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Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete analogues

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Abstract

We present a class of solutions of the two-dimensional Toda lattice equation, its fully discrete analogue and its ultra-discrete limit. These solutions demonstrate the existence of soliton resonance and web-like structure in discrete integrable systems such as differential-difference equations, difference equations and cellular automata (ultra-discrete equations).

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1. Introduction

The discretization of integrable systems is an important issue in mathematical physics. The most common situation is that in which some or all of the independent variables are discretized. A discretization process in which the dependent variables are also discretized in addition to the independent variables is known as 'ultra-discretization'. One of the most important ultra-discrete soliton systems is the so-called soliton cellular automaton (SCA) [12, 16, 17]. A general method to obtain the SCA from discrete soliton equations was proposed in [6, 18] and involves using an appropriate limiting procedure. Another issue which has received renewed interest in recent years is the phenomenon of soliton resonance, which was first discovered for the Kadomtsev–Petviashvili (KP) equation [8] (see also [7, 11]). More general resonant solutions possessing a web-like structure have recently been observed in a coupled KP (cKP) system [4, 5] and for the KP equation itself [1]. In particular, the Wronskian formalism was used in [1] to classify a class of resonant solutions of KP which also satisfy the Toda lattice hierarchy. It was also conjectured in [1] that resonance and web structure are not limited to KP and cKP, but rather they are a generic feature of integrable systems whose solutions can be expressed in terms of Wronskians.

The aim of this paper is to study soliton resonance and web structure in discrete soliton systems. In particular, by studying a class of soliton solutions of the two-dimensional Toda lattice (2DTL) equation, of its fully discrete version, and of their ultra-discrete analogue which was recently introduced by Nagai *et al* [9, 10], we show that an analogue to the class of solutions studied in [1] can be defined for all three of these systems, and that a similar type of resonant solutions with web-like structure is produced as a result in all three of these systems. To our knowledge, this is the first time that resonant behaviour and web structure have been observed in discrete soliton systems. These results also confirm that soliton resonance and web-like structure are general features of two-dimensional integrable systems whose solutions can be expressed via the determinant formalism.

2. The two-dimensional Toda lattice equation

We start by considering the two-dimensional Toda lattice (2DTL) equation,

$$\frac{\partial^2}{\partial x \partial t} Q_n = V_{n+1} - 2V_n + V_{n-1}, \tag{2.1}$$

with $Q_n(x, t) = \log[1 + V_n(x, t)]$. Equation (2.1) can be written in bilinear form

$$\frac{\partial^2 \tau_n}{\partial x \partial t} \tau_n - \frac{\partial \tau_n}{\partial t} \frac{\partial \tau_n}{\partial x} = \tau_{n+1} \tau_{n-1} - \tau_n^2 \tag{2.2}$$

through the transformation

$$V_n(x, t) = \frac{\partial^2}{\partial x \partial t} \log \tau_n(x, t). \tag{2.3}$$

It is well known that some solutions of the 2DTL equation can be written via the Casorati determinant form $\tau_n = \tau_n^{(N)}$ [3], with

$$\tau_n^{(N)} = \begin{vmatrix} f_n^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ \vdots & \ddots & \vdots \\ f_n^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}, \tag{2.4}$$

where $\left\{ f_n^{(1)}(x, t), \ldots, f_n^{(N)}(x, t) \right\}$ is a set of N linearly independent solutions of the linear equations

$$\frac{\partial f_n^{(i)}}{\partial x} = f_{n+1}, \qquad \frac{\partial f_n^{(i)}}{\partial t} = -f_{n-1}, \tag{2.5}$$

for $1 \leqslant i \leqslant N$. (Note that the superscript '(i)' does not denote differentiation here.) For example, a two-soliton solution of the 2DTL is obtained by the set $\{f^{(1)}, f^{(2)}\}$, with

$$f_n^{(i)} = \mathrm{e}^{\theta_n^{(2i-1)}} + \mathrm{e}^{\theta_n^{(2i)}}, \qquad i = 1, 2, \tag{2.6}$$

where the phases $\theta^{(j)}$ are given by linear functions of (n, x, t):

$$\theta_n^{(j)}(x, t) = n \log p_j + p_j x - \frac{1}{p_j} t + \theta_0^{(j)}, \qquad j = 1, \ldots, 4, \tag{2.7}$$

with $p_1 < p_2 < p_3 < p_4$. Equation (2.6) can be extended to the N-soliton solution by considering $\{f^{(1)}, \ldots, f^{(N)}\}$, with each $f^{(i)}$ defined according to equation (2.6).

On the other hand, solutions of the 2DTL equation can also be obtained by the set of τ functions $\left\{ \tau_n^{(N)} \,\middle|\, N = 1, \ldots, M \right\}$ with the choice of f-functions,

$$f_n^{(i)} = f_{n+i-1}, \qquad 1 < i \leqslant N \leqslant M, \tag{2.8}$$

with

$$f_n = \sum_{j=1}^{M} \mathrm{e}^{\theta_n^{(j)}}, \tag{2.9}$$

and with the phases $\theta_n^{(j)}$, $1 \leqslant j \leqslant M$, still given by equation (2.7). (Note that the meaning of N and M is the opposite of [1].) If the f-functions are chosen according to equation (2.8), the τ function $\tau_n^{(N)}$ is then given by the Hankel determinant

$$\tau_n^{(N)} = \begin{vmatrix} f_n & \cdots & f_{n+N-1} \\ \vdots & \ddots & \vdots \\ f_{n+N-1} & \cdots & f_{n+2N-2} \end{vmatrix}, \tag{2.10}$$

for $1 \leqslant N \leqslant M$. It should be noted that, even when the set of functions $\left\{f_n^{(i)}\right\}_{i=1}^{N}$ is chosen according to equation (2.8), no derivatives appear in the τ function, and therefore equation (2.10) cannot be considered a Wronskian in the same sense as for the KP equation (cf equation (1.9) in [1]). Nonetheless, this choice produces a similar outcome as in the KP equation. Indeed, similar to [1], we have the following:

Lemma 2.1. *Let f_n be given by equation (2.9), with $\theta_n^{(j)}$ ($j = 1, \ldots, M$) given by equation (2.7). Then, for $1 \leqslant N \leqslant M - 1$ the τ function defined by the Hankel determinant (2.10) has the form*

$$\tau_n^{(N)} = \sum_{1 \leqslant i_1 < \cdots < i_N \leqslant M} \Delta(i_1, \ldots, i_N) \exp\left(\sum_{j=1}^{N} \theta_n^{(i_j)}\right), \tag{2.11}$$

where $\Delta(i_1, \ldots, i_N)$ is the square of the van der Monde determinant,

$$\Delta(i_1, \ldots, i_N) = \prod_{1 \leqslant j < l \leqslant N} \left(p_{i_j} - p_{i_l}\right)^2.$$

Proof. Apply the Binet–Cauchy theorem to equation (2.10), as in [1]. □

An immediate consequence of lemma 2.1 is that the τ function $\tau_n^{(N)}$ is positive definite, and therefore all the solutions generated by it are non-singular. Like its analogue in the KP equation [1], the above τ function produces soliton solutions of resonant type with web structure. More precisely, in the next section we show that, like its analogue in the KP equation, the above τ function produces an (N_-, N_+)-soliton solution, that is, a solution with $N_- = M - N$ asymptotic line solitons as $n \to -\infty$ and $N_+ = N$ asymptotic line solitons as $n \to \infty$.

Before we turn our attention to resonant solutions, however, it is useful to take a look at the one-soliton solution of the 2DTL equation. Let us introduce the function

$$w_n(x, t) = \frac{\partial}{\partial x} \log \tau_n(x, t), \tag{2.12}$$

so that the solution of the 2DTL equation is given by

$$V_n(x, t) = \frac{\partial}{\partial t} w_n(x, t). \tag{2.13}$$

If $\tau_n = \mathrm{e}^{\theta_n^{(1)}} + \mathrm{e}^{\theta_n^{(2)}}$, with $\theta_n^{(i)}$ given by equation (2.7) and $p_1 < p_2$, then w_n is given by

$$w_n = \tfrac{1}{2}(p_1 + p_2) + \tfrac{1}{2}(p_1 - p_2) \tanh \tfrac{1}{2}\left(\theta_n^{(1)} - \theta_n^{(2)}\right)$$

$$\longrightarrow \begin{cases} p_1 & \text{as} \quad x \to \infty, \\ p_2 & \text{as} \quad x \to -\infty, \end{cases}$$

which leads to the one-soliton solution of the 2DTL equation:

$$V_n = -\frac{1}{4}(p_1 - p_2)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\operatorname{sech}^2 \frac{1}{2}\left(\theta_n^{(1)} - \theta_n^{(2)}\right). \tag{2.14}$$

In the x–n plane, this solution describes a plane wave $u = \Phi(\mathbf{k}\cdot\mathbf{x} - \omega t)$ with $\mathbf{x} = (x, n)$, having wavenumber vector $\mathbf{k} = (k_x, k_n)$ and frequency ω given by

$$\mathbf{k} = (p_1 - p_2, \log p_1 - \log p_2) =: \mathbf{k}_{1,2}, \qquad \omega = \frac{1}{p_1} - \frac{1}{p_2} =: \omega_{1,2}.$$

