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On the degenerate soliton solutions of the focusing nonlinear Schrödinger equation

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We characterize the N-soliton solutions of the focusing nonlinear Schrödinger (NLS) equation with degenerate velocities, i.e., solutions in which two or more soliton velocities are the same, which are obtained when two or more discrete eigenvalues of the scattering problem have the same real parts. We do so by employing the operator formalism developed by one of the authors to express the N-soliton solution of the NLS equation in a convenient form. First we analyze soliton solutions with fully degenerate velocities (a so-called multi-soliton group), clarifying their dependence on the soliton parameters. We then consider the dynamics of soliton groups interaction in a general N-soliton solution. We compute the long-time asymptotics of the solution and we quantify the interaction-induced position and phase shifts of each non-degenerate soliton as well as the interaction-induced changes in the center of mass and soliton parameters of each soliton group. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4977984]

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation,

$$iq_t + q_{xx} - 2\nu|q|^2 q = 0$$

[where subscripts x and t denote partial differentiation and $\nu = \mp 1$ denotes the focusing and defocusing cases, respectively], is well-known to be a ubiquitous model in physical applied mathematics which describes the modulations of weakly nonlinear dispersive wave trains in several different physical contexts. The equation also belongs to the class of infinite-dimensional completely integrable systems, which means that its initial value problem can be solved by the inverse scattering transform.[26] Moreover, the NLS equation with $\nu = -1$ (i.e., in the focusing case) admits N-soliton solutions, which describe elastic interactions among the individual solitons.

For the focusing NLS equation with simple eigenvalues, an N-soliton solution is uniquely identified by $4N$ real soliton parameters: the soliton amplitudes, $A_1, \ldots, A_N$, velocities $V_1, \ldots, V_N$, offsets $\xi_1, \ldots, \xi_N$, and phases, $\phi_1, \ldots, \phi_N$. In particular, the soliton velocity and soliton amplitude are respectively the real and imaginary parts of the discrete eigenvalue. (In contrast, for the defocusing NLS equation each soliton is completely specified by two real degrees of freedom: a real eigenvalue and a real norming constant.)

Of course the soliton solutions of the NLS equation have been extensively studied over the years due to their distinctive features and potential use in applications. In particular, many works have been devoted to the study of the soliton interactions and the long time asymptotic behavior of the solutions (e.g., see Refs. 1, 2, 7, 8, 12, and 26 and references therein). In most studies, however, the real parts of the discrete eigenvalues (i.e., the soliton velocities) are assumed to be pairwise distinct. (i.e., $V_i \neq V_j$ for $1 \leq i \neq j \leq N$.) Hereafter we refer to such solutions as "non-degenerate" soliton solutions. But one

can also consider soliton solutions for which this non-degeneracy condition is violated, i.e., solutions in which two or more of the discrete eigenvalues have the same real parts. We refer to such solutions as "degenerate" soliton solutions. The simplest degenerate solution is obtained by considering two simple discrete eigenvalues with the same real parts. Such a solution, which is well-known[12] and is sometimes referred to as a soliton "bound state," was studied by several researchers.[8,14,17] A special case of N-soliton solutions with higher degeneracy, obtained from an initial condition $q(x, 0) = A \operatorname{sech} x$, was studied from a spectral point of view in Ref. 18. Solutions with higher-order degeneracy were also numerically studied[6,16] and were used to characterize the dispersionless limit of the focusing NLS equation for a special class of initial conditions.[13] The behavior of multi-soliton solutions was also studied using perturbative methods.[3,8,10–12,15,24,25] Finally, degenerate 3-soliton solutions were also studied,[22] where their behavior and long-time asymptotics were characterized.

But a general characterization of soliton solutions with degenerate velocities has remained an open problem to the best of our knowledge. The purpose of this work is to address this problem and describe soliton solutions of the focusing NLS equation in which some or all of the soliton velocities are degenerate; i.e., $V_i = V_j$ for some $1 \le i \ne j \le N$. The main results of this work will be a description of the interaction among soliton groups and the calculation of explicit formulae for the position and phase shifts.

We should also note that the soliton solutions of the focusing NLS equation exist which correspond to high order zeros of the analytic scattering coefficients;[4,19,23,26] i.e., discrete eigenvalues with multiplicity higher than one. Such soliton solutions are referred to as "multi-pole" solutions. The simplest multi-pole solution, corresponding to a double eigenvalue (i.e., a "double-pole" solution), was studied by taking a limit of an appropriate 2-soliton solution with simple eigenvalues.[26] The more general case of multi-pole solutions with eigenvalues of higher multiplicity was recently studied by one of the authors.[22] In this work, however, we limit ourselves to studying simple-pole solutions, namely, solutions obtained from simple zeros of the analytic scattering coefficients.

The structure of this work is the following. In Section II we review the expression for the N-soliton solutions of the focusing NLS equation obtained via the operator formalism, the connection with the representation obtained from the inverse scattering transform, and the precise relation between the soliton parameters and the invariances of the NLS equation. In Section III we characterize fully degenerate soliton solutions (i.e., solutions in which all of the soliton velocities coincide). Then in Section IV, we compute the long-time asymptotic behavior of degenerate soliton solutions, and we quantify the interaction-induced parameter changes in each soliton group. Section V concludes this work with some final remarks. The proofs of all theorems, lemmas, etc., are confined to the appendices.

II. SOLITON SOLUTION FORMULAE FOR THE NLS EQUATION

A. Soliton solutions via the operator formalism

We begin by briefly recalling the expression of the N-soliton solution to the focusing NLS equation in the operator formalism. It was shown in Ref. 20 that the N-soliton solution of the focusing NLS equation,

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{2.1}$$

can be written as

$$q(x,t) = 1 - \det\begin{pmatrix} I - L_0 & -L \\ L^* & I \end{pmatrix} \Big/ \det\begin{pmatrix} I & -L \\ L^* & I \end{pmatrix}, \tag{2.2}$$

where asterisk denotes the complex conjugate,

$$L(x,t) = \exp(Ax + iA^2 t)\, C, \qquad L_0(x,t) = \exp(Ax + iA^2 t)\, \mathbf{c}\mathbf{a}^T, \tag{2.3}$$

superscript T denotes the matrix transpose, A is an $N \times N$ nonsingular matrix, $\mathbf{a}$ and $\mathbf{c}$ are the arbitrary nonzero vectors, and C is the unique solution of the Sylvester equation

$$AC + CA^* = \mathbf{c}\mathbf{a}^T. \tag{2.4}$$

It was also shown in Ref. 20 that if we take

$$A = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N) \qquad \mathbf{a} = (1, 1, \ldots, 1)^T \qquad \mathbf{c} = (e^{\beta_1}, e^{\beta_2}, \ldots, e^{\beta_N})^T,$$

the solution (2.2) is equivalent to

$$q(x,t)=\Big[\sum_{j=1}^{N} l_j+\sum_{k=1}^{N-1}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\cdots<i_k}}^{N}\sum_{\substack{j_1,\dots,j_{k+1}=1\\ j_1<\cdots<j_{k+1}}}^{N}\tilde{p}\big(^{i_1,\dots,i_k}_{j_1,\dots,j_{k+1}}\big)\Big]\Big/\Big[1+\sum_{k=1}^{N}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\cdots<i_k}}^{N}\sum_{\substack{j_1,\dots,j_k=1\\ j_1<\cdots<j_k}}^{N}\tilde{p}\big(^{i_1,\dots,i_k}_{j_1,\dots,j_k}\big)\Big] \tag{2.5}$$

where

$$\tilde{p}\big(^{i_1,\dots,i_k}_{j_1,\dots,j_\lambda}\big)=\prod_{\mu=1}^{k} l^*_{i_\mu}\prod_{\nu=1}^{\lambda} l_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(\alpha^*_{i_\mu}-\alpha^*_{i_\nu})^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{\lambda}(\alpha_{j_\mu}-\alpha_{j_\nu})^2\Big/\prod_{\mu=1}^{k}\prod_{\nu=1}^{\lambda}(\alpha^*_{i_\mu}+\alpha_{j_\nu})^2, \tag{2.6a}$$

$$l_j(x,t)=\exp(\alpha_j x+i\alpha_j^2 t+\beta_j). \tag{2.6b}$$

Throughout this paper, we assume that $\mathrm{Re}\,\alpha_n>0$ for all $n=1,\dots,N$ and $\alpha_n\neq\alpha_{n'}$ for all $n\neq n'$ without loss of generality.

In the simplest case of $N=1$, where $A=\alpha$, $\mathbf{a}=1$, and $\mathbf{c}=\mathrm{e}^{\beta}$, the solution of Eq. (2.4) is simply $C=\exp(\beta)/(\alpha+\alpha^*)$. Then $L(x,t)=\exp(\alpha x+i\alpha^2 t+\beta)/(\alpha+\alpha^*)$, $L_0(x,t)=\exp(\alpha x+i\alpha^2 t+\beta)$ and Eq. (2.2) yields the one-soliton solution of the NLS equation in the operator formalism as

$$q(x,t)=\frac{l(x,t)}{1+|l(x,t)/(\alpha+\alpha^*)|^2} \tag{2.7}$$

with

$$l(x,t)=\mathrm{e}^{z(x,t)}\qquad z(x,t)=\alpha x+i\alpha^2 t+\beta\,.$$

For later reference, we also write down the general expression for the 2-soliton solution in the operator formalism

$$q(x,t)=\frac{l_1+l_2+\frac{(\alpha_1-\alpha_2)^2}{(\alpha_1^*+\alpha_1)^2(\alpha_1^*+\alpha_2)^2}|l_1|^2 l_2+\frac{(\alpha_1-\alpha_2)^2}{(\alpha_2^*+\alpha_1)^2(\alpha_2^*+\alpha_2)^2}l_1|l_2|^2}{1+\frac{|l_1|^2}{(\alpha_1^*+\alpha_1)^2}+\frac{l_1^* l_2}{(\alpha_1^*+\alpha_2)^2}+\frac{l_1 l_2^*}{(\alpha_2^*+\alpha_1)^2}+\frac{|l_2|^2}{(\alpha_2^*+\alpha_2)^2}+\frac{(\alpha_1^*-\alpha_2^*)^2(\alpha_1-\alpha_2)^2|l_1|^2|l_2|^2}{(\alpha_1^*+\alpha_1)^2(\alpha_1^*+\alpha_2)^2(\alpha_2^*+\alpha_1)^2(\alpha_2^*+\alpha_2)^2}}. \tag{2.8}$$

B. Soliton parameters and solution degeneracy

It is useful to relate the solution parameters appearing in the solution via the operator formalism to those appearing in the solution via inverse scattering transform (IST).

