b) Are there concrete physical systems that are governed exactly by the general system presented in this work? (c) Does the system (2.1) have a universal character (meaning that they appear in different physical settings like the nonlinear Schrödinger equation, the Korteweg–de Vries equation and the three-wave interaction equations), or are they tied to a specific physical context (like the Maxwell–Bloch equations and the equations of stimulated Raman scattering)? (d) Do the IVPs and IBVPs that are solvable by IST and its generalizations have practical relevance?

We hope that the above questions and the results of this paper will stimulate further work on these systems in the near future.

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