The soliton parameters $(\mathbf{k}, \omega)$ satisfy the dispersion relation $\omega k_x + 2\cosh k_n - 2 = 0$. The above one-soliton solution (2.14) is referred to as a line soliton, since in the x–n plane it is localized around the (contour) line $\theta_n^{(1)} = \theta_n^{(2)}$. Since in this paper we are interested in the pattern of soliton solutions in the x–n plane, we will refer to $c = \mathrm{d}x/\mathrm{d}n$ as the *velocity* of the line soliton in the x direction. (That is, $c = 0$ indicates the direction of the positive n-axis.) For the soliton solution in equation (2.14), this velocity is $c_{1,2}$, where

$$c_{i,j} = -(\log p_i - \log p_j)/(p_i - p_j). \tag{2.15}$$

3. Resonance and web structure in the two-dimensional Toda lattice equation

We first consider $(N_-, 1)$-soliton solutions, i.e., solutions obtained when $N = 1$. In particular, we start with (2, 1)-soliton solutions (i.e., $N = 1$ and $M = 3$), whose τ function is given by

$$\tau_n = \mathrm{e}^{\theta_n^{(1)}} + \mathrm{e}^{\theta_n^{(2)}} + \mathrm{e}^{\theta_n^{(3)}},$$

with $p_1 < p_2 < p_3$ without loss of generality. The corresponding function w_n describes the confluence of two shocks: two shocks for $n \to -\infty$ (each corresponding to a line soliton for V_n) with velocities $c_{1,2}$ and $c_{2,3}$ merge into a single shock for $n \to \infty$ with velocity $c_{1,3}$, with $c_{i,j}$ given by equation (2.15) in all cases. This Y-shape interaction represents a resonance of three line solitons. The resonance conditions for three solitons with wavenumber vectors $\{\mathbf{k}_{i,j} \mid 1 \leqslant i < j \leqslant 3\}$ and frequencies $\{\omega_{i,j} \mid 1 \leqslant i < j \leqslant 3\}$ are given by

$$\mathbf{k}_{1,2} + \mathbf{k}_{2,3} = \mathbf{k}_{1,3}, \qquad \text{and} \qquad \omega_{1,2} + \omega_{2,3} = \omega_{1,3}, \tag{3.1}$$

which are trivially satisfied by those line solitons. We should point out that this solution is also the resonant case of the ordinary two-soliton solution of the 2DTL equation. As mentioned earlier, ordinary two-soliton solutions are given by the $N = 2\tau$ function (2.4) with (2.6). The explicit form of the $\tau_n^{(2)}$ function is

$$\tau_n^{(2)} = (p_1 - p_3)\,\mathrm{e}^{\theta_n^{(1,3)}} + (p_1 - p_4)\,\mathrm{e}^{\theta_n^{(1,4)}} + (p_2 - p_3)\,\mathrm{e}^{\theta_n^{(2,3)}} + (p_2 - p_4)\,\mathrm{e}^{\theta_n^{(2,4)}},$$

where for brevity we introduced the notation $\theta_n^{(i,j)} = \theta_n^{(i)} + \theta_n^{(j)}$, and where $\theta_n^{(i)}$ is given by equation (2.7), as before. Note that if $p_2 = p_3$, the $\tau_n^{(2)}$ function can be written as

$$\tau_n^{(2)} = \mathrm{e}^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(4)}}\left[(p_1 - p_3)\Delta\,\mathrm{e}^{-\theta_n^{(4)}} + (p_1 - p_4)\,\mathrm{e}^{-\theta_n^{(2)}} + (p_2 - p_4)\,\mathrm{e}^{-\theta_n^{(1)}}\right],$$

where $\Delta = \exp\left(\theta_0^{(3)} - \theta_0^{(2)}\right) =$ constant. Since the exponential factor $\mathrm{e}^{\theta_n^{(1)}+\theta_n^{(2)}+\theta_n^{(4)}}$ gives zero contribution to the solution $V_n = \partial_t\partial_x \log \tau_n^{(2)}$, the above $\tau_n^{(2)}$ function is equivalent to a (2, 1)-soliton solution except for the signs of the phases (more precisely, it is a (1, 2)-soliton). Note also that the condition $p_2 = p_3$ is nothing else but the resonance condition, and it describes the limiting case of an infinite phase shift in the ordinary two-soliton solution, where the phase shift between the solitons as $n \to \pm\infty$ is given by

$$\delta = (p_1 - p_3)(p_2 - p_4)/[(p_2 - p_3)(p_1 - p_4)]. \tag{3.2}$$

The resonance process for the $(N_-, 1)$-soliton solutions of the 2DTL equation can be expressed as a generalization of the confluence of shocks discussed above (cf [1]).

We next consider more general (N_-, N_+)-soliton solutions. Following [1], we can describe the asymptotic pattern of the solution in the general case $N \neq 1$ by introducing a local coordinate frame (ξ, n) in order to study the asymptotics for large $|n|$, with

$$x = cn + \xi.$$

The phase functions $\theta_n^{(i)}$ in f in equation (2.7) then become

$$\theta_n^{(i)} = p_i\xi + \eta_i(c)n + \theta_0^{(i)}, \qquad \text{for} \quad i = 1, \ldots, M,$$

with

$$\eta_i(c) := p_i(c + (1/p_i)\log p_i).$$

Without loss of generality, we assume an ordering for the parameters $\{p_i \mid i = 1, \ldots, M\}$: $0 < p_1 < p_2 < \cdots < p_M$. Then one can easily show that the lines $\eta = \eta_i(c)$ are in general position; that is, each line $\eta = \eta_i(c)$ intersects with all other lines at $M - 1$ distinct points in the c–η plane; in other words, only *two* lines meet at each intersection point.

The goal is now to find the dominant exponential terms in the $\tau_n^{(N)}$ function (2.11) for $n \to \pm\infty$ as a function of the velocity c. First note that if only one exponential is dominant, then $w_n = \partial_x \log \tau_n^{(N)}$ is just a constant, and therefore the solution $V_n = \partial_t w_n$ is zero. Then, nontrivial contributions to V_n arise when one can find *two* exponential terms which dominate over the others. Note that because the intersections of the η_i are always pairwise, three or more terms cannot make a dominant balance for large $|n|$. In the case of $(N_-, 1)$-soliton solutions, it is easy to see that at each c the dominant exponential term for $n \to \infty$ is provided by only η_1 and/or η_M, and therefore there is only one shock ($N_+ = 1$) moving with velocity $c_{1,M}$ corresponding to the intersection point of η_1 and η_M. On the other hand, as $n \to -\infty$, each term η_j can become dominant for some c, and at each intersection point $\eta_j = \eta_{j+1}$ the two exponential terms corresponding to η_j and η_{j+1} give a dominant balance; therefore there are $N_- = M - 1$ shocks moving with velocities $c_{j,j+1}$ for $j = 1, \ldots, M - 1$.

In the general case, $N \neq 1$, the $\tau_n^{(N)}$ function in (2.11) involves exponential terms having combinations of phases. In this case the exponential terms that make a dominant balance can be found using the same methods as in [1]. Let us first define the *level of intersection* of the $\eta_i(c)$. Note that $c_{i,j}$ in equation (2.15) identifies the intersection point of $\eta_i(c)$ and $\eta_j(c)$, i.e., $\eta_i(c_{i,j}) = \eta_j(c_{i,j})$.

Definition 3.1. *We define the level of intersection, denoted by $\sigma_{i,j}$, as the number of other η_l that are larger than $\eta_i(c_{i,j}) = \eta_j(c_{i,j})$ at $c = c_{i,j}$. That is,*

$$\sigma_{i,j} := |\{\eta_l \mid \eta_l(c_{i,j}) > \eta_i(c_{i,j}) = \eta_j(c_{i,j})\}|.$$

We also define $I(n)$ as the set of pairs (η_i, η_j) having the level $\sigma_{i,j} = n$, namely

$$I(n) := \{(\eta_i, \eta_j) \mid \sigma_{i,j} = n, \text{for}\, i < j\}.$$

The level of intersection lies in the range $0 \leqslant \sigma_{i,j} \leqslant M - 2$. Note also that the total number of pairs (η_i, η_j) is

$$\binom{M}{2} = \frac{1}{2}M(M-1) = \sum_{n=0}^{M-2} |I(n)|.$$

One can show that

Lemma 3.2. *The set $I(n)$ is given by*

$$I(n) = \{(\eta_i, \eta_{M-n+i-1}) \mid i = 1, \ldots, n+1\}.$$

Proof. From the assumption $q_1 < q_2 < \cdots < q_M$, we have the following inequality at $c = c_{i,j}$ (i.e. $\eta_i = \eta_j$) for $i < j$,

$$\eta_{i+1}, \ldots, \eta_{j-1} < \eta_i = \eta_j < \eta_1, \ldots, \eta_{i-1}, \eta_{j+1}, \ldots, \eta_M.$$

Then taking $j = M - n - 1$ leads to the assertion of the lemma. □

Now define $N_+ = N$ and $N_- = M - N$. The above lemma indicates that, for each intersecting pair (η_i, η_j) with the level $N_- - 1$ $(N_+ - 1)$, there are $N_+ - 1$ terms η_l which are smaller (larger) than $\eta_i = \eta_j$. Then the sum of those $N_+ - 1$ terms with either η_i or η_j provides two dominant exponents in the $\tau_n^{(N)}$ function for $n \to -\infty (n \to \infty)$ (see more detail in the proof of theorem 3.3). Note also that $|I(N_\pm - 1)| = N_\mp$. Now we can state our main theorem:

Theorem 3.3. *Let w_n be defined by equation (2.12), with $\tau_n^{(N)}$ given by equation (2.11). Then w_n has the following asymptotics for $n \to \pm\infty$:*

(i) For $n \to -\infty$ and $x = c_{i,N_+ + i} n + \xi$ for $i = 1, \ldots, N_-$,

$$w_n \longrightarrow \begin{cases} K_i(+,-) := \sum_{j=i+1}^{N+i} p_j & \text{as} \quad \xi \to \infty, \\ K_i(-,-) := \sum_{j=i}^{N+i-1} p_j & \text{as} \quad \xi \to -\infty. \end{cases}$$

(ii) For $n \to \infty$ and $x = c_{i,N_- + i} n + \xi$ for $i = 1, \ldots, N_+$,

$$w_n \longrightarrow \begin{cases} K_i(+,+) := \sum_{j=1}^{i-1} p_j + \sum_{j=1}^{N-i+1} p_{M-j+1} & \text{as} \quad \xi \to \infty, \\ K_i(-,+) := \sum_{j=1}^{i} p_j + \sum_{j=1}^{N-1} p_{M-j+i} & \text{as} \quad \xi \to -\infty. \end{cases}$$

where $c_{i,j}$ is given by equation (2.15).