Starting from Eq. (2.7), noting that $|\alpha^*+\alpha|^2=4\mathrm{Re}^2\alpha$, the solution can be written as

$$q(x,t)=\frac{\mathrm{e}^{z(x,t)}}{1+\mathrm{e}^{2\mathrm{Re}z(x,t)-2\ln(2\mathrm{Re}\alpha)}}=\mathrm{Re}\alpha\,\mathrm{e}^{i\mathrm{Im}z(x,t)}\mathrm{sech}\left[\mathrm{Re}z(x,t)-\ln(2\mathrm{Re}\alpha)\right], \tag{2.9a}$$

$$\mathrm{Re}z(x,t)=\mathrm{Re}\alpha\,x-2\mathrm{Re}\alpha\mathrm{Im}\alpha\,t+\mathrm{Re}\beta, \tag{2.9b}$$

$$\mathrm{Im}z(x,t)=\mathrm{Im}\alpha\,x+(\mathrm{Re}^2\alpha-\mathrm{Im}^2\alpha)\,t+\mathrm{Im}\beta. \tag{2.9c}$$

One can compare the above expression for the one-soliton solution with the classical representation,[2] namely, $q(x,t)=q_{1s}(x,t;A,V,\xi,\phi)$, where

$$q_{1s}(x,t;A,V,\xi,\phi)=A\,\mathrm{sech}[A(x-2Vt-\xi)]\,\mathrm{e}^{i[Vx+(A^2-V^2)t+\phi]}. \tag{2.10}$$

From Eq. (2.10), it is evident that A is the solution amplitude and inverse width, V is the soliton velocity, ξ and ϕ are the initial displacement and overall phase. Comparing Eqs. (2.9) and (2.10), we get

$$\alpha=A+iV,\qquad \beta=\ln(2A)-A\xi+i\phi.$$

Or, written in another way

$$A=\mathrm{Re}\,\alpha,\qquad V=\mathrm{Im}\,\alpha,\qquad \xi=\left[\ln(2\mathrm{Re}\alpha)-\mathrm{Re}\beta\right]/(\mathrm{Re}\,\alpha),\qquad \phi=\mathrm{Im}\,\beta. \tag{2.11}$$

Equation (2.11) provides the desired "translation table" between the soliton parameters appearing in the IST formalism[2] and the operator formalism.[19] Note that $i\alpha^* = V + iA$ is (up to a possible factor of 2 depending on the specific scaling chosen) the discrete eigenvalue of the scattering problem for the focusing NLS equation. Also recall that, by assumption, $A_n > 0\ \forall n = 1, \ldots, N$.

Remark 1. One can sort the discrete eigenvalues (or equivalently the α_n) so that the corresponding soliton velocities are in non-decreasing order:

$$V_1 \le V_2 \le \cdots \le V_N\,. \tag{2.12}$$

Without loss of generality, we will assume that this has been done and that Eq. (2.12) *holds throughout this work.*

Definition 2. We say that a soliton velocity V_j is degenerate if $V_j = V_{j+1}$ for some $j = 1, \ldots, N-1$. Moreover, we say the degeneracy is of order m if $V_j = V_{j+1} = \cdots = V_{j+m-1}$. Finally, we say that a soliton solution is degenerate if some of the soliton velocities are degenerate.

Remark 3. In the literature, the label "degenerate" is occasionally used to denote solutions of Eq. (2.1) *obtained from higher-order zeros of the analytic scattering coefficients. Importantly, such a definition of degenerate solutions is not equivalent to the one used in this work. In the formalism of Ref. 20, such solutions are obtained by taking A in Eqs.* (2.3) and (2.4) *to have a non-trivial Jordan block structure. All the solutions discussed in this work originate from scattering data with simple zeros, corresponding to diagonal matrices A with distinct entries.*

C. Relation between soliton parameters, NLS invariances and conserved quantities

Recall that soliton interactions result in position and phase shifts for solutions with non-degenerate velocities. In order to generalize those results to degenerate solutions, it will be useful to determine exactly how the NLS invariances affect the parameters of any arbitrary N-soliton solution. We do so in this section.

Throughout this section, it will be convenient to write down explicitly the dependence of the solution $q(x, t)$ on the parameters $\beta_1, \ldots, \beta_N$ (or equivalently $\xi_1, \ldots, \xi_N$ and $\phi_1, \ldots, \phi_N$). That is, we write

$$q(x,t) = q_N(x,t,\beta_1,\beta_2,\cdots,\beta_N) = q_N(x,t,\xi_1,\xi_2,\cdots,\xi_N,\phi_1,\phi_2,\cdots,\phi_N).$$

We also fix the following notations for future use:

$$l_n(x,t) = \exp[z_n(x,t)] \qquad z_n(x,t) = \alpha_n x + i\alpha_n^2 t + \beta_n \tag{2.13a}$$

$$\alpha_n = A_n + iV_n, \qquad \beta_n = \ln(2A_n) - A_n\xi_n + i\phi_n \tag{2.13b}$$

$$\mathrm{Re} z_n = A_n x - 2A_n V_n t + \ln 2A_n - A_n\xi_n \qquad \mathrm{Im} z_n = V_n x + (A_n^2 - V_n^2)t + \phi_n \tag{2.13c}$$

$$\alpha_{i,j} = (\alpha_i - \alpha_j)^2/(\alpha_i + \alpha_j^*)^2, \qquad 1 \le i,j \le N, \qquad i \ne j \tag{2.13d}$$

with $A_n > 0$, $V_n, \xi_n, \phi_n \in \mathbb{R}$, for all $n = 1, 2, \cdots, N$.

A. Phase rotations

Recall that, if $q(x, t)$ solves the NLS equation, so does $e^{ic}q(x, t)$ for all $c \in \mathbb{R}$. We next show that the following identity holds:

$$e^{ic} q_N(x,t,\beta_1 - ic, \beta_2 - ic, \cdots, \beta_N - ic) = q_N(x,t,\beta_1,\beta_2,\cdots,\beta_N), \qquad \forall x,t \in \mathbb{R}. \tag{2.14}$$

To prove this identity, we first notice that $e^{ic} l_n(x, t, \beta_n - ic) = l_n(x, t, \beta_n)\ \forall n = 1, \ldots, N$. We also note that:

(i) Every term in the numerator of Eq. (2.5) contains a product of $k + 1$ terms out of $\{l_n(x, t, \beta_n)\}_{n=1}^N$ and k terms out of $\{l_n^*(x, t, \beta_n)\}_{n=1}^N$ for $k = 0, \ldots, N - 1$. This means that setting $\beta_n \mapsto \beta_n - ic$ will produce a total phase translation of e^{-ic} in the numerator.

(ii) Every term in the denominator contains a product of k terms out of $\{l_n(x,t,\beta_n)\}_{n=1}^N$ and k terms out of $\{l_n^*(x,t,\beta_n)\}_{n=1}^N$ for $k=0,1,\ldots,N$. Thus, the product is unaffected by setting $\beta_n \mapsto \beta_n - ic$.

Thus, changing the phase of the whole solution by c is equivalent to add c to the phase parameter ϕ_n of every soliton (or equivalently adding ic to each β_n).

B. Space-time translations

If $q(x,t)$ solves the NLS equation, so does $q(x-x_0, t-t_0)$ for all $x_0, t_0 \in \mathbb{R}$. We want to show that for any choice of $(x_0,t_0) \in \mathbb{R}^2$, there exist constants $\beta_1^0,\ldots,\beta_N^0$ such that

$$q(x-x_0, t-t_0, \beta_1-\beta_1^0, \beta_2-\beta_2^0, \ldots, \beta_N-\beta_N^0) = q(x,t,\beta_1,\beta_2,\ldots,\beta_N), \qquad \forall x,t \in \mathbb{R}.$$

First, we prove a preliminary result. From the solution representation (2.5), the following should be obvious:

Lemma 4. If $l_n(x,t,\beta_n) = l_n(x-x_0, t-t_0, \beta_n-\beta_n^0)$ for all $1 \le n \le N$ and for all $x,t \in \mathbb{R}$, the equality $q_N(x,t,\beta_1,\ldots,\beta_N) = q_N(x-x_0, t-t_0, \beta_1-\beta_1^0,\ldots,\beta_N-\beta_N^0)$ holds for all $x,t \in \mathbb{R}$.

The converse of Lemma 4 is nontrivial. On the other hand, using Lemma 4, in Subsection 1 of the Appendix we prove the main result of this section:

Theorem 5. *The equality*

$$\begin{aligned} &q_N(x,t,\xi_1,\ldots,\xi_N,\phi_1,\ldots,\phi_N) \\ &\qquad = q_N(x-x_0, t-t_0, \xi_1-\xi_1^0,\ldots,\xi_N-\xi_N^0, \phi_1-\phi_1^0,\ldots,\phi_N-\phi_N^0) \end{aligned}$$

holds for all $x,t \in \mathbb{R}$ and for all $x_0, t_0, \xi_1^0,\ldots,\xi_N^0, \phi_1^0,\ldots,\phi_N^0$, if

$$T\mathbf{s} = \mathbf{0} \tag{2.15}$$

where T and $\mathbf{s}$ are respectively the $2N \times 2(N+1)$ matrix and the $2(N+1)$-component vector

$$T = \begin{pmatrix} \mathbf{1}_N & -2\mathbf{V} & -I_N & O_N \\ \mathbf{V} & \mathbf{C} & O_N & I_N \end{pmatrix}, \qquad \mathbf{s} = \begin{pmatrix} x_0 \\ t_0 \\ \boldsymbol{\xi} \\ \boldsymbol{\phi} \end{pmatrix},$$

I_N and O_N are the $N \times N$ identity and zero matrices, $\mathbf{1}_N = (1,\ldots,1)^T$,

$$\boldsymbol{\xi} = (\xi_1^0,\ldots,\xi_N^0)^T, \quad \boldsymbol{\phi} = (\phi_1^0,\ldots,\phi_N^0)^T,$$
$$\mathbf{V} = (V_1,\ldots,V_N)^T, \qquad \mathbf{C} = (A_1^2 - V_1^2,\ldots,A_N^2 - V_N^2)^T.$$

Obviously, $\operatorname{rank}(T) = 2N$. So we have 2 independent variables among the entries of $\mathbf{s}$.

Theorem 5 shows that, in general, to obtain any N-soliton solution, we can choose arbitrarily any two among the entries of $\mathbf{s}$ and assign them any values, then the others will be determined automatically. For a special case, if one performs a spatio-temporal shift by x_0 and t_0, respectively, the solution of the system (2.15) reduces simply to

$$\boldsymbol{\xi} = \mathbf{1}_N x_0 - 2\mathbf{V} t_0 \qquad \boldsymbol{\phi} = -\mathbf{V} x_0 - \mathbf{C} t_0. \tag{2.16}$$

In component form, the solution of the system (2.16) is

$$\xi_j^0 = x_0 - 2V_j t_0, \qquad \phi_j^0 = -V_j x_0 - (A_j^2 - V_j^2) t_0, \qquad 1 \le j \le N.$$

In particular, for degenerate eigenvalues (i.e., when $V_j = \cdots = V_{j+m}$) the above formula implies that the position shifts of the soliton parameters are the same (i.e., $\xi_j^0 = \cdots = \xi_{j+m}^0$).

C. Galilean transformations

If $q(x, t)$ is a solution, so is $e^{i(V_0 x - V_0^2 t)} q(x - 2V_0 t, t)$ for $V_0 \in \mathbb{R}$. One can show that this kind of transformation is equivalent to $l_n(x, t, A_n, V_n) \to l_n(x, t, A_n, V_n + V_0)$ for all $n = 1, 2, \ldots, N$, meaning that it changes all solitons' velocity parameters by V_0 simultaneously. The proof uses similar arguments as those used in the calculations about the phase rotations, so it is omitted for brevity.

D. Scaling transformations

If $q(x, t)$ is a solution, so is $cq(cx, c^2 t)$ for $c \in \mathbb{R}$. The relation between this transformation and the soliton parameters is given by the following, which is proved in Subsection 1 of the Appendix:

Lemma 6. For all $c > 0$, the identity $cq(cx, c^2 t, A_n, V_n, \xi_n, \phi_n) = q(x, t, cA_n, cV_n, \xi_n/c, \phi_n)$ holds.