Proof. First note that at the point $\eta_i = \eta_{N+i}$, i.e., $(\eta_i, \eta_{N+i}) \in I(N_- - 1)$, from lemma 3.2 we have the inequality,

$$\underbrace{\eta_{i+1}, \eta_{i+2}, \ldots, \eta_{i+N-1}}_{N-1} < \eta_i = \eta_{N+i}.$$

This implies that, for $c = -(\log p_{N+i} - \log p_i)/(p_{N+i} - p_i)$, the following two exponential terms in the $\tau_n^{(N)}$ function in lemma 2.1,

$$\exp\left(\sum_{j=i}^{N+i-1} \theta_n^{(j)}\right), \qquad \exp\left(\sum_{j=i+1}^{N+i} \theta_n^{(j)}\right),$$

provide the dominant terms for $n \to -\infty$. Note that the condition $\eta_i = \eta_{N+i}$ leads to $c = c_{i,N+i} = -(\log p_{N+i} - \log p_i)/(p_{N+i} - p_i)$. Thus the function w_n can be approximated by the following form along $x = c_{i,N+i} n + \xi$ for $n \to -\infty$:

$$\begin{aligned} w_n &\sim \frac{\partial}{\partial \xi} \log(\Delta_i(-,-)\, \mathrm{e}^{K_i(-,-)\xi} + \Delta_i(+,-)\, \mathrm{e}^{K_i(+,-)\xi}) \\ &= \frac{K_i(-,-)\Delta_i(-,-)\, \mathrm{e}^{K_i(-,-)\xi} + K_i(+,-)\Delta_i(+,-)\, \mathrm{e}^{K_i(+,-)\xi}}{\Delta_i(-,-)\, \mathrm{e}^{K_i(-,-)\xi} + \Delta_i(+,-)\, \mathrm{e}^{K_i(+,-)\xi}}, \\ &= \frac{K_i(-,-)\Delta_i(-,-)\, \mathrm{e}^{(p_i - p_{N+i})\xi} + K_i(+,-)\Delta_i(+,-)}{\Delta_i(-,-)\, \mathrm{e}^{(p_i - p_{N+i})\xi} + \Delta_i(+,-)}, \end{aligned}$$

where

$$\Delta_i(-,-) = \Delta(i,\ldots,N+i-1)\exp\left(\sum_{j=i}^{N+i-1}\theta_0^{(j)}\right)$$

$$\Delta_i(+,-) = \Delta(i+1,\ldots,N+i)\exp\left(\sum_{j=i+1}^{N+i}\theta_0^{(j)}\right).$$

Now, from $p_i < p_{N+i}$ it is obvious that w_n has the desired asymptotics as $\xi \to \pm\infty$ for $n \to -\infty$.

Similarly, for the case of $(\eta_i, \eta_{N_-+i}) \in I(N_+ - 1)$ we have the inequality

$$\eta_i = \eta_{N_-+i} < \underbrace{\eta_1, \eta_2, \ldots, \eta_{i-1}, \eta_{N_-+i+1}\ldots, \eta_M}_{N-1}.$$

Then the dominant terms in the $\tau_n^{(N)}$ function on $x = c_{i,N_-+i}n + \xi$ for $n \to \infty$ are given by the exponential terms

$$\exp\left(\sum_{j=1}^{i}\theta_n^{(j)} + \sum_{j=1}^{N-i}\theta_n^{(M-j+1)}\right), \qquad \exp\left(\sum_{j=1}^{i-1}\theta_n^{(j)} + \sum_{j=1}^{N-i+1}\theta_n^{(M-j+1)}\right).$$

Then, following the same argument as before, we obtain the desired asymptotics as $\xi \to \pm\infty$ for $n \to \infty$.

For other values of c, that is for $c \neq c_{i,N_++i}$ and $c \neq c_{i,N_-+i}$, just one exponential term is dominant, and thus w_n approaches a constant as $|n| \to \infty$. This completes the proof. $\square$

Theorem 3.3 determines the complete structure of asymptotic patterns of the solutions $V_n(x,t)$ given by (2.10). Indeed, theorem 3.3 can be summarized as follows: as $n \to -\infty$, the function w_n has N_- jumps, moving with velocities c_{j,N_++j} for $j = 1,\ldots,N_-$; as $n \to \infty$, w_n has N_+ jumps, moving with velocities c_{i,N_-+i} for $i = 1,\ldots,N_+$. Since each jump represents a line soliton for $V_n(x,t)$, the whole solution therefore represents an (N_-, N_+)-soliton. The velocity of each of the asymptotic line solitons in the (N_-, N_+)-soliton is determined from the c–η graph of the levels of intersections. As an example, in figure 1 we show a (2, 1)-soliton solution (also called a Y-shape solution, or a Y-junction), a (2, 2)-soliton solution, a (2, 3)-soliton solution and a (3, 3)-soliton solution.

Note that, given a set of M phases (as determined by the parameters p_i for $i = 1,\ldots,M$), the same graph can be used for any (N_-, N_+)-soliton with $N_- + N_+ = M$. In particular, if $M = 2N$, we have $N_+ = N_- = N$, and theorem 3.3 implies that the velocities of the N incoming solitons are equal to those of the N outgoing solitons. In the case of the ordinary multi-soliton solution of the 2DTL equation, the τ function (2.4) does not contain all the possible combinations of phases and therefore the theorem should be modified. However, the key idea of using the levels of intersection for the asymptotic analysis is still applicable. In fact, by considering the $\tau_n^{(N)}$ function given by the Casorati determinant (2.4) with $f_i = \mathrm{e}^{\theta_n^{(2i-1)}} + \mathrm{e}^{\theta_n^{(2i)}}$ for $i = 1,\ldots,N$ and $p_1 < p_2 < \cdots < p_{2N}$, one can find the asymptotic velocities for the ordinary N-soliton solutions as $c_{2i-1,2i} = -(\log p_{2i} - \log p_{2i-1})/(p_{2i} - p_{2i-1})$. Note that these velocities are different from those of the resonant N-soliton solutions.

Note also that even when $N_+ = N_- = N$, the interaction pattern of resonant soliton solutions differs from that of ordinary N-soliton solutions. As seen from figure 1, the resonant solutions of the 2DTL obtained from equation (2.10) are very similar to the solitons of the KP and coupled KP equation [1, 4, 5], where such solutions were called 'spider-web' solitons.

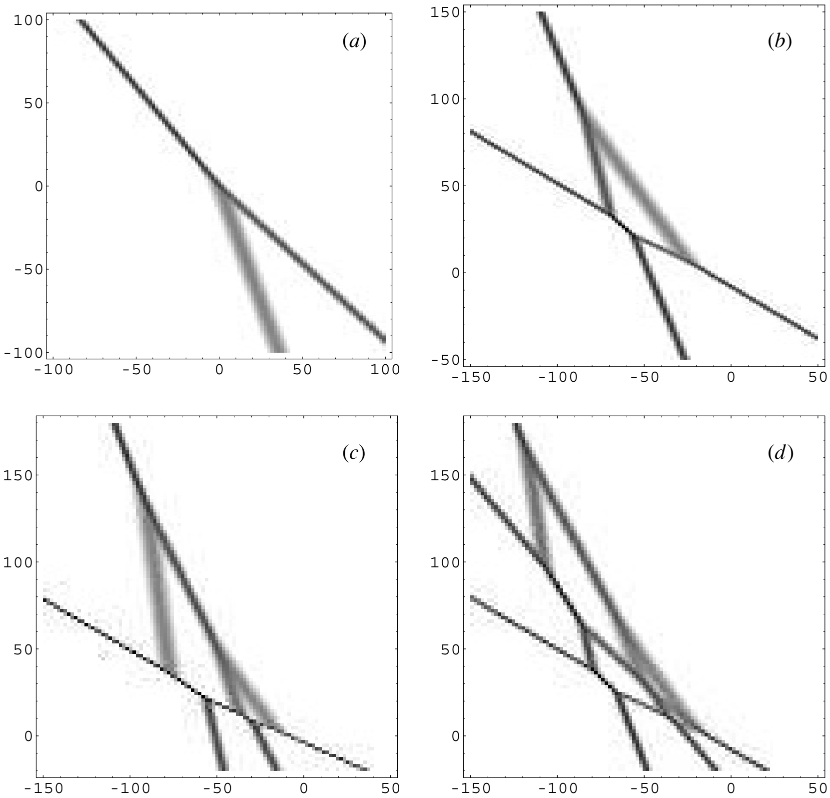

Figure 1. Resonant solutions of the two-dimensional Toda lattice: (*a*) (2, 1)-soliton solution (i.e., a Y-junction) at $t = 0$, with $N = 1$, $M = 3$, $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 2$; (*b*) (2, 2)-soliton solution at $t = 14$, with $N = 2$, $M = 4$, $p_1 = 1/8$, $p_2 = 1/2$, $p_3 = 1$ and $p_4 = 4$; (*c*) (3, 2)-soliton solution at $t = 10$, with $N = 2$, $M = 5$, $p_1 = 1/10$, $p_2 = 1/5$, $p_3 = 1/2$, $p_4 = 1$ and $p_5 = 6$; (*d*) (3, 3)-soliton solution at $t = 14$, with $N = 3$, $M = 6$, $p_1 = 1/10$, $p_2 = 1/4$, $p_3 = 1/2$ and $p_4 = 1$, $p_5 = 2$, $p_6 = 4$. In all cases the horizontal axis is n and the vertical axis is x, and each figure is a plot of $q(n, x, t)$ in logarithmic greyscale. Note that the values of n in the horizontal axis are discrete.

(In contrast, an ordinary N-soliton solution produces a simple pattern of N intersecting lines.) The web structure manifests itself in the number of bounded regions, the number of vertices and the number of intermediate solitons, which are respectively $(N_- - 1)(N_+ - 1)$, $2N_-N_+ - M$ and $3N_-N_+ - 2M$ for an (N_-, N_+)-soliton solution [1]. (In contrast, an ordinary N-soliton solution has $(N-1)(N-2)/2$ bounded regions and $N(N-1)/2$ interaction vertices.) Finally, it should be noted that, as in the KP equation, only the Y-shape solution is a travelling wave solution. All other resonant solutions (as well as ordinary N-soliton solutions with $N \geqslant 3$) have a time-dependent shape, as shown in [1].

4. The fully discrete 2D Toda lattice equation

The 2DTL equation (2.1) is a differential-difference evolution equation, since only one of the independent variables is discrete, while the other two are continuous. Hereafter, we refer to equation (2.1) as a semi-continuous case. We now consider a fully discrete analogue of the

2DTL equation (2.1), namely

$$\begin{aligned}&\Delta_l^+\Delta_m^- Q_{l,m,n} = V_{l,m-1,n+1} - V_{l+1,m-1,n} - V_{l,m,n} + V_{l+1,m,n-1},\\ &V_{l,m,n} = (\delta\kappa)^{-1}\log[1+\delta\kappa(\exp Q_{l,m,n}-1)],\end{aligned} \tag{4.1}$$

with $l, m, n \in \mathbb{Z}$, l and m being the discrete analogues of the time t and space x coordinates, respectively, and where Δ_l^+ and Δ_m^- are the forward and backward difference operators defined by

$$\Delta_l^+ f_{l,m,n} = \frac{f_{l+1,m,n} - f_{l,m,n}}{\delta}, \tag{4.2}$$

$$\Delta_m^- f_{l,m,n} = \frac{f_{l,m,n} - f_{l,m-1,n}}{\kappa}. \tag{4.3}$$

Equation (4.1), which is the discrete analogue of equation (2.1), can be written in bilinear form [2] in a manner similar to equation (2.2):

$$\left(\Delta_l^+\Delta_m^-\tau_{l,m,n}\right)\tau_{l,m,n} - \left(\Delta_l^+\tau_{l,m,n}\right)\Delta_m^-\tau_{l,m,n} = \tau_{l,m-1,n+1}\tau_{l+1,m,n-1} - \tau_{l+1,m-1,n}\tau_{l,m,n}, \tag{4.4}$$

with $Q_{l,m,n}$ related to $\tau_{l,m,n}$ by the transformation $V_{l,m,n} = \Delta_l^+\Delta_m^-\log\tau_{l,m,n}$, i.e.,

$$Q_{l,m,n} = \log\frac{\tau_{l+1,m+1,n-1}\tau_{l,m,n+1}}{\tau_{l+1,m,n}\tau_{l,m+1,n}}. \tag{4.5}$$

Note that $Q_{l,m,n} = \log[1 + (\mathrm{e}^{\delta\kappa V_{l,m,n}} - 1)/\delta\kappa]$. Special solutions of equation (4.4) (which is the discrete analogue of equation (2.2)) are obtained when the τ function $\tau_{l,m,n}$ is expressed in terms of a Casorati determinant $\tau_{l,m,n} = \tau_{l,m,n}^{(N)}$ as [2]

$$\tau_{l,m,n}^{(N)} = \begin{vmatrix} f_{l,m,n}^{(1)} & f_{l,m,n+1}^{(1)} & \cdots & f_{l,m,n+N-1}^{(1)} \\ f_{l,m,n}^{(2)} & f_{l,m,n+1}^{(2)} & \cdots & f_{l,m,n+N-1}^{(2)} \\ \vdots & \vdots & & \vdots \\ f_{l,m,n}^{(N)} & f_{l,m,n+1}^{(N)} & \cdots & f_{l,m,n+N-1}^{(N)} \end{vmatrix}, \tag{4.6}$$

where each of the functions $\left\{f_{l,m,n}^{(i)}, i = 1, 2, \ldots, N\right\}$ satisfies the following discrete dispersion relations:

$$\Delta_l^+ f_{l,m,n} = f_{l,m,n+1}, \tag{4.7}$$

$$\Delta_m^- f_{l,m,n} = -f_{l,m,n-1}. \tag{4.8}$$

If we take as a solution for equations (4.7) and (4.8) the functions

$$f_{l,m,n}^{(i)} = \phi(p_i) + \phi(q_i), \tag{4.9}$$

with

$$\phi(p) = p^n(1+\delta p)^l(1+\kappa p^{-1})^{-m}, \tag{4.10}$$

the τ function (4.6) yields an N-soliton solution for the discrete 2DTL equation (4.4).

As in the semi-continuous 2DTL, however, solutions of equation (4.4) can also be obtained when we consider the τ function defined by the Hankel determinant

$$\tau_{l,m,n}^{(N)} = \begin{vmatrix} f_{l,m,n} & f_{l,m,n+1} & \cdots & f_{l,m,n+N-1} \\ f_{l,m,n+1} & f_{l,m,n+2} & \cdots & f_{l,m,n+N} \\ \vdots & \vdots & & \vdots \\ f_{l,m,n+N-1} & f_{l,m,n+N} & \cdots & f_{l,m,n+2N-2} \end{vmatrix}, \tag{4.11}$$

where

$$f_{l,m,n} = \sum_{i=1}^{M} \alpha_i \phi(p_i), \tag{4.12}$$

which corresponds to choosing

$$f^{(i)}_{l,m,n} = f_{l,m,n+i-1}, \tag{4.13}$$

for $i = 1, \dots, N$. Without loss of generality, we can label the parameters p_i so that $0 < p_1 < p_2 < \cdots < p_{M-1} < p_M$. Then, as in the semi-continuous 2DTL, we have the following:

Lemma 4.1. *Let $f_{l,m,n}$ be given by equation (4.13). Then, for $1 \leqslant N \leqslant M-1$, the τ function defined by the Hankel determinant (4.11) has the form*

$$\tau^{(N)}_n = \sum_{1\leqslant i_1<\cdots<i_N\leqslant M} \Delta(i_1, \dots, i_N) \prod_{j=1}^{N} \alpha_{i_j} \phi\big(p_{i_j}\big) \tag{4.14}$$

where $\Delta(i_1, \dots, i_N)$ is the square of the van der Monde determinant,

$$\Delta(i_1, \dots, i_N) = \prod_{1\leqslant j<l\leqslant N} \big(p_{i_j} - p_{i_l}\big)^2. \tag{4.15}$$

Proof. Again, the result follows by applying the Binet–Cauchy theorem to the Hankel determinant (4.11). □

Unlike its counterpart in the semi-continuous 2DTL equation, the τ function in equation (4.11) cannot be written in terms of a Wronskian, since no derivatives appear. However, as in the semi-continuous 2DTL equation, the τ function thus defined is positive definite, and therefore all the solutions generated by it are non-singular. In the next section we show that, like its analogue in the semi-continuous 2DTL equation, the above τ function produces soliton solutions of resonant type with web structure, and we conjecture that, like in the continuous case, an (N_-, N_+)-soliton with $N_- = M - N$ and $N_+ = N$ is created.

Like with the semi-continuous 2DTL equation, however, before discussing resonant solutions it is convenient to first look at one-soliton solutions of the fully discrete 2DTL equation. Let us introduce the analogue of equation (2.12), namely the function

$$w_{l,m,n} = \log \frac{\tau_{l,m+1,n-1}}{\tau_{l,m,n}}, \tag{4.16}$$

so that the solution of the discrete 2DTL equation is given by

$$Q_{l,m,n} = \log \frac{\tau_{l+1,m+1,n-1}\tau_{l,m,n+1}}{\tau_{l+1,m,n}\tau_{l,m+1,n}} = w_{l+1,m,n} - w_{l,m,n+1}. \tag{4.17}$$

It is also useful to rewrite the function $\phi(p_i)$ in equations (4.10), (4.12) as $\phi(p_i) = \mathrm{e}^{\theta^{(i)}_{l,m,n}}$, where

$$\theta^{(i)}_{l,m,n} = n \log p_i - m \log\big(1 + \kappa p_i^{-1}\big) + l \log(1 + \delta p_i) + \theta^{(i)}_0. \tag{4.18}$$

If $\tau_n = \mathrm{e}^{\theta^{(1)}_{l,m,n}} + \mathrm{e}^{\theta^{(2)}_{l,m,n}}$, with $p_1 < p_2$, then $w_{l,m,n}$ is given by

$$\begin{aligned} w_{l,m,n} &= \log \tfrac{1}{2}\Big(p_1^{-1}\big(1 + \kappa p_1^{-1}\big)^{-1} + p_2^{-1}\big(1 + \kappa p_2^{-1}\big)^{-1}\Big) \\ &\quad + \tfrac{1}{2}\Big(p_1^{-1}\big(1 + \kappa p_1^{-1}\big)^{-1} - p_2^{-1}\big(1 + \kappa p_2^{-1}\big)^{-1}\Big) \tanh \tfrac{1}{2}\big(\theta^{(1)}_{l,m,n} - \theta^{(2)}_{l,m,n}\big) \\ &\longrightarrow \begin{cases} -\log p_1 - \log\big(1 + \kappa p_1^{-1}\big) & \text{as} \quad n \to \infty, \\ -\log p_2 - \log\big(1 + \kappa p_2^{-1}\big) & \text{as} \quad n \to -\infty, \end{cases} \end{aligned}$$

which leads to the one-soliton solution of the discrete 2DTL equation. In the n–m plane, this solution describes a plane wave $\exp(Q_{l,m,n}) = \Phi(\mathbf{k}\cdot\mathbf{x} - \omega l)$ with $\mathbf{x} = (n, m)$, having wavenumber vector $\mathbf{k} = (k_n, k_m)$ and frequency ω given by

$$\mathbf{k} = \left(\log p_1 - \log p_2, -\log\left(1 + \kappa p_1^{-1}\right) + \log\left(1 + \kappa p_2^{-1}\right)\right) =: \mathbf{k}_{1,2},$$
$$\omega = -\log(1 + \delta p_1) + \log(1 + \delta p_2) =: \omega_{1,2}.$$

The soliton parameters $(\mathbf{k}, \omega)$ now satisfy the discrete dispersion relation $(e^{-\omega} - 1)(1 - e^{-k_m}) = \delta\kappa(e^{-k_m+k_n} + e^{-\omega-k_n} - e^{-\omega-k_m} - 1)$. The one-soliton solution (4.17) is referred to as a line soliton since, like its semi-continuous analogue, it is localized around the (contour) line $\theta^{(1)}_{l,m,n} = \theta^{(2)}_{l,m,n}$ in the n–m plane. Again, we will refer to $c = \mathrm{d}n/\mathrm{d}m$ as the velocity of the line soliton in the n direction. For the above line soliton solution, this velocity is given by $c_{1,2}$, where now

$$c_{i,j} = \left(\log\left(1 + \kappa p_i^{-1}\right) - \log\left(1 + \kappa p_j^{-1}\right)\right)\big/(\log p_i - \log p_j). \tag{4.19}$$

5. Resonance and web structure in the discrete 2D Toda lattice equation

As in the semi-continuous case, we first consider $(N_-, 1)$-soliton solutions, i.e., solutions obtained in the case $N = 1$, and in particular we start from (2, 1)-soliton solutions (i.e., the case $N = 1$ and $M = 3$), whose τ function is given by

$$\tau_n = e^{\theta^{(1)}_{l,m,n}} + e^{\theta^{(2)}_{l,m,n}} + e^{\theta^{(3)}_{l,m,n}},$$

with $\theta^{(i)}_{l,m,n}$ given by equation (4.18), and where $p_1 < p_2 < p_3$ without loss of generality. As in the continuous case, this solution describes the confluence of two shocks: two shocks for $m \to -\infty$ (each corresponding to a line soliton for $Q_{l,m,n}$) with velocities $c_{1,2}$ and $c_{1,3}$ merge into a single shock for $m \to \infty$ with velocity $c_{1,3}$, where $c_{i,j}$ is given by equation (4.19) in all cases. This Y-shape interaction represents a resonance of three line solitons. The resonance conditions for three solitons with the wavenumber vectors $\{\mathbf{k}_{i,j} \mid 1 \leqslant i < j \leqslant 3\}$ and the frequencies $\{\omega_{i,j} \mid 1 \leqslant i < j \leqslant 3\}$ are still given by equation (3.1), and again are trivially satisfied by those line solitons. Furthermore, this solution is also the resonant case of the ordinary two-soliton solution of the discrete 2DTL equation, arising in the limit of an infinite phase shift. The resonance process for the $(N_-, 1)$-soliton solutions of the discrete 2DTL equation can be expressed as a generalization of the confluence of shocks discussed earlier.