E. Conserved quantities and center of mass

It will be useful to recall some well known results[2] for the solutions of the NLS equation. Recall that the total mass $\bar{m}$, momentum $\bar{p}$, and the center of mass (CoM) $\bar{\xi}$ of a solution of the NLS equation are defined respectively as

$$\bar{m} = \int_{-\infty}^{\infty} |q(x,t)|^2 \mathrm{d}x, \qquad \bar{p} = 2\int_{-\infty}^{\infty} \mathrm{Im}(q^* q_x)\,\mathrm{d}x, \qquad \bar{\xi} = \frac{1}{\bar{m}} \int_{-\infty}^{\infty} x|q(x,t)|^2 \mathrm{d}x\,. \tag{2.17}$$

The total mass and momentum are conserved quantities of the motion, namely, $d\bar{m}/dt = d\bar{p}/dt = 0$. For the center of mass, instead, direct calculation shows $d\bar{\xi}/dt = \bar{p}/\bar{m}$. Therefore, we can write CoM $\bar{\xi}$ as

$$\bar{\xi}(t) = \frac{\bar{p}}{\bar{m}} t + \bar{\xi}(0)\,. \tag{2.18}$$

Notice that the right hand side of this law is linear in time. Also, for pure soliton solutions, the quantities in Eq. (2.17) assume very simple expressions: The total mass, the total momentum, and the CoM of an N-soliton solution of the focusing NLS equation are given by

$$\bar{m} = 2\sum_{n=1}^{N} A_n, \qquad \bar{p} = 4\sum_{n=1}^{N} A_n V_n, \qquad \bar{\xi}(t) = \frac{2\sum_{n=1}^{N} A_n V_n}{\sum_{n=1}^{N} A_n}\, t + \bar{\xi}(0)\,. \tag{2.19}$$

Note that the result holds for arbitrary N-soliton solutions, even for solutions having degenerate soliton velocities.

III. SOLITON SOLUTIONS WITH FULLY DEGENERATE VELOCITIES

We begin by characterizing 2-soliton solutions with degenerate velocities. This is the first step to analyze N-soliton solutions with at most doubly degenerate velocities, since their asymptotic behavior is simply the sum of that of several 1-soliton solutions and degenerate 2-soliton solutions. (This result will be provided in Section IV.). After the characterization of degenerate 2-soliton solutions, we will move on to the discussion of N-soliton solutions with degeneracy of order N (cf. Section III C). The results in this section will allow us to characterize the asymptotics for arbitrary N-soliton solutions.

A. Degenerate 2-soliton solutions: Polar solution form

Let $N = 2$ and $V_1 = V_2 = V, A_1 \neq A_2$. Without loss of generality we take $A_1 < A_2$. In Subsection 2 of the Appendix, we show that the general expression (2.8) for the 2-soliton solution can be written in a more convenient form

$$q_{d2s}(x,t) = A(x,t)\, e^{iZ(x,t)}, \tag{3.1}$$

with $A(x, t)$ and $Z(x, t)$ given by

$$A(x,t) = B(x,t)\,\mathrm{sech}[P(x,t)], \qquad B(x,t) = \frac{A_2^2 - A_1^2}{2|A_1 \tanh L_1(x,t) - A_2 \tanh L_2(x,t)|}, \tag{3.2a}$$

$$P(x,t) = \ln|\Xi(x,t)| + \ln 2B(x,t) - \ln(A_2^2 - A_1^2), \qquad Z(x,t) = \arg[\Xi(x,t)], \tag{3.2b}$$

where

$$L_n(x,t) = A_n x - 2A_n V t - A_n \xi_n + \ln\frac{A_2 - A_1}{A_1 + A_2}, \qquad n = 1, 2, \tag{3.3a}$$

$$\Xi(x,t) = A_1 e^{i[(A_1^2 - V^2)t + \phi_1]}\operatorname{sech} L_1(x,t) + A_2 e^{i[(A_2^2 - V^2)t + \phi_2]}\operatorname{sech} L_2(x,t). \tag{3.3b}$$

The expression (3.1) for the solution is similar to Eq. (2.10) for the 1-soliton solution, in the sense that $A(x, t)$ and $Z(x, t)$ are both real. Note that $A(x, t)$ and $Z(x, t)$ remain finite for all $x, t \in \mathbb{R}$.[20]

In the following section, we will use Eq. (3.1) to study the long-time asymptotics of doubly degenerate soliton solutions. Before we do so, however, it is useful to characterize the solution.

B. Degenerate 2-soliton solutions: Parameter dependence

Now we start by discussing the parameter dependence of the degenerate 2-soliton solutions. In Subsection 3 of the Appendix we prove the following:

Theorem 7. *The CoM $\bar{\xi}$ of a degenerate 2-soliton solution with velocity V is given by*

$$\bar{\xi}(t) = 2Vt + \frac{1}{A_1 + A_2}\left(A_1\xi_1 + A_2\xi_2 + 2\ln\frac{A_1 + A_2}{A_2 - A_1}\right). \tag{3.4}$$

Let us define the separation parameter w for a degenerate 2-soliton solution as

$$w = \xi_1 - \xi_2 + \xi_{12}, \qquad \xi_{12} = \left(\frac{1}{A_2} - \frac{1}{A_1}\right)\ln\frac{A_2 - A_1}{A_1 + A_2}. \tag{3.5}$$

The solution (3.1) depends on A_n, V, ϕ_n, and ξ_n for $n = 1, 2$. Making use of the Galilean transformation, we let $V = 0$ without loss of generality for the remainder of this section. Also, by making use of the space translation invariance (cf. Section II C), without loss of generality we take ξ_1 and ξ_2 so that

$$A_1\xi_1 + A_2\xi_2 = 2\ln\frac{A_2 - A_1}{A_1 + A_2}. \tag{3.6}$$

By Theorem 7, this ensures that $\bar{\xi}(t) = 0$. Finally, by making use of phase rotations, we take $\phi_1 + \phi_2 = 0$ without loss of generality. Then we introduce the parameter ϕ as

$$\phi = \phi_1 - \phi_2, \tag{3.7}$$

Using Eqs. (3.5)–(3.7) and the fact that $\phi_1 + \phi_2 = 0$, we obtain

$$\xi_j = (-1)^{j+1}\frac{A_{3-j}}{A_1 + A_2}w + \frac{1}{A_j}\ln\frac{A_2 - A_1}{A_1 + A_2}, \qquad \phi_j = (-1)^{j+1}\phi/2, \tag{3.8}$$

for $j = 1, 2$. Therefore, the shape of the above solution to the NLS equation depends only on the four real parameters A_1, A_2, w, and ϕ (instead of seven). We express this by writing the degenerate 2-soliton solution with zero overall phase and zero CoM as

$$q_{d2s}(x,t;w,\phi) = A(x,t;w,\phi)\, e^{iZ(x,t;w,\phi)}, \tag{3.9}$$

with $A(x,t;w,\phi)$ and $Z(x,t;w,\phi)$ given by

$$A(x,t;w,\phi) = B(x;w)\operatorname{sech}[P(x,t;w,\phi)], \qquad B(x;w) = \frac{A_2^2 - A_1^2}{2|A_1\tanh L_1(x;w) - A_2\tanh L_2(x;w)|},$$
$$P(x,t;w,\phi) = \ln|\Xi(x,t;w,\phi)| + \ln 2B(x;w) - \ln(A_2^2 - A_1^2), \qquad Z(x,t;w,\phi) = \arg[\Xi(x,t;w,\phi)],$$
$$\Xi(x,t;w,\phi) = A_1 e^{i(A_1^2 t + \phi/2)}\operatorname{sech} L_1(x;w) + A_2 e^{i(A_2^2 t - \phi/2)}\operatorname{sech} L_2(x;w),$$

where

$$L_n(x;w) = A_n x + (-1)^n\frac{A_1 A_2}{A_1 + A_2}w, \qquad n = 1, 2.$$

In particular,

$$|\Xi(x,t;w,\phi)|^2 = A_1^2\operatorname{sech}^2 L_1 + A_2^2\operatorname{sech}^2 L_2 + 2A_1A_2\operatorname{sech} L_1\operatorname{sech} L_2\cos[(A_1^2 - A_2^2)t + \phi].$$

Remark 8. Note that the phase difference $\phi = \phi_1 - \phi_2$ only determines a temporal shift of $|q_{d2s}(x, t)|$ and does not affect its shape. Therefore, the shape of degenerate 2-soliton solutions of the NLS equation is only determined by 3 real parameters: the soliton amplitudes A_1 and A_2 and the additional separation parameter w.

Figure 1 shows 9 soliton solutions with different choices of soliton parameters. Recall that without loss of generality we have taken $V = 0$ (which can always be done via a Galilean transformation). Note that the spatial and temporal windows differ from one column to another.

We now further examine the temporal dependence of the degenerate solution (3.9). First, we give the following lemma, which is proved in Subsection 3 of the Appendix:

Lemma 9. The following properties hold for the solution (3.9):

(i) $|q_{d2s}(x, t)|$ *is a periodic function of t, with period*

$$T = 2\pi/(A_2^2 - A_1^2)\,. \tag{3.10}$$

(ii) *For any fixed x, there are exactly two critical points for $|q_{d2s}|$ in each time period:*

$$t_1 \equiv -\phi/(A_1^2 - A_2^2) \mod T, \qquad t_2 \equiv (\pi - \phi)/(A_1^2 - A_2^2) \mod T\,.$$

We should point out that Eq. (3.10) was first obtained in Ref. 17. Note that the solution $q_{d2s}(x, t)$ with arbitrary parameters is not always periodic in time (cf. the discussion following Theorem 11).

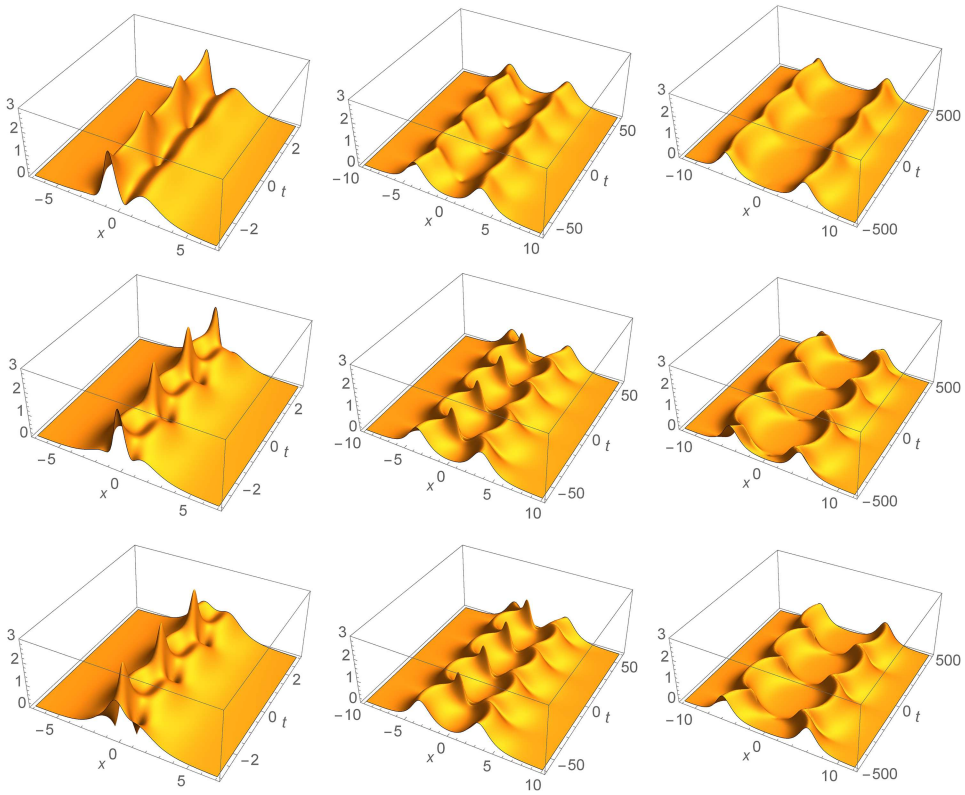

FIG. 1. The absolute values of nine degenerate 2-soliton solutions with different solution parameters and zero velocities. The left column: $A_1 = 1$ and $A_2 = 2$. The center column: $A_1 = 1$ and $A_2 = 1.1$. The right column: $A_1 = 1$ and $A_2 = 1.01$. The top row: $w = 1$, $\phi = 0$. The center row: $w = 0.2$, $\phi = 0$. The bottom row: $w = 0.2$, $\phi = \pi/2$. This figure shows how the parameters affect the shape of the solution. It is easy to see that the period increases as the difference $|A_1 - A_2|$ decreases as shown in Eq. (3.10) (Notice the different time windows in three columns.). The separation of the two peaks changes in one solution as w changes (Notice the difference between the first two rows). Also, the parameter ϕ only determines a temporal shift.

The first results in Lemma 9 are illustrated in Fig. 1. In light of these considerations, we can restrict our attention to one time period.

The top row of Fig. 2 shows the contour plots of the absolute value of two degenerate 2-soliton solutions. The red line and blue line indicate the two critical points, while the black line corresponds to a generic value of time. The bottom row of Fig. 2 then shows the profile of $|q_{d2s}(x, t)|$ for the same two solutions at those values of time. From Fig. 2, one can also observe that: (i) When $t = t_1 + nT$ for $n \in \mathbb{Z}$, the 2 peaks of the solution are furthest apart; (ii) When $t = t_2 + nT$ for $n \in \mathbb{Z}$, the 2 peaks are closest to each other (or, in extreme cases as in Fig. 2(right), they merge into a single peak).

We now discuss the spatial locations of the peaks in a degenerate 2-soliton solution. Interestingly, it is easy to show that the family of degenerate 2-soliton solutions of the NLS equation possesses an extra symmetry

$$q_{d2s}(x, t; w, \phi) = q_{d2s}(-x, t; -w, \phi), \qquad \forall w \in \mathbb{R}. \tag{3.11}$$

This shows that we only need to consider the case $w \geq 0$. One should notice that, for general values of $w \neq 0$, the two solutions appearing on either side of Eq. (3.11) are different but are mirror images with each other with respect to $x = 0$ for any fixed time t (cf. Fig. 3 left and right).

In the special case $w = 0$, the two solutions in Eq. (3.11) coincide. Thus this special solution is even. In fact, the two peaks merge into one at certain values of time (cf. Fig. 3 center). One can also easily show that the condition $w = 0$ is not only sufficient in order for the solution to be even but also necessary. To see this, recall that, when $w = 0$, the function $\tanh L_n(x, t)$ is odd with respect to x for $n = 1, 2$ and $\mathrm{sech} L_n(x, t)$ is even with respect to x for $n = 1, 2$. Thus by inspecting solution (3.1), one can show the expected result.

We now discuss the opposite case to that of an even solution, namely, the limit as $w \to \infty$. We will show that as $w \to \infty$ (i.e., $|\xi_1 - \xi_2| \to \infty$ from Eq. (3.5)): (i) the degenerate 2-soliton solution can

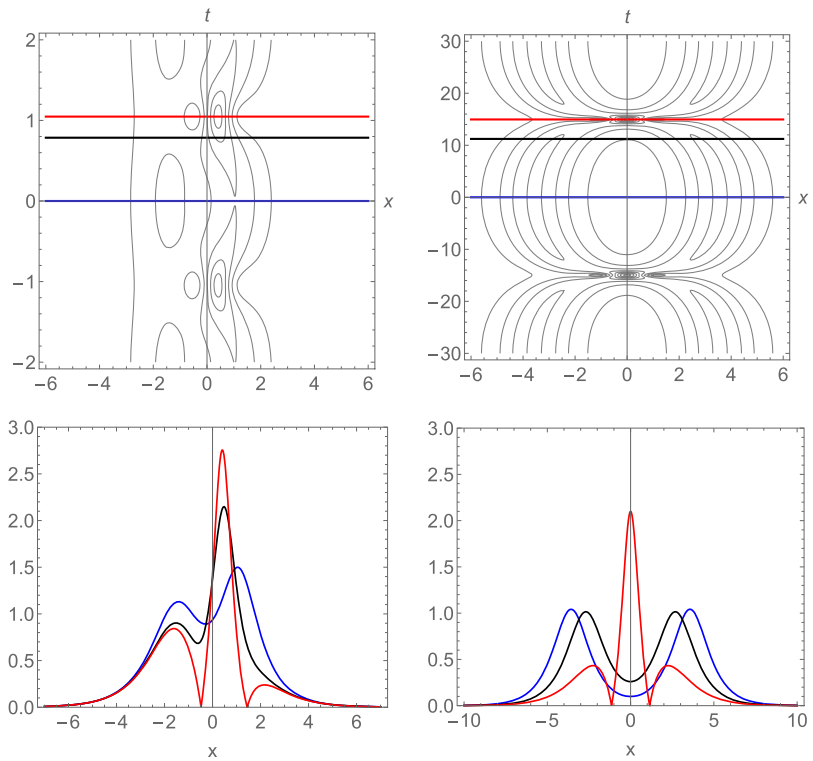

FIG. 2. Contour plots (top) and spatial profiles (bottom) of the absolute values of two degenerate 2-soliton solutions with $V = 0$. The left column: $A_1 = 1, A_2 = 2, w = -1/2 \ln 3$ and $\phi = 0$. The right column: A symmetric solution [see text for details] centered at the origin with $A_1 = 1, A_2 = 1.1, w = 0$ and $\phi = 0$. Colored lines correspond to different values of time: $t = t_1$ (blue), $t = \frac{1}{4}t_1 + \frac{3}{4}t_2$ (black) and $t = t_2$ (red). The bottom plots show the solution profiles at the corresponding values of time.

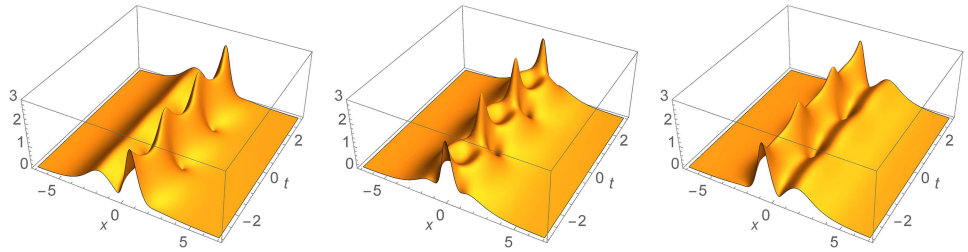

FIG. 3. The absolute values of three degenerate 2-soliton solutions with $V = 0$ and same amplitude and phase parameters ($A_1 = 1$, $A_2 = 2$ and $\phi = 0$), but different values of w, illustrating that symmetry (3.11). Left: $w = -1$. Center: $w = 0$. Right: $w = 1$. The center solution is self-symmetric, while the solution on the left and the one on the right are mirror images of each other with respect to the line $x = 0$.

be regarded as a sum of two 1-soliton solutions. (This statement is similar to the results of long-time asymptotics of the non-degenerate 2-soliton solutions. Since in the latter case, as one takes $t \to \pm\infty$ the two peaks get far away from each other, and the whole solution can be seen as a sum of two 1-soliton solutions.), (ii) the separation between 2 peaks is of order w.

Theorem 10. *As $w \to \infty$, the solution $q_{d2s}(x, t)$ can be written as a sum of two 1-soliton solutions,*

$$q_{d2s}(x,t) = q_1(x,t) + q_2(x,t) + o(1),$$

with each soliton given by ($n = 1, 2$)

$$q_n(x,t) = A_n \operatorname{sech}\left[A_n(x - x_n)\right] \mathrm{e}^{i(A_n^2 t + \phi_n)}, \qquad x_n = \frac{(-1)^{n+1}}{A_n}\left(\frac{A_1 A_2}{A_1 + A_2} w + \ln \frac{A_1 + A_2}{A_2 - A_1}\right). \tag{3.12}$$

The proof of the above theorem is in Subsection 3 of the Appendix. From Eq. (3.12) it is easy to see that the separation between two solitons, which is simply the difference between the displacements of each soliton, is given by

$$|x_1 - x_2| = w + \left(\frac{1}{A_1} + \frac{1}{A_2}\right) \ln \frac{A_1 + A_2}{A_2 - A_1}.$$

Notice that in the limit, the value of w itself is the separation between the two peaks. This is why we called w the separation parameter.

C. Fully degenerate *N*-soliton solutions

We now generalize our characterization of 2-soliton solutions to N-soliton solutions with N degenerate velocities ($N \geq 3$), i.e., $V_1 = \cdots = V_N = V$. We assume that $A_1 < A_2 < \cdots < A_N$. We can always do this by relabeling. The following two results concern the time dependence and the spatial behavior of such solutions, respectively:

Theorem 11. *Let $q(x,t)$ be a fully degenerate N-soliton solution of the NLS equation. If the squared soliton amplitudes A_j^2 and the squared velocity V^2 are all commensurate, namely, if*

$$V^2 = cm_0, \qquad A_j^2 = cm_j, \qquad j = 1, \ldots, N, \tag{3.13}$$

for some constant $c > 0$, with $m_j \in \mathbb{N}$ for $j = 0, \ldots, N$, the solution $q(x + 2Vt, t)$ is periodic in time.

Several remarks are in order: (i) Even with Eq. (3.13) satisfied, the critical points in time of the resulting periodic N-soliton solutions do not have simple expressions. (ii) The proof also holds for the case $N = 2$, meaning that degenerate 2-soliton solutions are not necessarily periodic, even though their modulus are. (iii) Note that the condition in Theorem 11 is only a sufficient one. (iv) At the same time, it is easy to find fully degenerate N-soliton solutions which are not periodic, not even in modulus. For example, such a non-periodic solution can be obtained by taking $A_1 = 1$, $A_2 = 2$, and $A_3 = \pi$.

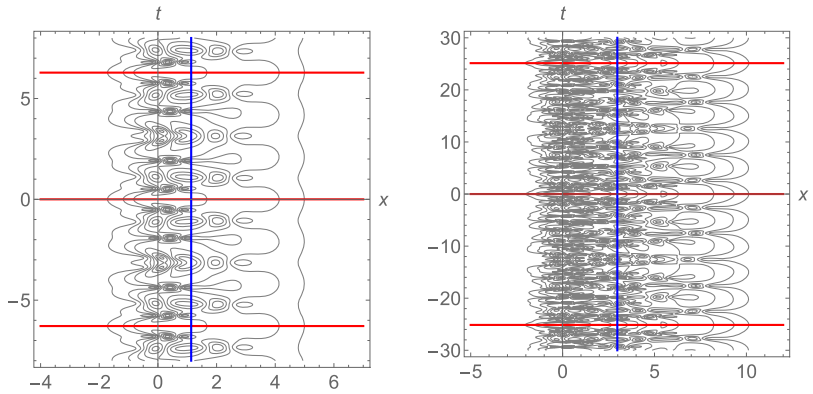

FIG. 4. Contour plots of the absolute values of one 3-soliton solution (left) and one 5-soliton solution (right). The red lines denote integer multiples of the temporal period, while the blue vertical lines indicate the center of mass. Left: $A_j = j$ and $V_j = \xi_j = \phi_j = 0$, for $j = 1, 2, 3$, resulting in a temporal period of 2π. Right: $A_j = 1 + (j - 1)/2$ and $V_j = \xi_j = \phi_j = 0$, for $j = 1, \ldots, 5$, resulting in a temporal period of 8π.