Next we consider more general (N_-, N_+)-soliton solutions. Following [1] and the semi-continuous case, we can describe the asymptotic pattern of the solution by introducing a local coordinate frame (ξ, m) in order to study the asymptotics for large $|m|$ with

$$n = cm + \xi.$$

Then the phase functions $\theta^{(i)}_{l,m,n}$ become

$$\theta^{(i)}_{l,m,n} = \xi \log p_i + \eta_i(c)m + \theta^{(i)}_0, \qquad \text{for} \quad i = 1, \ldots, M,$$

with

$$\eta_i(c) := \log p_i\left(c - \log\left(1 + \kappa p_i^{-1}\right)\big/\log p_i\right).$$

Without loss of generality, we assume an ordering for the parameters $\{p_i \mid i = 1, \ldots, M\}$: $0 < p_1 < p_2 < \cdots < p_M$. Then, as in the semi-continuous case, one can easily show that the lines $\eta = \eta_i(c)$ are in general position. As before, the goal is then to find the dominant exponential terms in the $\tau^{(N)}_{l,m,n}$ function (4.14) for $m \to \pm\infty$ as a function of the velocity c. First note that if only one exponential is dominant, then $w_{l,m,n} = \log\left(\tau^{(N)}_{l,m+1,n-1}\big/\tau^{(N)}_{l,m,n}\right)$ is just a constant, and therefore the solution $Q_{l,m,n} = w_{l+1,m,n} - w_{l,m,n+1}$ is zero. Then, as in the semi-continuous case, nontrivial contributions to $Q_{l,m,n}$ arise when one can find *two* exponential terms which dominate over the others. Also, as in the semi-continuous case, since the intersections of the η_i are always pairwise, three or more terms cannot make a dominant balance for large $|m|$. For $(N_-, 1)$-soliton solutions, it is easy to see that at each c the dominant exponential term for $m \to \infty$ is provided by only η_1 and/or η_M, and therefore there is only one shock ($N = 1$) moving with velocity $c_{1,M}$ corresponding to the intersection point of η_1 and η_M. On the other hand, as $m \to -\infty$, each term η_j can become dominant for some c, and at each intersection point $\eta_j = \eta_{j+1}$ the two exponential terms corresponding to η_j and η_{j+1} give a dominant balance; therefore there are $N_- = M - 1$ shocks moving with velocities $c_{j,j+1}$ for $j = 1, \ldots, M - 1$.

In the general case, $N \neq 1$, the $\tau^{(N)}_{l,m,n}$ function in (4.14) involves exponential terms having combinations of phases, and two exponential terms that make a dominant balance, can be found in a similar way as in the semi-continuous case. We define again the *level of intersection* of the $\eta_i(c)$. Again, $c_{i,j}$ in equation (4.19) identifies the intersection point of $\eta_i(c)$ and $\eta_j(c)$, i.e., $\eta_i(c_{i,j}) = \eta_j(c_{i,j})$.

Definition 5.1. *We define the level of intersection, denoted by $\sigma_{i,j}$, as the number of other η_l that at $c = c_{i,j}$ are larger than $\eta_i(c_{i,j}) = \eta_j(c_{i,j})$. That is,*

$$\sigma_{i,j} := |\{\eta_l \mid \eta_l(c_{i,j}) > \eta_i(c_{i,j}) = \eta_j(c_{i,j})\}|.$$

We also define $I(n)$ as the set of pairs (η_i, η_j) having the level $\sigma_{i,j} = n$, namely

$$I(n) := \{(\eta_i, \eta_j) \mid \sigma_{i,j} = n, \text{for } i < j\}.$$

As in the semi-continuous case, one can then show the following:

Lemma 5.2. *The set $I(n)$ is given by*

$$I(n) = \{(\eta_i, \eta_{M-n+i-1}) \mid i = 1, \ldots, n+1\}.$$

Proof. See the proof of lemma 3.2. □

As in the semi-continuous case, let $N_+ = N$ and $N_- = M - N$. The above lemma indicates that, for each intersecting pair (η_i, η_j) with the level $N_- - 1$ $(N - 1)$, there are $N_+ - 1$ terms η_l which are smaller (larger) than $\eta_i = \eta_j$. Then the sum of those $N_+ - 1$ terms with either η_i or η_j provides two dominant exponents in the $\tau^{(N)}_{l,m,n}$ function for $m \to -\infty (m \to \infty)$. We then have the following:

Theorem 5.3. *Let $w_{l,m,n}$ be a function defined by equation (4.16), with $\tau^{(N)}_{l,m,n}$ given by equation (4.14). Then $w_{l,m,n}$ has the following asymptotics for $m \to \pm\infty$:*

(i) For $m \to -\infty$ and $n = c_{i,N_++i}m + \xi$ for $i = 1, \ldots, N_-$,

$$w_{l,m,n} \longrightarrow \begin{cases} K_i(+,-) := \sum_{j=i+1}^{N+i} \log p_j & \text{as} \quad \xi \to \infty, \\ K_i(-,-) := \sum_{j=i}^{N+i-1} \log p_j & \text{as} \quad \xi \to -\infty. \end{cases}$$

(ii) For $m \to \infty$ and $n = c_{i,N_-+i}m + \xi$ for $i = 1, \ldots, N_+$,

$$w_{l,m,n} \longrightarrow \begin{cases} K_i(+,+) := \sum_{j=1}^{i-1} \log p_j + \sum_{j=1}^{N-i+1} \log p_{M-j+1} & \text{as} \quad \xi \to \infty, \\ K_i(-,+) := \sum_{j=1}^{i} \log p_j + \sum_{j=1}^{N-1} \log p_{M-j+i} & \text{as} \quad \xi \to -\infty, \end{cases}$$

where $c_{i,j}$ is given by equation (4.19).

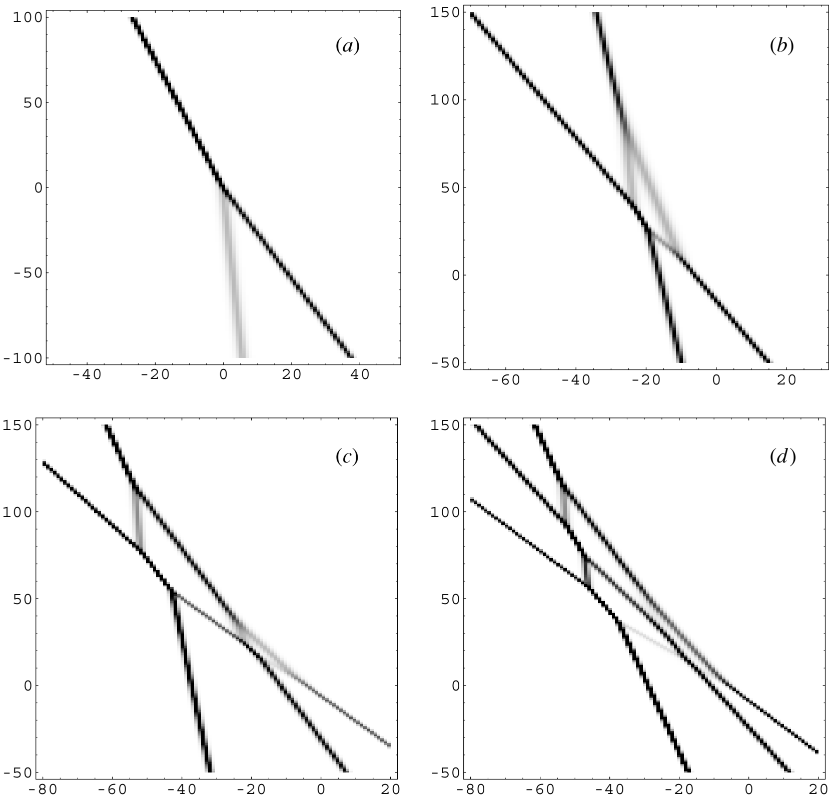

Figure 2. Resonant solutions of the fully discrete two-dimensional Toda lattice: (*a*) (2, 1)-soliton solution (i.e., a Y-junction) at $l = 0$, with $p_1 = 1/10$, $p_2 = 1/2$, $p_3 = 10$, (*b*) (2, 2)-soliton solution at $l = 40$, with $p_1 = 1/10$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 15$; (*c*) (3, 2)-soliton solution at $l = 80$, with $p_1 = 1/20$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 10$, $p_5 = 60$; (*d*) (3, 3)-soliton solution at $l = 80$, with $p_1 = 1/20$, $p_2 = 1/2$, $p_3 = 2$, $p_4 = 10$, $p_5 = 20$, $p_6 = 120$. In all cases $\delta = \kappa = 1/4$; the horizontal axis is n and the vertical axis is n and each figure is a plot of $Q_{l,m,n}$ in logarithmic greyscale. Note that the values of both m and n in the horizontal and vertical axes are discrete.