Theorem 12. *The CoM $\bar{\xi}$ of a fully degenerate N-soliton solution is given by*

$$\bar{\xi}(t) = 2Vt + \frac{1}{\sum_{j=1}^{N} A_j}\left(\sum_{j=1}^{N} A_j\xi_j + 2\sum_{1\le s<j\le N} \ln\frac{A_s + A_j}{|A_s - A_j|}\right).$$

The proofs of both theorems are in Subsection 4 of the Appendix. Two periodic solutions are shown in Fig. 4 where the red lines separate different periods and the blue lines indicate the CoM of each solution.

IV. LONG-TIME ASYMPTOTICS FOR SOLITON SOLUTIONS WITH ARBITRARY VELOCITIES

We now use a similar approach to the one used in Refs. 5, 9, and 21 to characterize the long-time asymptotics of general soliton solutions of the NLS equation. We first introduce the approach in a simpler setting. Namely, in Section IV A we begin to discuss the case of non-degenerate solutions. (Of course all the results of Section IV A are well-known.) Then we move on to degenerate N-soliton solutions in Sections IV B and IV C. We will use the following:

Definition 13. Let $M \le N$ be the number of soliton groups in the solution, i.e., the number of distinct soliton velocities among $V_1, \ldots, V_N$. Also, let d_m be the degree of degeneracy of each soliton group, namely, the total number of eigenvalues with identical velocities. Finally, let n_m be the index of the first eigenvalue in the m-th soliton group.

According to Def. 13, the distinct velocities are identified by the set $\{V_{n_m}\}_{m=1,\ldots,M}$. The mth soliton group moves as a whole with velocity V_{n_m}. That is, the soliton velocities are labeled as

$$V_{n_1} = \cdots = V_{n_1+d_1-1} < V_{n_2} = \cdots = V_{n_2+d_2-1} < \cdots < V_{n_M} = \cdots = V_{n_M+d_M-1},$$

where $n_1 = 1$, $n_M + d_M - 1 = N$, and $A_{n_l+s} \ne A_{n_l+s'}$ for $s \ne s'$. Note that the d_m's satisfy the relation $d_1 + \cdots + d_M = N$. It is convenient to separate the solitons (or soliton groups if they are degenerate) into 2 sets, corresponding to non-degenerate and degenerate velocities. Namely, with some abuse of terminology, we will refer to the eigenvalue and velocity of a soliton as *degenerate*, if the soliton belongs to a group with $d_m > 1$, and *non-degenerate* otherwise. The solitons (or soliton groups) in each set are labeled, respectively, by the index sets $S_1 = \{n_m | d_m = 1\}$ and $S_2 = \{n_m \mid d_m > 1\}$. In other words, the set S_1 contains the indices of all non-degenerate solitons and set S_2 those of all degenerate soliton groups. (For example, for a 5-soliton solution with $V_1 = V_2 < V_3 = V_4 < V_5$, there are 3 soliton groups, with $d_1 = d_2 = 2$, $d_3 = 1$, $n_1 = 1$, $n_2 = 3$, and $n_3 = 5$. The distinct velocities are V_1,V_3 and V_5. The index sets are $S_1 = \{5\}$, and $S_2 = \{1,3\}$.)

A. Long-time asymptotics for non-degenerate soliton solutions

We first compute the long-time asymptotics for a 2-soliton solution (2.8) with non-degenerate velocities. Let $x = 2Vt + y$ where V and y are the arbitrary real parameters. Then for the function $l_j(x, t)$ defined in Eq. (2.6b) we have

$$|l_j(2Vt + y, t)| = \exp\left[2A_j(V - V_j)t + A_j y + \ln(2A_j) - A_j\xi_j\right], \qquad j = 1, 2,$$

implying

$$|l_j(2Vt + y, t)| = \begin{cases} O(\mathrm{e}^{2A_j(V-V_j)t}), & V \neq V_j, \\ \exp(A_j y + \ln(2A_j) - A_j\xi_j), & V = V_j, \end{cases} \qquad t \to \pm\infty \quad j = 1, 2\,. \tag{4.1}$$

Using the above expression, in Subsection 5 of the Appendix we prove the following:

Theorem 14. *As $t \to \pm\infty$ with $x = 2V_jt + y$ and $y = O(1)$, for $j = 1,2$,*

$$q(x, t) = A_j \mathrm{e}^{i[V_jx+(A_j^2-V_j^2)t+\phi_j^\pm]}\mathrm{sech}[A_j(x - 2V_jt - \xi_j^\pm)] + o(1), \qquad t \to \pm\infty, \tag{4.2a}$$

where the constants $\phi_j^\pm$ and $\xi_j^\pm$ are given in terms of the soliton parameters as

$$\begin{pmatrix} \xi_1^+ \\ \xi_1^- \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 - \dfrac{1}{A_1}\ln\dfrac{(A_1 - A_2)^2 + (V_1 - V_2)^2}{(A_1 + A_2)^2 + (V_1 - V_2)^2} \end{pmatrix}, \qquad \begin{pmatrix} \phi_1^+ \\ \phi_1^- \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_1 + 2\arctan\dfrac{2A_2(V_1 - V_2)}{A_1^2 - A_2^2 + (V_1 - V_2)^2} \end{pmatrix}, \tag{4.2b}$$

$$\begin{pmatrix} \xi_2^+ \\ \xi_2^- \end{pmatrix} = \begin{pmatrix} \xi_2 - \dfrac{1}{A_2}\ln\dfrac{(A_1 - A_2)^2 + (V_1 - V_2)^2}{(A_1 + A_2)^2 + (V_1 - V_2)^2} \\ \xi_2 \end{pmatrix}, \qquad \begin{pmatrix} \phi_2^+ \\ \phi_2^- \end{pmatrix} = \begin{pmatrix} \phi_2 + 2\arctan\dfrac{2A_1(V_2 - V_1)}{A_2^2 - A_1^2 + (V_2 - V_1)^2} \\ \phi_2 \end{pmatrix}. \tag{4.2c}$$

Theorem 14 implies that the whole solution can be written asymptotically as

$$q(x, t) = \sum_{j=1}^{2} q_{1s}(x, t; A_j, V_j, \xi_j^\pm, \phi_j^\pm) + o(1), \qquad t \to \pm\infty,$$

where q_{1s} was defined in Eq. (2.10). The interaction-induced soliton asymptotic positions $\xi_1^\pm$ and $\xi_2^\pm$ are shown in Fig. 5 (left). Next we generalize the above results to arbitrary soliton solutions with non-degenerate velocities.

Theorem 15. *For any non-degenerate N-soliton solution q(x,t), the following asymptotic behaviors hold:*

$$q(x, t) = \sum_{n=1}^{N} A_n \mathrm{sech}\left[A_n(x - 2V_nt - \xi_n^\pm)\right] \mathrm{e}^{i\left[V_nx+(A_n^2-V_n^2)t+\phi_n^\pm\right]} + o(1), \qquad t \to \pm\infty,$$

where $\xi_n^\pm$ and $\phi_n^\pm$ are defined as

$$\xi_n^- = \xi_n - \frac{1}{A_n}\sum_{s=n+1}^{N} \ln\frac{(A_n - A_s)^2 + (V_n - V_s)^2}{(A_n + A_s)^2 + (V_n - V_s)^2}, \tag{4.3a}$$

$$\phi_n^- = \phi_n + 2\sum_{s=n+1}^{N} \arctan\frac{2A_s(V_n - V_s)}{A_n^2 - A_s^2 + (V_n - V_s)^2}, \tag{4.3b}$$

$$\xi_n^+ = \xi_n - \frac{1}{A_n}\sum_{s=1}^{n-1} \ln\frac{(A_n - A_s)^2 + (V_n - V_s)^2}{(A_n + A_s)^2 + (V_n - V_s)^2}, \tag{4.3c}$$

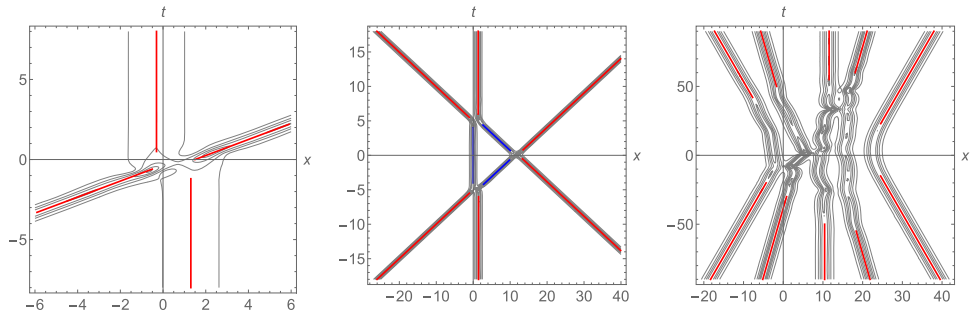

FIG. 5. Contour plot of the absolute values of three non-degenerate soliton solutions, together with straight lines (in red or blue) denoting the asymptotic position of the center of mass of each soliton before and after the interaction. The left: A 2-soliton solution with parameters $A_1 = 1$, $A_2 = 2$, $V_1 = 0$, $V_2 = 1$, $\xi_1 = \ln 2 - 1$, $\xi_2 = \ln 2$, and $\phi_1 = \phi_2 = 0$, The center: A 3-soliton solution with $A_1 = 1.8$, $A_2 = 2$, $A_3 = 2.3$, $V_1 = -1$, $V_2 = 0$, $V_3 = 1$, $\xi_1 = 10$, $\xi_2 = 0$, $\xi_3 = 10$, and $\phi_1 = \phi_2 = \phi_3 = 0$. The right: A 5-soliton solution with parameters: $A_1 = A_3 = A_5 = 1$, $A_2 = 1.1$, $A_4 = 1.2$, $V_1 = -0.1$, $V_2 = -0.05$, $V_3 = 0$, $V_4 = 0.05$, $V_5 = 0.1$, $\xi_1 = \ln 2$, $\xi_2 = -2.010\,49$, $\xi_3 = \ln 2 - 1$, $\xi_4 = -0.103\,776$, $\xi_5 = \ln 2 - 1$, $\phi_j = 0$ for $j = 1, \ldots, 4$ and $\phi_5 = -6$. One can clearly see the position shifts of each soliton in one soliton solution after the interactions.

$$\phi_n^+ = \phi_n + 2\sum_{s=1}^{n-1} \arctan \frac{2A_s(V_n - V_s)}{A_n^2 - A_s^2 + (V_n - V_s)^2}\,. \tag{4.3d}$$

Note that if $n = N$ or $n = 1$, the sum $\sum_{s=n+1}^{N}$ or $\sum_{s=1}^{n-1}$ appeared in Theorem 15 is defined as zero respectively. The proof is in Subsection 5 of the Appendix. The position shifts resulting from a 3-soliton interaction or a 5-soliton interaction are shown in Fig. 5 center or right respectively.