Proof. Once the obvious modifications are made, the proof proceeds exactly like in the semi-continuous case, namely theorem 3.3. □

Like its counterpart in the semi-continuous case, theorem 5.3 determines the complete structure of asymptotic patterns of the solutions $Q_{l,m,n}$ given by (4.11). Indeed, theorem 5.3 can be summarized as follows: as $m \to -\infty$, the function $w_{l,m,n}$ has N_- jumps, moving with velocities c_{j,N_++j} for $j = 1, \ldots, N_-$; as $m \to \infty$, $w_{l,m,n}$ has N_+ jumps, moving with velocities c_{i,N_-+i} for $i = 1, \ldots, N_+$. Since each jump represents a line soliton of $Q_{l,m,n}$, the whole solution therefore represents an (N_-, N_+)-soliton. The velocity of each of the asymptotic line solitons in the (N_-, N_+)-soliton is determined from the c–η graph of the levels of intersections. Note that, given a set of M phases (as determined by the parameters p_i for $i = 1, \ldots, M$), the same graph can be used for any (N_-, N_+)-soliton with $N_- + N_+ = M$. As an example, in figure 2 we show a (2, 1)-soliton solution, a (2, 2)-soliton solution, a (2, 3)-soliton solution and a (3, 3)-soliton solution.

In particular, if $M = 2N$, we have $N_+ = N_- = N$, and theorem 5.3 implies that the velocities of the N incoming solitons are equal to those of the N outgoing solitons. In the case of the ordinary multi-soliton solutions of the discrete 2DTL equation, the τ function (4.6) does not contain all the possible combinations of phases and therefore theorem 5.3 should be modified. However, as in the semi-continuous case, the idea of using the levels of intersection is still applicable, and one can find that the asymptotic velocities for the ordinary N-solitons generated by the Casorati determinant (4.6) with $f_i = \mathrm{e}^{\theta_n^{(2i-1)}} + \mathrm{e}^{\theta_n^{(2i)}}$ $(i = 1, \ldots, N)$ and $p_1 < p_2 < \cdots < p_{2N}$ are $c_{2i-1,2i} = \left(\log\left(1+\kappa p_{2i}^{-1}\right) - \log\left(1+\kappa p_{2i-1}^{-1}\right)\right) / (\log p_{2i} - \log p_{2i-1})$. Note that, like in the semi-continuous case, these velocities are different from those of the resonant N-soliton solutions.

The resonant solutions of the fully discrete 2DTL provide the basis for the construction of the resonant solution of the ultra-discrete 2DTL, as is shown in the next two sections.

6. The ultra-discrete two-dimensional Toda lattice

We now turn our attention to an ultra-discrete analogue of the 2DTL equation. Using equations (4.2) and (4.3), we first write the 2DTL equation (4.4) in bilinear form as

$$(1 - \delta\kappa)\tau_{l+1,m,n}\tau_{l,m+1,n} - \tau_{l+1,m+1,n}\tau_{l,m,n} + \delta\kappa\tau_{l,m,n+1}\tau_{l+1,m+1,n-1} = 0. \tag{6.1}$$

We define the difference operator Δ' as

$$\Delta' = \mathrm{e}^{-\partial_n}\left(\Delta_n^+ - \Delta_l^+\right)\left(\Delta_n^+ - \Delta_m^+\right), \tag{6.2}$$

where from here on the symbols Δ_l^+, Δ_m^+ and Δ_n^+ will be used to denote the difference operators

$$\Delta_l^+ = \mathrm{e}^{\partial_l} - 1, \qquad \Delta_m^+ = \mathrm{e}^{\partial_m} - 1, \qquad \Delta_n^+ = \mathrm{e}^{\partial_n} - 1, \tag{6.3}$$

and where the shift operators e^{∂_l}, e^{∂_m} and e^{∂_n} are defined by $\mathrm{e}^{\partial_n} f_{l,m,n} = f_{l,m,n+1}$, etc. That is,

$$\Delta' f_{l,m,n} = f_{l+1,m+1,n-1} + f_{l,m,n+1} - f_{l+1,m,n} - f_{l,m+1,n}. \tag{6.4}$$

Using equations (6.3) and (6.2), we can rewrite equation (6.1) as

$$(1 - \delta\kappa) + \delta\kappa \exp[\Delta' \log \tau_{l,m,n}] = \exp\left[\Delta_l^+\Delta_m^+ \log \tau_{l,m,n}\right], \tag{6.5}$$

which becomes, taking a logarithm and applying Δ' (assuming $\delta\kappa \neq 1$),

$$\Delta' \log\left[1 + \frac{\delta\kappa}{1 - \delta\kappa} \exp(\Delta' \log \tau_{l,m,n})\right] = \Delta_l^+\Delta_m^+\Delta' \log \tau_{l,m,n}. \tag{6.6}$$

We now take an ultra-discrete limit of equation (6.6) following equation (2.1) [9, 10]. This is accomplished by choosing the lattice intervals as

$$\delta_\epsilon = \mathrm{e}^{-r/\epsilon}, \qquad \kappa_\epsilon = \mathrm{e}^{-s/\epsilon}, \tag{6.7}$$

where $r, s \in \mathbb{Z}_{\geqslant 0}$ are some predetermined integer constants, and by defining

$$v_{l,m,n}^\epsilon = \Delta'\epsilon \log \tau_{l,m,n}^\epsilon. \tag{6.8}$$

Taking the limit $\epsilon \to 0^+$ in equation (6.6) and noting that $\lim_{\epsilon\to 0^+} \epsilon \log(1+\mathrm{e}^{X/\epsilon}) = \max(0, X)$, we then obtain

$$\Delta_l^+\Delta_m^+ v_{l,m,n} = \Delta' \max(0, v_{l,m,n} - r - s), \tag{6.9}$$

where $v_{l,m,n} = \lim_{\epsilon\to 0^+} v_{l,m,n}^\epsilon$. That is, using equation (6.8),

$$v_{l,m,n} = \Delta' \lim_{\epsilon\to 0^+} \epsilon\tau_{l,m,n}^\epsilon. \tag{6.10}$$

Equation (6.9) is the ultra-discrete analogue of the 2DTL equation and can be considered a cellular automaton in the sense that $v_{l,m,n}$ takes on integer values.

Let us briefly discuss ordinary soliton solutions of the ultra-discrete 2DTL equation (6.9). As shown in [9, 10], soliton solutions for the ultra-discrete 2DTL equation (6.9) are obtained by an ultra-discretization of the soliton solution of the discrete 2DTL equation (4.4). For example, a one-soliton solution for equation (4.4) is given by

$$\tau_{l,m,n} = 1 + \eta_1, \tag{6.11}$$

with

$$\eta_i = \alpha_i \frac{\phi(p_i)}{\phi(q_i)}, \tag{6.12}$$

and where

$$\phi(p) = p^n (1 + \delta p)^l (1 + \kappa p^{-1})^{-m} \tag{6.13}$$

as before. We introduce a new dependent variable

$$\rho^{\epsilon}_{l,m,n} = \epsilon \log \tau_{l,m,n}, \tag{6.14}$$

and new parameters P_1, Q_1, $A_1 \in \mathbb{Z}$ as

$$\mathrm{e}^{P_1/\epsilon} = p_1, \qquad \mathrm{e}^{Q_1/\epsilon} = q_1, \qquad \mathrm{e}^{A_1/\epsilon} = \alpha_1. \tag{6.15}$$

Taking the limit $\epsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max(0, \Theta_1), \tag{6.16}$$

where

$$\begin{aligned} \Theta_i = A_i + n(P_i - Q_i) + l\{\max(0, P_i - r) - \max(0, Q_i - r)\} \\ + m\{\max(0, -Q_i - s) - \max(0, -P_i - s)\}, \end{aligned} \tag{6.17}$$

with $\mathrm{e}^{-r/\epsilon} = \delta$ and $\mathrm{e}^{-s/\epsilon} = \kappa$ as before, and where $\rho_{l,m,n} = \lim_{\epsilon\to 0^+} \rho^{\epsilon}_{l,m,n}$. According to equations (6.10) and (6.14), the one-soliton solution for equation (6.9) is then given by

$$v_{l,m,n} = \rho_{l+1,m+1,n-1} + \rho_{l,m,n+1} - \rho_{l+1,m,n} - \rho_{l,m+1,n}. \tag{6.18}$$

Using a similar procedure we can construct a two-soliton solution. Equation (4.4) admits a two-soliton solution given by

$$\tau_{l,m,n} = 1 + \eta_1 + \eta_2 + \theta_{12}\eta_1\eta_2, \tag{6.19}$$

with

$$\theta_{12} = \frac{(p_2 - p_1)(q_1 - q_2)}{(q_1 - p_2)(q_2 - p_1)}, \tag{6.20}$$

and where $\eta_i = \alpha_i \phi(p_i)/\phi(q_i)$ as before ($i = 1, 2$). In order to take the ultra-discrete limit of the above solution, we suppose without loss of generality that the soliton parameters satisfy the inequality

$$0 < p_1 < p_2 < q_2 < q_1. \tag{6.21}$$

Introducing again the dependent variable $\rho^{\epsilon}_{l,m,n} = \epsilon \log \tau_{l,m,n}$, as well as integer parameters P_i, Q_i and A_i as

$$\mathrm{e}^{P_i/\epsilon} = p_i, \qquad \mathrm{e}^{Q_i/\epsilon} = q_i, \qquad \mathrm{e}^{A_i/\epsilon} = \alpha_i \tag{6.22}$$

($i = 1, 2$), and taking the limit of small ϵ, we obtain

$$\rho_{l,m,n} = \max(0, \Theta_1, \Theta_2, \Theta_1 + \Theta_2 + P_2 - Q_2), \tag{6.23}$$

where Θ_i ($i = 1, 2$) was defined in equation (6.17), with $\rho_{l,m,n} = \lim_{\epsilon\to 0^+} \rho^{\epsilon}_{l,m,n}$ again, and where $v_{l,m,n}$ is obtained from $\rho_{l,m,n}$ using equation (6.18). Note that $P_1 < P_2 < Q_2 < Q_1$.