B. Long-time asymptotics for *N*-soliton solutions with at most double degeneracy

We are now ready to discuss the long-time asymptotics of degenerate solutions. We begin by calculating the long-time asymptotics of an N-soliton solution (for $N > 2$) that includes components with doubly degenerate velocities. As before, we assume that the soliton velocities are ordered according to Eq. (2.12). We will use the same notation as in Def. 13. Thus, in this section we have $d_m \in \{1, 2\}$ for all $m = 1, \ldots, M$. In Subsection 6 of the Appendix we prove the following:

Theorem 16. *The asymptotics of q(x, t) are given by*

$$q(x,t) = \sum_{n_m \in S_1} A_{n_m} \operatorname{sech}\left[A_{n_m}(x - 2V_{n_m}t - \xi_{n_m}^{\pm})\right] \mathrm{e}^{i\left[V_{n_m}x + (A_{n_m}^2 - V_{n_m}^2)t + \phi_{n_m}^{\pm}\right]}$$
$$+ \sum_{n_m \in S_2} A_{n_m}^{\pm}(x,t)\, \mathrm{e}^{iZ_{n_m}^{\pm}(x,t)} + o(1), \qquad t \to \pm\infty, \tag{4.4}$$

uniformly with respect to x, where the asymptotic parameters of the non-degenerate solitons are

$$\xi_{n_m}^- = \xi_{n_m} - \frac{1}{A_{n_m}} \sum_{j=n_m+1}^{N} \ln \frac{(A_{n_m} - A_j)^2 + (V_{n_m} - V_j)^2}{(A_{n_m} + A_j)^2 + (V_{n_m} - V_j)^2},$$

$$\xi_{n_m}^+ = \xi_{n_m} - \frac{1}{A_{n_m}} \sum_{j=1}^{n_m-1} \ln \frac{(A_{n_m} - A_j)^2 + (V_{n_m} - V_j)^2}{(A_{n_m} + A_j)^2 + (V_{n_m} - V_j)^2},$$

$$\phi_{n_m}^- = \phi_{n_m} + 2 \sum_{j=n_m+1}^{N} \arctan \frac{2A_j(V_{n_m} - V_j)}{A_{n_m}^2 - A_j^2 + (V_{n_m} - V_j)^2}$$

$$\phi_{n_m}^+ = \phi_{n_m} + 2 \sum_{j=1}^{n_m-1} \arctan \frac{2A_j(V_{n_m} - V_j)}{A_{n_m}^2 - A_j^2 + (V_{n_m} - V_j)^2},$$

while those of the degenerate soliton groups are

$$A^{\pm}_{n_m}(x,t) = B^{\pm}_{n_m}(x,t)\mathrm{sech}\left[P^{\pm}_{n_m}(x,t)\right], \quad B^{\pm}_{n_m}(x,t) = \frac{|A^2_{n_m} - A^2_{n_m+1}|}{2|A_{n_m}\tanh L^{\pm}_{n_m}(x,t) - A_{n_m+1}\tanh L^{\pm}_{n_m+1}(x,t)|},$$

$$P^{\pm}_{n_m}(x,t) = \ln|\Xi^{\pm}_{n_m}(x,t)| + \ln 2B^{\pm}_{n_m}(x,t) - \ln|A^2_{n_m} - A^2_{n_m+1}|, \qquad Z^{\pm}_{n_m}(x,t) = arg[\Xi^{\pm}_{n_m}(x,t)],$$

$$\Xi^{\pm}_{n_m}(x,t) = A_{n_m}\mathrm{e}^{iImz^{\pm}_{n_m}(x,t)}\mathrm{sech}L^{\pm}_{n_m}(x,t) + A_{n_m+1}\mathrm{e}^{iImz^{\pm}_{n_m+1}(x,t)}\mathrm{sech}L^{\pm}_{n_m+1}(x,t),$$

where, for $s = n_m, n_m + 1$,

$$Imz^-_s(x,t) = V_s x + (A_s^2 - V_s^2)t + \phi_s + 2\sum_{l=n_m+2}^{N}\arctan\frac{2A_l(V_s - V_l)}{A_s^2 - A_l^2 + (V_s - V_l)^2},$$

$$Imz^+_s(x,t) = V_s x + (A_s^2 - V_s^2)t + \phi_s + 2\sum_{l=1}^{n_m-1}\arctan\frac{2A_l(V_s - V_l)}{A_s^2 - A_l^2 + (V_s - V_l)^2},$$

$$L^-_s(x,t) = A_s(x - 2V_s t - \xi_s) + \ln\left|\frac{A_{n_m} - A_{n_m+1}}{A_{n_m} + A_{n_m+1}}\right| + \sum_{l=n_m+2}^{N}\ln\frac{(A_s - A_l)^2 + (V_s - V_l)^2}{(A_s + A_l)^2 + (V_s - V_l)^2},$$

$$L^+_s(x,t) = A_s(x - 2V_s t - \xi_s) + \ln\left|\frac{A_{n_m} - A_{n_m+1}}{A_{n_m} + A_{n_m+1}}\right| + \sum_{l=1}^{n_m-1}\ln\frac{(A_s - A_l)^2 + (V_s - V_l)^2}{(A_s + A_l)^2 + (V_s - V_l)^2}.$$

Corollary 17. As $t \to \pm\infty$, *the CoM of the* n_m*-th 2-soliton group is given by:*

$$\bar{\xi}^{\pm}_{n_m} = 2V_{n_m}t + \frac{1}{A_{n_m} + A_{n_m+1}}\left(A_{n_m}\xi^{\pm}_{n_m} + A_{n_m+1}\xi^{\pm}_{n_m+1} + 2\ln\left|\frac{A_{n_m+1} + A_{n_m}}{A_{n_m+1} - A_{n_m}}\right|\right) + o(1).$$

As an example, Fig. 6 illustrates the asymptotic behavior of the center of mass for each soliton group for various multi-soliton solutions with double degeneracy before and after the interaction, confirming the results of Theorem 16. Note that the period of a degenerate 2-soliton group remains the same before and after the interaction.

The total position shift for any soliton with a non-degenerate velocity is

$$\xi^+_{n_m} - \xi^-_{n_m} = \frac{1}{A_{n_m}}\sum_{s=1}^{N}{}'(-1)^{\sigma_s}\ln\frac{(A_{n_m} - A_s)^2 + (V_{n_m} - V_s)^2}{(A_{n_m} + A_s)^2 + (V_{n_m} - V_s)^2},$$

where the prime indicates that the term $s = n_m$ is omitted, and $\sigma_s = 1$ for $s = 1, \ldots, n_m - 1$ and $\sigma_s = 0$ for $s = n_{m+1}, \ldots, N$. This formula agrees with the usual expression obtained for non-degenerate solutions. A similar result is found for the phase shift. *Therefore, the interactions of a non-degenerate soliton are unaffected by whether the other solitons velocities are degenerate.*

A similar result also holds for the position shift for a degenerate soliton group, except that the other eigenvalues in the same group are also excluded from the summation. The expressions for $\bar{\xi}^{\pm}_{n_m}$ are determined in Eq. (A8). Therefore the total position shift for the mth degenerate soliton group is

$$\bar{\xi}^+_{n_m} - \bar{\xi}^-_{n_m} = \frac{1}{A_{n_m} + A_{n_m+1}}\sum_{s=1}^{N}{}''(-1)^{\sigma'_s}\sum_{l=0}^{1}\ln\frac{(A_{n_m+l} - A_s)^2 + (V_{n_m} - V_s)^2}{(A_{n_m+l} + A_s)^2 + (V_{n_m} - V_s)^2},$$

where the double primes indicate that the terms $s = n_m$, $n_m + 1$ are omitted, and $\sigma'_s = 1$ for $s = 1, \ldots, n_m - 1$, $\sigma'_s = 0$ for $s = n_m + 2, \ldots, N$. By using the definition (3.5) of the separation parameter w, we are able to calculate its asymptotic values $w^{\pm}_{n_m}$ as $t \to \pm\infty$ for each degenerate 2-soliton group

$$w^+_{n_m} = w_{n_m} - \sum_{s=0}^{1}\frac{(-1)^s}{A_{n_m+s}}\sum_{s'=1}^{n_m-1}\ln\frac{(A_{n_m+s} - A_{s'})^2 + (V_{n_m} - V_{s'})^2}{(A_{n_m+s} + A_{s'})^2 + (V_{n_m} - V_{s'})^2}, \tag{4.5a}$$

$$w^-_{n_m} = w_{n_m} - \sum_{s=0}^{1}\frac{(-1)^s}{A_{n_m+s}}\sum_{s'=n_m+2}^{N}\ln\frac{(A_{n_m+s} - A_{s'})^2 + (V_{n_m} - V_{s'})^2}{(A_{n_m+s} + A_{s'})^2 + (V_{n_m} - V_{s'})^2}. \tag{4.5b}$$

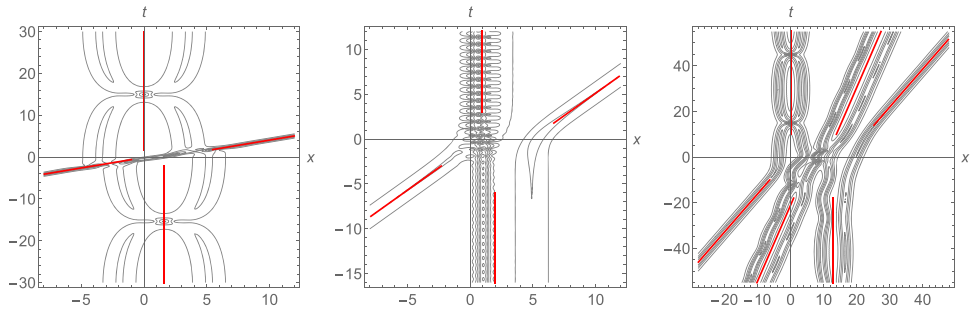

FIG. 6. Contour plots of the absolute values of soliton solutions of the focusing NLS equation with double degeneracy, together with straight lines (in red) denoting the asymptotic behavior of the CoM of each soliton group before and after the interaction. Left: A degenerate 3-soliton solution with $A_1 = 1$, $A_2 = 1.1$, $A_3 = 2$, $V_1 = V_2 = 0$, $V_3 = 1$, $\xi_1 = -\ln 21$, $\xi_2 = -\frac{10}{11}\ln 21$, $\xi_3 = \ln 2 - 1/2$, $\phi_1 = \phi_2 = \phi_3 = 0$. Center: Same for a degenerate 3-soliton solution with $A_1 = 1$, $A_2 = 3$, $A_3 = 1$, $V_1 = V_2 = 0$, $V_3 = 0.5$, $\xi_1 = \xi_3 = \ln 2$, $= \xi_2 = \ln 6/3$, and $\phi_1 = \phi_2 = \phi_3 = 0$. Each of these solutions has two soliton groups, one of which is degenerate. Right: Same for a degenerate 5-soliton solution with $A_1 = A_3 = A_5 = 1$, $A_2 = 1.1$, $A_4 = 1.2$, $V_1 = V_2 = 0$, $V_3 = V_4 = 0.15$, $V_5 = 0.3$, $\xi_1 = -3.044\,52$, $\xi_2 = -2.767\,75$, $\xi_3 = \ln 2 - 1$, $\xi_4 = -0.103\,776$, $\xi_5 = \ln 2 - 1$, $\phi_j = 0$ for $1 \le j \le 4$ and $\phi_5 = -6$. In this case the solution has three soliton groups, two of which are degenerate. These graphs show that the soliton group in each soliton solution may change its shape, i.e., w changes, after the interactions. This is an essential difference between the degenerate soliton solutions and normal soliton solutions. In the latter solutions, each soliton remains its shape after interactions (only temporal and position shifts happens).

We therefore see that, for each 2-soliton group, the separation parameter is affected by the interaction with all other solitons or soliton groups. In other words, *the soliton interactions change not only the overall phase and position of a degenerate two-soliton group but also its shape*. This result clearly differs from the case of non-degenerate solitons, since in that case the shape of any solitons is fully determined by their amplitudes and velocities, which are invariant under soliton-interactions.