More in general, starting from equations (4.6) and (4.9) (with $0 < p_1 < p_2 < \cdots < p_M < q_M < q_{N-1} < \cdots < q_1$) and repeating the same construction, one obtains the N-soliton solution of the ultra-discrete 2DTL equation (6.9) as [10]

$$\rho_{l,m,n} = \max_{\mu=0,1}\left[\sum_{1\leqslant i\leqslant M}\mu_i\Theta_i + \sum_{1\leqslant i<i'\leqslant M}\mu_i\mu_{i'}(P_{i'} - Q_{i'})\right] \tag{6.24}$$

where $\max_{\mu=0,1}$ indicates maximization over all possible combinations of the integers $\mu_i = 0, 1$, with $i = 1, \ldots, M$. Again, $v_{l,m,n}$ is obtained from $\rho_{l,m,n}$ via equation (6.18).

Ordinary soliton solutions corresponding to the above choices were presented in [9, 10]. In the next section we show how this basic construction can be generalized to obtain resonant soliton solutions.

7. Resonance and web structure in the ultra-discrete 2D Toda lattice equation

Following [1], we now construct more general solutions of the ultra-discrete 2DTL equation (6.9) which display soliton resonance and web structure.

We first consider the case of a (2, 1)-soliton for equation (4.4), which is given by

$$\tau_{l,m,n} = \xi_1 + \xi_2 + \xi_3, \tag{7.1}$$

where

$$\xi_i = \alpha_i\phi(p_i) \tag{7.2}$$

($i = 1, 2, 3$), with

$$\phi(p) = p^n(1+\delta p)^l(1+\kappa p^{-1})^{-m} \tag{7.3}$$

as before, and where again we take $0 < p_1 < p_2 < p_3$. As in the previous section, we introduce the new dependent variable

$$\rho^{\epsilon}_{l,m,n} = \epsilon\log\tau_{l,m,n}, \tag{7.4}$$

and new parameters as

$$\mathrm{e}^{P_i/\epsilon} = p_i, \qquad \mathrm{e}^{A_i/\epsilon} = \alpha_i \tag{7.5}$$

($i = 1, 2, 3$), with $\mathrm{e}^{-r/\epsilon} = \delta$ and $\mathrm{e}^{-s/\epsilon} = \kappa$ as before. Taking the limit $\epsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max(R_1, R_2, R_3) \tag{7.6}$$

where $\rho_{l,m,n} = \lim_{\epsilon\to 0^+}\rho^{\epsilon}_{l,m,n}$ as before, but where now

$$R_i = A_i + nP_i + l\max(0, P_i - r) - m\max(0, -P_i - s) \tag{7.7}$$

($i = 1, 2, 3$). Figure 3 shows that this solution, which again can be called a (2, 1)-soliton, is a Y-shape solution. Note however that the (2, 1)-soliton in figure 3 looks like a (1, 2)-soliton, in the sense that there are two solitons for large positive m and only one for large negative m. In general, an (N_-, N_+)-soliton of the discrete 2DTL equation (4.1) leads to an (N_+, N_-)-soliton of equation (6.9) when taking the ultra-discrete limit. We also note that, interestingly, an L-shape solution can be obtained instead of a Y-shape solution for different solution parameters. An example of such an L-shape soliton is shown in figure 4. No analogue of this solution exists in the 2DTL and its fully discrete version.

Next, we consider the case of a (2, 2)-soliton for equation (4.4) following [1]. Let us consider the following τ function,

$$\tau_{l,m,n} = \begin{vmatrix} f_{l,m,n} & f_{l,m,n+1} \\ f_{l,m,n+1} & f_{l,m,n+2} \end{vmatrix}, \tag{7.8}$$

```
 m ..2.....2.....     m ..2....11.....     m ..2....2......     m ..2...2.......
   ...2....2.....       ...2...11.....       ...2...2......       ...2..2.......
   ....2...2.....       ....2..11.....       ....2..2......       ....2.2.......
 5 ....2...2.....     5 ....2..11.....     5 ....2..2......     5 ....2.2.......
   .....2..2.....       .....2.11.....       .....2.2......       .....22.......
   ......2.2.....       ......211.....       ......22......       ......4.......
   ......2.2.....       ......211.....       ......22......       ......4.......
   .......22.....       .......31.....       .......5......       .......4......
 0 ........4.....     0 .......13.....     0 .......31.....     0 .......5......
   ........4.....       ........5.....       ........5.....       .......13.....
   .........4....       ........22....       ........4.....       ........5.....
   .........5....       .........5....       .........4....       ........22....
   .........13...       .........31...       .........5....       .........5....
-5 ..........5...    -5 ..........5...    -5 .........13...    -5 .........31...
   ..........22..       ..........4...       ..........5...       ..........5...
   ...........5..       ...........4..       ..........22..       ..........4...
          0     n              0     n              0     n              0     n
```

Figure 3. A (2, 1)-resonant soliton solution (i.e., a Y-junction) for the ultradiscrete 2DTL equation (6.9), with $P_1 = -5, P_2 = 1, P_3 = 4, r = 3, s = 1, A_1 - A_3 = 1, A_2 - A_3 = 1$. From left to right, the four plots represent the solution respectively at $l = 0, l = 2, l = 4$ and $l = 6$. The dots indicate zero values of $v_{l,m,n}$.

```
 m ........4.....     m .......13.....     m .......31.....     m .......4......
   ........4.....       .......13.....       .......31.....       .......4......
   ........4.....       .......13.....       .......31.....       .......4......
 5 ........4.....     5 .......13.....     5 .......31.....     5 .......4......
   ........4.....       .......13.....       .......31.....       .......4......
   ........4.....       .......13.....       .......31.....       .......4......
   ........4.....       .......13.....       .......31.....       .......4......
   ........4.....       .......11.....       .......31.....       .......4......
 0 ........4.....     0 .......13.....     0 .......31.....     0 .......5......
   ........4.....       ........5.....       ........5.....       .......13.....
   .........4....       ........22....       ........4.....       ........5.....
   .........5....       .........5....       .........4....       ........22....
   .........13...       .........31...       .........5....       .........5....
-5 ..........5...    -5 ..........5...    -5 .........13...    -5 .........31...
   ..........22..       ..........4...       ..........5...       ..........5...
   ...........5..       ...........4..       ..........22..       ..........4...
          0     n              0     n              0     n              0     n
```

Figure 4. An L-shape (2, 1)-resonant soliton solution for equation (6.9), with $P_1 = -5, P_2 = -1, P_3 = 4, r = 3, s = 1, A_1 - A_3 = 1, A_2 - A_3 = 1$. From left to right, the four plots represent the solution respectively at $l = 0, l = 2, l = 4$ and $l = 6$.

where

$$f_{l,m,n} = \xi_1 + \xi_2 + \xi_3 + \xi_4, \tag{7.9}$$

where ξ_i $(i = 1, \ldots, 4)$ is again defined as in equation (7.3), and where $0 < p_1 < p_2 < p_3 < p_4$ holds. We introduce again the new parameters $e^{P_k/\epsilon} = p_k$ and $e^{A_k/\epsilon} = \alpha_k$ $(k = 1, \ldots, 4)$ and the new dependent variable $\rho^{\epsilon}_{l,m,n} = \epsilon \log \tau_{l,m,n}$. Taking the limit $\epsilon \to 0^+$, we then obtain

$$\rho_{l,m,n} = \max_{1 \leqslant i < j \leqslant 4} (K_{ij} + 2P_j), \tag{7.10}$$

where $\rho_{l,m,n} = \lim_{\epsilon \to 0^+} \rho^{\epsilon}_{l,m,n}$, as before, and

$$K_{ij} = R_i + R_j, \tag{7.11}$$

and with R_j given by equation (7.7) as before. Figure 5 shows the temporal evolution of a (2, 2)-soliton solution. Note the appearance of a hole in figure 5.

Like in the 2DTL (2.1) and its fully discrete version (4.1), we now consider more general resonant solutions for the ultra-discrete 2DTL (6.9). We start from the general τ function

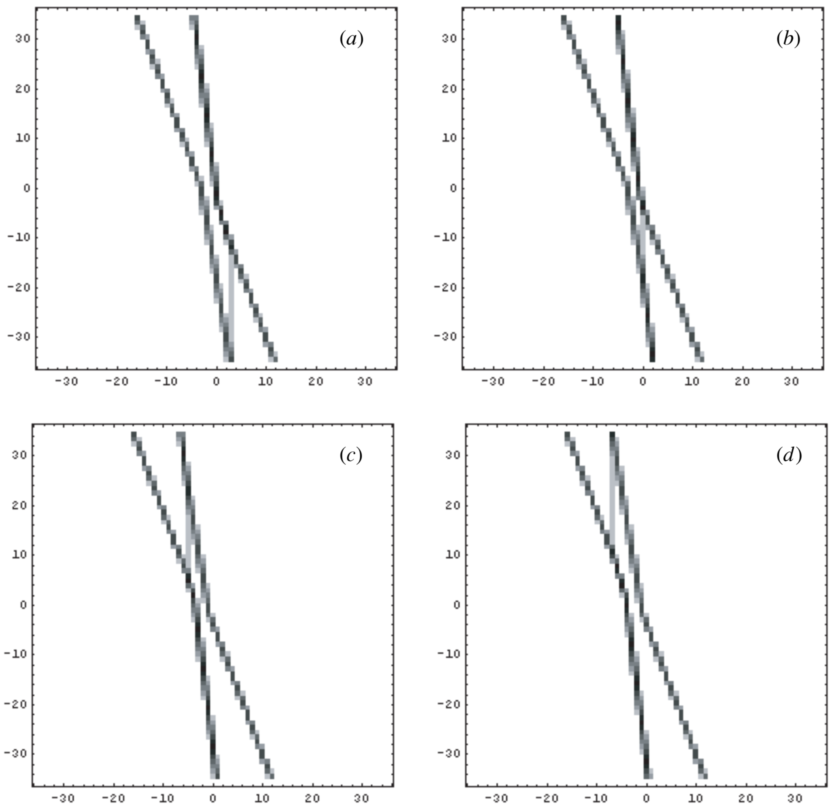

Figure 5. Snapshots illustrating the temporal evolution of a (2, 2)-resonant soliton solution for equation (6.9), with $P_1 = -7, P_2 = -5, P_3 = 1, P_4 = 3, r = 4, s = 2, A_1 = -10, A_2 = -6, A_3 = 0, A_4 = 2$: (*a*) $l = -12$; (*b*) $l = -7$; (*c*) $l = 4$; (*d*) $l = 7$. As in figures 3 and 4, the horizontal axis is n and the vertical axis is m. Since the interaction extends over a wider range of values of m and n, the solution is now plotted in greyscale, in a similar way as in figures 1 and 2; the values of $v_{l,m,n}$ however are still discrete, as in figures 3 and 4.