C. Long-time asymptotics for *N*-soliton solutions with arbitrary degeneracy

We are finally ready to discuss the asymptotics of general N-soliton solutions with arbitrary degeneracy. We will again use the same notation as in Def. 13. Using this setup, in Subsection 7 of the Appendix we prove the main result of this paper, which generalizes Theorem 16 to arbitrary simple pole soliton solutions of the focusing NLS equation:

Theorem 18. *Let q(x,t) be an N-soliton solution of the focusing NLS equation with arbitrary degeneracy. The long-time asymptotic behavior of the solution is given by*

$$q(x,t) = \sum_{m=1}^{M} q_m^{\pm}(x,t) + o(1), \qquad t \to \pm\infty, \tag{4.6}$$

where $q_m^{\pm}(x,t)$ is a solution defined as in Eq. (2.5) describing a d_m-soliton group with degree of degeneracy d_m, soliton amplitudes A_{n_m+s}, velocity V_{n_m}, and parameters $\xi_{n_m+s}^{\pm}, \phi_{n_m+s}^{\pm}$ for $s = 0, 1, \ldots, n_m + d_m - 1$. The asymptotic parameters are given by

$$\xi_{n_m+s}^{-} = \xi_{n_m+s} - \frac{1}{A_{n_m+s}} \sum_{s'=n_m+d_m}^{N} \ln \frac{(A_{n_m+s} - A_{s'})^2 + (V_{n_m} - V_{s'})^2}{(A_{n_m+s} + A_{s'})^2 + (V_{n_m} - V_{s'})^2},$$

$$\phi_{n_m+s}^{-} = \phi_{n_m+s} + 2 \sum_{s'=n_m+d_m}^{N} \arctan \frac{2A_{s'}(V_{n_m} - V_{s'})}{A_{n_m+s}^2 - A_{s'}^2 + (V_{n_m} - V_{s'})^2},$$

$$\xi_{n_m+s}^{+} = \xi_{n_m+s} - \frac{1}{A_{n_m+s}} \sum_{s'=1}^{n_m-1} \ln \frac{(A_{n_m+s} - A_{s'})^2 + (V_{n_m} - V_{s'})^2}{(A_{n_m+s} + A_{s'})^2 + (V_{n_m} - V_{s'})^2},$$

$$\phi_{n_m+s}^{+} = \phi_{n_m+s} + 2 \sum_{s'=1}^{n_m-1} \arctan \frac{2A_{s'}(V_{n_m} - V_{s'})}{A_{n_m+s}^2 - A_{s'}^2 + (V_{n_m} - V_{s'})^2}.$$

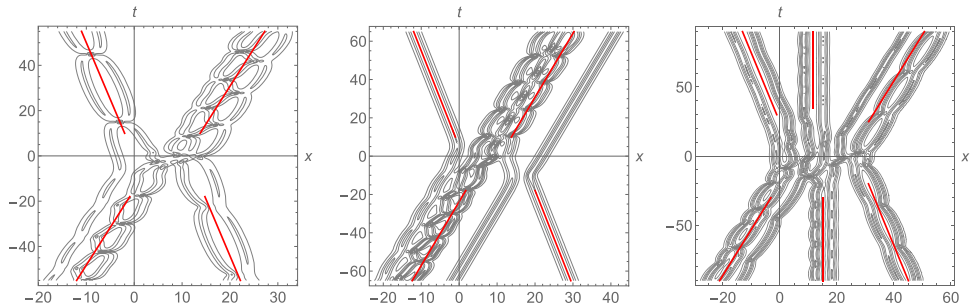

FIG. 7. Contour plots of the absolute values of soliton solutions of the focusing NLS equation with higher degeneracy, together with straight lines (in red) denoting the asymptotic behavior of the CoM of each soliton group before and after the interaction. Left: A 5-soliton solution with degrees of degeneracy {2,3}, obtained for $A_1 = 1$, $A_2 = 1.1$, $A_3 = 1$, $A_4 = 1.2$, $A_5 = 1.4$, $V_1 = V_2 = -0.1$, $V_3 = V_4 = V_5 = 0.15$, $\xi_1 = -3.044\,52$, $\xi_2 = -2.767\,75$, $\xi_3 = -1$, $\xi_4 = 0$ and $\xi_5 = 2$. Center: A 5-soliton solution with degrees of degeneracy {1,4}, obtained for $A_1 = 1$, $A_2 = 1.1$, $A_3 = 1$, $A_4 = 1.2$, $A_5 = 1.4$, $V_1 = -0.1$, $V_2 = V_3 = V_4 = V_5 = 0.15$, $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = -1$, $\xi_4 = 0$ and $\xi_5 = 2$. Right: A 7-soliton solution with degrees of degeneracy {2,2,3}, obtained for $A_1 = 1$, $A_2 = 1.1$, $A_3 = 1$, $A_4 = 1.2$, $A_5 = 1$, $A_6 = 1.1$, $A_7 = 1.2$, $V_1 = V_2 = -0.1$, $V_3 = V_4 = 0$, $V_5 = V_6 = V_7 = 0.15$, $\xi_1 = 1$, $\xi_2 = 3$, $\xi_3 = -1$, $\xi_4 = 0$ and $\xi_5 = \xi_6 = \xi_7 = 2$. All phase parameters ϕ_j are zeros in all of these solutions.

If the degenerate N-soliton solution contains some degenerate 2-soliton groups, a result of the asymptotic separation parameter $w^{\pm}$ of the 2-soliton groups will be obtained from the above theorem. It turns out that the formulas are exactly the same as Eq. (4.5).

From Theorem 18 we have immediately:

Corollary 19. As $t \to \pm\infty$, the CoM of the m-th soliton group $q_m^{\pm}(x,t)$ is given by

$$\bar{\xi}_m^{\pm} = 2V_{n_m}t + \frac{1}{\sum_{s=n_m}^{n_m+d_m-1} A_s}\left(\sum_{s=n_m}^{n_m+d_m-1} A_s\xi_s^{\pm} + 2\sum_{n_m\le s<s'\le n_m+d_m-1} \ln\left|\frac{A_s+A_{s'}}{A_s-A_{s'}}\right|\right) + o(1)\,. \tag{4.7}$$

(If $d_m = 1$, Eq. (4.7) simply yields $\bar{\xi}_m^{\pm} = 2V_{n_m}t + \xi_m^{\pm}$.) The shapes of any d_m-soliton groups will also be changed by the interactions similar to the situation discussed in Section IV B, which is different from the situation for non-degenerate solitons. Figure 7 illustrates the above results by showing three degenerate soliton solutions with degree of degeneracy larger than two. As in Fig. 6, the red lines indicate the CoM of each soliton group before and after the interactions. Importantly, from Theorem 18 we also have the following:

Theorem 20. *For any N-soliton simple-pole solution of the NLS equation, the soliton shifts (i.e., the position shifts and the phase shifts) are independent of the collision order, no matter how high the degree of degeneracy is.*

The equivalent result of Theorem 20 for non-degenerate solutions was of course well-known and follows from the formula (4.3) for the position and phase shifts.

V. CONCLUSIONS

We characterized the long time behavior of simple-pole N-soliton solutions of the focusing NLS equation with arbitrary degeneracy in the soliton velocities. We first considered two-soliton solutions with degenerate velocities ($V_1 = V_2$); we expressed such solutions in a convenient polar form, and we showed that such kind of solutions is completely determined (up to phase rotations, spatio-temporal translations, and Galilean invariance) by four real parameters: the soliton amplitudes A_1 and A_2, the "shape parameter" w, and a temporal offset ϕ. We then characterized various aspects of these solutions, including the behavior of the center of mass, the modulus of the solutions period and critical points in time, mirror solutions, and special symmetric two-soliton solutions. We also proved that the degenerate 2-soliton solutions decomposes into a sum of two 1-soliton solutions if the

separation between the two peaks is large, which is a similar result to the one of long-time asymptotics of non-degenerate 2-soliton solutions. We then considered N-soliton solutions with full degeneracy, i.e., $V_1 = \cdots = V_N$, and we gave an exact formula for the center of mass of such solutions and gave the conditions for periodicity.

We next considered the long-time asymptotics of N-soliton solutions with double degeneracy and that of N-soliton solutions with arbitrary degeneracy respectively. In both cases, we obtained exact formulae for the position and phase shifts for each soliton group (i.e., each portion of the solution arising from eigenvalues with a degenerate velocity) and each soliton (i.e., each portion of the solution arising from eigenvalues with a non-degenerate velocity) as $t \to \pm\infty$.

Importantly, we also showed that the shape parameter w of degenerate 2-soliton groups is affected by the soliton interactions, implying that 2-soliton groups change their shapes as a result of their interactions with other solitons or other soliton groups. We determined exactly the change of the separation parameter w for 2-soliton groups in an arbitrary N-soliton solution.

Two questions arise from the above discussion: (i) How many independent shape parameters determine the shape of a degenerate d-soliton group with $d \geq 3$? (ii) How are the values of these parameters affected by the interactions? A reasonable guess regarding the first question is that one needs total $3d - 2$ parameters to characterize a d-soliton group. However we could not prove this result at the present time, and therefore we leave these questions for future work.

We should point out that generically speaking, soliton solutions with degenerate velocities are unlikely to be robust under random perturbations, since arbitrary perturbations are likely to cause small changes in all of the soliton parameters (including the discrete eigenvalues) and thereby break the degeneracy among the soliton velocities. Nonetheless, special classes of perturbations may exist that preserve the soliton degeneracy. Whether such a class indeed exists, and whether it can be characterized, is a further interesting question.

It would also be interesting to see whether the results of this work can be combined with those of Refs. 20 and 23 to generalize them to multi-pole solutions of the focusing NLS equation. One possible way to do so could be to consider a suitable limit of a degenerate soliton solution to the case in which some or all of the amplitudes also coincide, which produces a multi-pole solution. We suspect, however, that, as in the case of non-degenerate solutions, such limiting procedures would be prohibitively complicated except in the simplest of cases, and that a more fruitful approach would be to start from the beginning with a non-trivial Jordan block structure for the matrix A in the formalism of Ref. 20 in the case in which some of the soliton parameters α_j have the same imaginary part.

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APPENDIX: PROOFS

In this appendix, we give the proofs of the results presented in the main text. We will take the principal branch of the complex logarithm, mapping following identities, which will be used in some of the calculations

$$\operatorname{Re}\ln\alpha_{i,j} = \ln|\alpha_{i,j}| = \ln\frac{(A_i - A_j)^2 + (V_i - V_j)^2}{(A_i + A_j)^2 + (V_i - V_j)^2},$$
$$\operatorname{Im}\ln\alpha_{i,j} = \arg\alpha_{i,j} = 2\arctan\frac{2A_j(V_i - V_j)}{A_i^2 - A_j^2 + (V_i - V_j)^2},$$

with $\alpha_{i,j}$ as in Eq. (2.13d).