defined in equation (4.11), and introduce again the parameters $e^{P_k/\epsilon} = p_k$ and $e^{A_k/\epsilon} = \alpha_k$ $(k = 1, 2, \ldots, M)$ and the variable $\rho^{\epsilon}_{l,m,n} = \epsilon \log \tau_{l,m,n}$, together with $e^{-r/\epsilon} = \delta$ and $e^{-s/\epsilon} = \kappa$. Taking the limit $\epsilon \to 0^+$, we then obtain the following solution of the ultra-discrete 2DTL (6.9):

$$\rho_{l,m,n} = \max_{1\leqslant i_1<\cdots<i_N\leqslant M} \left[\sum_{j=1}^{N} R_{i_j} + 2\sum_{j=2}^{N} (j-1) P_{i_j} \right], \tag{7.12}$$

where again $\lim_{\epsilon\to 0^+} \rho^{\epsilon}_{l,m,n} = \rho_{l,m,n}$, with the maximum being taken among all possible combinations of the indices i_j $(j = 1, \ldots, N)$, and where once more we have

$$R_i = A_i + nP_i + l\max(0, P_i - r) - m\max(0, -P_i - s). \tag{7.13}$$

Equation (7.12) produces complicated soliton solutions displaying resonance and web structure. As an example, in figures 6 and 7 we show some snapshots of the time evolution of a (3, 3)-resonant soliton solution and a (4, 4)-resonant soliton solution. Indeed, we conjecture that, similar to its counterparts for the 2DTL and in its fully discrete analogue, equation (7.12) yields the (N_-, N_+)-soliton solution of the ultra-discrete 2DTL equation (6.9), with $N_+ = N$

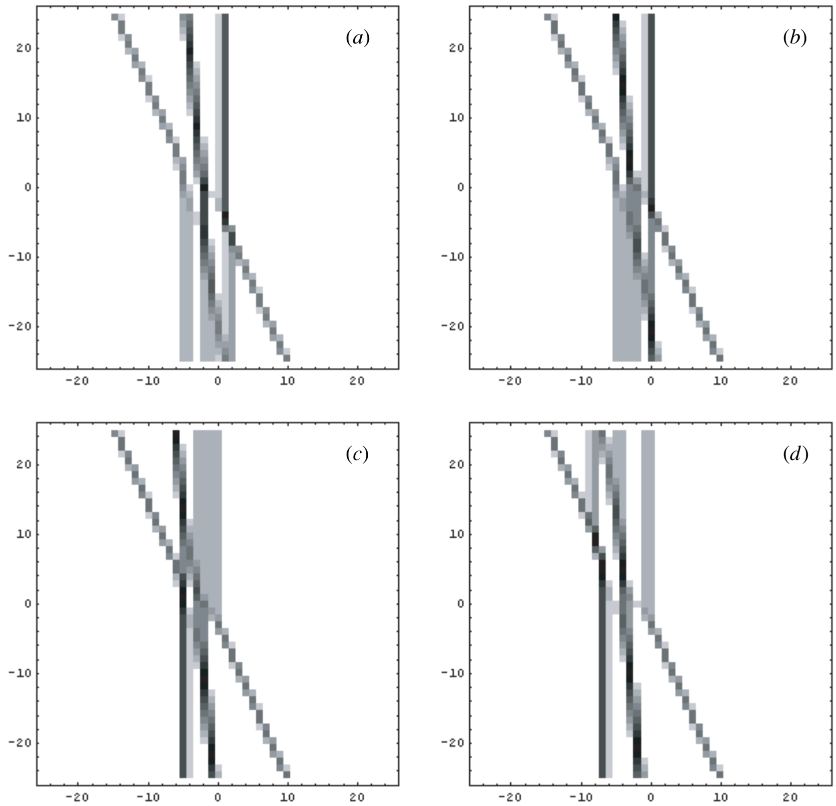

Figure 6. Snapshots illustrating the temporal evolution of a (3, 3)-resonant soliton solution for equation (6.9), with $P_1 = -10, P_2 = -7, P_3 = -5, P_4 = -1, P_5 = 4, P_6 = 5$, $r = 7, s = 4, A_1 = -8, A_2 = -6, A_3 = 0, A_4 = 2, A_5 = 4, A_6 = 7$: (*a*) $l = -15$, (*b*) $l = -10$, (*c*) $l = 0$, (*d*) $l = 10$.

and $N_- = M - N$. Unlike the semi-continuous and fully discrete cases, however, we were unable to prove this conjecture using the techniques introduced in [1]. In this respect, it should be noted that solutions of the ultra-discrete 2DTL arise as a result of the properties of the maximum function, and therefore their study might require the use of techniques from tropical algebraic geometry, which is a subject of current research [13–15].

It should also be noted that the interaction patterns in the ultra-discrete system differ somewhat from their analogues in the semi-continuous and fully discrete cases. In particular, low-amplitude interaction arms may disappear when considering the ultra-discrete limit. Furthermore, the specific interaction patterns in the ultra-discrete limit depend on the value of the parameters r and s and different kinds of solutions may appear for different values of r and s. In particular, large values of r and s tend to result in the production of several vertical solitons, as shown in figures 6 and 7. In order to preserve the soliton count in these cases, all the outgoing vertical solitons should be counted as one, as should the incoming ones. In this sense, a set of outgoing or incoming vertical lines can be considered as a bound state of several solitons. A full characterization of these phenomena and their parameter dependence is however outside the scope of this work.

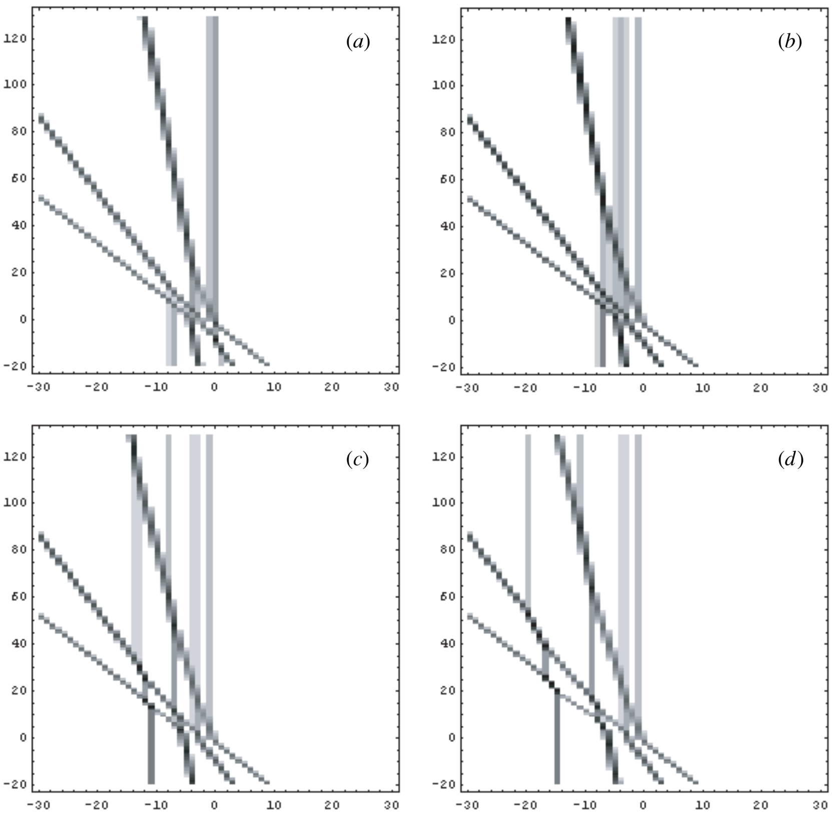

Figure 7. Snapshots illustrating the temporal evolution of a (4, 4)-resonant soliton solution for equation (6.9), with $P_1 = -15$, $P_2 = -12$, $P_3 = -9$, $P_4 = -3$, $P_5 = 1$, $P_6 = 1$, $P_7 = 4$, $P_8 = 7$, $r = 7$, $s = 4$, $A_1 = -8$, $A_2 = -6$, $A_3 = 0$, $A_4 = 2$, $A_5 = 4$, $A_6 = 7$, $A_7 = 8$, $A_8 = 10$: (*a*) $l = -10$, (*b*) $l = 0$, (*c*) $l = 10$, (*d*) $l = 20$.

8. Conclusions

We have demonstrated the existence of soliton resonance and web structure in discrete soliton systems by presenting a class of solutions of the two-dimensional Toda lattice (2DTL) equation, its fully discrete analogue and their ultra-discrete limit. Soliton resonance and web structure had been previously found for nonlinear partial differential equations such as the KP and cKP systems. Note that the 2DTL is a differential-difference equation, its fully discrete version is a difference equation and its ultra-discrete limit is a cellular automaton; therefore our findings show that resonance and web structure phenomena are rather general features of two-dimensional integrable systems whose solutions are expressed in determinant form.

A full characterization of the solutions, including the study of asymptotic amplitudes and velocities and the resonance condition, was provided both in the semi-continuous and in the fully discrete case. Their analogue in the ultra-discrete case, together with an analysis of the intermediate patterns of interactions, is out of the scope of this work, and remains as a problem for further research. Of particular interest is the ultra-discrete 2DTL, where new types of solutions such as the L-shape soliton shown in figure 4 appear.

Finally, we note that the class of solutions presented in this work is just one of the possible choices that yield resonance and web structure. Just like with the KP and cKP equations, the class of soliton solutions of each of the systems we have considered (namely, the 2DTL and its fully discrete and its ultra-discrete analogues) is much wider, and includes also partially resonant solutions. The solutions described in this work represent the extreme case in which all of the interactions among the various solitons are resonant, whereas ordinary soliton solutions represent the opposite case where none of the interactions among the solitons are resonant. Inbetween these two situations, a number of intermediate cases exist in which only some of the interactions are resonant. As in the case of the KP equation, the study of these partially resonant solutions remains an open problem.

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