1. NLS invariances and soliton parameters

Proof of Theorem 5. By Lemma 4, it is sufficient to require the following identity to hold, (Recall that l_n is defined in Eq. (2.13b).):

$$\alpha_n(x-x_0)+i\alpha_n^2(t-t_0)+\beta_n-\beta_n^0=\alpha_n x+i\alpha_n^2 t+\beta_n+2ik_n\pi, \qquad \forall 1\le n\le N,\ k_n\in\mathbb{Z}.$$

After separating the real and imaginary parts, a system of $2N$ linear equations in the $2N+2$ unknowns (i.e., x_0, t_0, $\xi_1,\ldots,\xi_N$, and $\phi_1,\ldots,\phi_N$) is obtained as

$$A_n x_0-2A_nV_nt_0+\mathrm{Re}\beta_n^0=0, \qquad V_nx_0+(A_n^2-V_n^2)t_0+\mathrm{Im}\beta_n^0=2k_n\pi, \qquad k_n\in\mathbb{Z}.$$

Writing the system in terms of ξ_n^0 and ϕ_n^0, where $\beta_n^0=-A_n\xi_n^0+i\phi_n^0$, we have

$$x_0-2V_nt_0-\xi_n^0=0, \qquad V_nx_0+(A_n^2-V_n^2)t_0+\phi_n^0=2k_n\pi, \qquad k_n\in\mathbb{Z}.$$

Since that $V_nx_0+(A_n^2-V_n^2)t_0+\phi_n^0$ only appears in the exponential function $\exp[i(V_nx_0+(A_n^2-V_n^2)t_0+\phi_n^0)]$, we choose $k_n=0$ for simplicity and write this system in matrix form as follows:

$$T\mathbf{s}=\mathbf{0},$$

where $\mathbf{s}$ and T are defined in Eq. (2.15). □

Proof of Lemma 6. The solution $cq(cx,c^2t)$ is

$$cq(cx,c^2t)=\frac{\sum_{j=1}^{N}cl_j(cx,c^2t)+\sum_{k=1}^{N-1}\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\cdots<i_k}}^{N}\sum_{\substack{j_1,\ldots,j_{k+1}=1\\ j_1<\cdots<j_{k+1}}}^{N}c\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(cx,c^2t)}{1+\sum_{k=1}^{N}\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\cdots<i_k}}^{N}\sum_{\substack{j_1,\ldots,j_k=1\\ j_1<\cdots<j_k}}^{N}\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_k\end{smallmatrix}\right)(cx,c^2t)},$$

$$\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_\lambda\end{smallmatrix}\right)=\prod_{\mu=1}^{k}l_{i_\mu}^*\prod_{\nu=1}^{\lambda}l_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(\alpha_{i_\mu}^*-\alpha_{i_\nu}^*)^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{\lambda}(\alpha_{j_\mu}-\alpha_{j_\nu})^2\Big/\Big[\prod_{\mu=1}^{k}\prod_{\nu=1}^{\lambda}(\alpha_{i_\mu}^*+\alpha_{j_\nu})^2\Big],$$

$$l_j(x,t)=\exp(\alpha_jcx+i\alpha_j^2ct+\beta_j).$$

Now, there are three parts we need to examine: $cl_j(cx,c^2t)$, $c\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(cx,c^2t)$, and $\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_k\end{smallmatrix}\right)(cx,c^2t)$.

(i) For $cl_j(cx,c^2t)$, we can rewrite it as $cl_j(cx,c^2t)=\exp[(c\alpha_j)x+i(c\alpha_j)^2t+\beta_j+\ln c]$. Notice that

$$c\alpha_j=cA_j+icV_j, \qquad \beta_j+\ln c=\ln 2cA_j-cA_j(\xi_j/c)+i\phi_j.$$

Therefore $cl_j(cx,c^2t,A_j,V_j,\xi_j,\phi_j)=l_j(x,t,cA_j,cV_j,\xi_j/c,\phi_j)$, for $j=1,2,\ldots,N$.

(ii) For the part $c\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(cx,c^2t)$,

$$\begin{aligned}c\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(cx,c^2t)&=c\prod_{\mu=1}^{k}l_{i_\mu}^*\prod_{\nu=1}^{k+1}l_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(\alpha_{i_\mu}^*-\alpha_{i_\nu}^*)^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k+1}(\alpha_{j_\mu}-\alpha_{j_\nu})^2\Big/\Big[\prod_{\mu=1}^{k}\prod_{\nu=1}^{k+1}(\alpha_{i_\mu}^*+\alpha_{j_\nu})^2\Big]\\ &=\prod_{\mu=1}^{k}cl_{i_\mu}^*\prod_{\nu=1}^{k+1}cl_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(c\alpha_{i_\mu}^*-c\alpha_{i_\nu}^*)^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k+1}(c\alpha_{j_\mu}-c\alpha_{j_\nu})^2\Big/\Big[\prod_{\mu=1}^{k}\prod_{\nu=1}^{k+1}(c\alpha_{i_\mu}^*+c\alpha_{j_\nu})^2\Big].\end{aligned}$$

Thus by using the result from the case (i), we know that in the parts $cl_{i_\mu}^*$ and cl_{j_ν}, c can be absorbed into the $l_{i_\mu}^*$ and l_{j_ν} respectively. Therefore, the following identity holds:

$$c\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(cx,c^2t,A_n,V_n,\xi_n,)=\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_{k+1}\end{smallmatrix}\right)(x,t,cA_n,cV_n,\xi_n/c).$$

(iii) Lastly let us consider the term $\tilde{p}\left(\begin{smallmatrix}i_1,\ldots,i_k\\ j_1,\ldots,j_k\end{smallmatrix}\right)(cx,c^2t)$,

$$\tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_k}(cx,c^2t)=\prod_{\mu=1}^{k}l_{i_\mu}^*\prod_{\nu=1}^{k}l_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(\alpha_{i_\mu}^*-\alpha_{i_\nu}^*)^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(\alpha_{j_\mu}-\alpha_{j_\nu})^2\Big/\Big[\prod_{\mu=1}^{k}\prod_{\nu=1}^{k}(\alpha_{i_\mu}^*+\alpha_{j_\nu})^2\Big]$$
$$=\prod_{\mu=1}^{k}cl_{i_\mu}^*\prod_{\nu=1}^{k}cl_{j_\nu}\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(c\alpha_{i_\mu}^*-c\alpha_{i_\nu}^*)^2\prod_{\substack{\mu,\nu=1\\ \mu<\nu}}^{k}(c\alpha_{j_\mu}-c\alpha_{j_\nu})^2\Big/\Big[\prod_{\mu=1}^{k}\prod_{\nu=1}^{k}(c\alpha_{i_\mu}^*+c\alpha_{j_\nu})^2\Big].$$

Therefore using a similar argument as the one used in the case (ii), the following identity holds:

$$\tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_k}(cx,c^2t,A_n,V_n,\xi_n)=\tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_k}(x,t,cA_n,cV_n,\xi_n/c).$$

The proof is therefore completed. □

2. Simplified expression for doubly-degenerate soliton solutions

In this case $V_1=V_2$, the general expression (2.8) for the 2-soliton solution reduces to

$$q_{d2s}(x,t)=\frac{l_1+l_2+\frac{(A_1-A_2)^2}{4A_1^2(A_1+A_2)^2}|l_1|^2l_2+\frac{(A_1-A_2)^2}{4A_2^2(A_1+A_2)^2}l_1|l_2|^2}{1+\frac{|l_1|^2}{4A_1^2}+\frac{l_1^*l_2+l_1l_2^*}{(A_1+A_2)^2}+\frac{|l_2|^2}{4A_2^2}+\frac{(A_1-A_2)^4}{16A_1^2A_2^2(A_1+A_2)^4}|l_1|^2|l_2|^2}.$$

First we perform the following change of variables:

$$S=\frac{(l_1l_2)^{\frac{1}{2}}}{\sqrt{A_1A_2}}=\frac{\mathrm{e}^{(z_1+z_2)/2}}{\sqrt{A_1A_2}},\qquad T=\left(\frac{l_1}{l_2}\right)^{\frac{1}{2}}\sqrt{\frac{A_2}{A_1}}=\mathrm{e}^{(z_1-z_2)/2}\sqrt{\frac{A_2}{A_1}},$$

where $l_j^{1/2}=\exp\left(z_j/2\right)$ (cf. Eq. (2.6b)), so that

$$l_1=A_1ST,\qquad l_2=A_2S/T.$$

We can then rewrite the solution as follows:

$$q_{d2s}(x,t)=\frac{A_1ST+A_2S/T+\frac{A_2(A_1-A_2)^2}{4(A_1+A_2)^2}|S|^2ST^*+\frac{A_1(A_1-A_2)^2}{4(A_1+A_2)^2}|S|^2S/T^*}{1+\frac{1}{4}|ST|^2+A_1A_2\frac{T^*/T+T/T^*}{(A_1+A_2)^2}|S|^2+\frac{1}{4}|\frac{S}{T}|^2+\frac{(A_1-A_2)^4}{16(A_1+A_2)^4}|S|^4}.$$

After dividing every term in the fraction by $|S|^2$, denoting $\Delta=(A_2-A_1)/[2(A_1+A_2)]$ and noticing that $S/T^*=\mathrm{e}^{\mathrm{Re}z_2+i\mathrm{Im}z_1}/A_2$, $ST^*=\mathrm{e}^{\mathrm{Re}z_1+i\mathrm{Im}z_2}/A_1$, the solution reduces to

$$\begin{aligned}q_{d2s}(x,t)&=\frac{A_1\Delta\mathrm{e}^{i\mathrm{Im}z_1}\cosh L_2+A_2\Delta\mathrm{e}^{i\mathrm{Im}z_2}\cosh L_1}{\Delta^2\cosh(L_1+L_2)+\frac{1}{4}\cosh(L_1-L_2)+(\frac{1}{4}-\Delta^2)\cos(\mathrm{Im}z_1-\mathrm{Im}z_2)}\\&=\frac{A_1\Delta\mathrm{e}^{i\mathrm{Im}z_1}\mathrm{sech}L_1+A_2\Delta\mathrm{e}^{i\mathrm{Im}z_2}\mathrm{sech}L_2}{(\Delta^2+\frac{1}{4})+(\Delta^2-\frac{1}{4})\tanh L_1\tanh L_2+(\frac{1}{4}-\Delta^2)\cos(\mathrm{Im}z_1-\mathrm{Im}z_2)\mathrm{sech}L_1\mathrm{sech}L_2}.\end{aligned}\tag{A1}$$

One should notice that the last term in the denominator can be written as

$$\begin{aligned}&\left(\frac{1}{4}-\Delta^2\right)\cos(\mathrm{Im}z_1-\mathrm{Im}z_2)\mathrm{sech}L_1\mathrm{sech}L_2\\&=\frac{1}{2(A_1+A_2)^2}\left(\left|A_1\mathrm{e}^{i\mathrm{Im}z_1}\mathrm{sech}L_1+A_2\mathrm{e}^{i\mathrm{Im}z_2}\mathrm{sech}L_2\right|^2+A_1^2\mathrm{tanh}^2L_1+A_2^2\mathrm{tanh}^2L_2-A_1^2-A_2^2\right).\end{aligned}$$

Plugging this into the denominator of Eq. (A1), it reduces to

$$\begin{aligned}&\left(\Delta^2+\frac{1}{4}\right)+\left(\Delta^2-\frac{1}{4}\right)\tanh L_1\tanh L_2+\left(\frac{1}{4}-\Delta^2\right)\cos(\mathrm{Im}z_1-\mathrm{Im}z_2)\mathrm{sech}L_1\mathrm{sech}L_2\\&=\frac{1}{2(A_1+A_2)^2}(A_1\tanh L_1-A_2\tanh L_2)^2+\frac{1}{2(A_1+A_2)^2}|A_1\mathrm{e}^{i\mathrm{Im}z_1}\mathrm{sech}L_1+A_2\mathrm{e}^{i\mathrm{Im}z_2}\mathrm{sech}L_2|^2.\end{aligned}$$

After combining all the parts, the solution $q_{d2s}(x,t)$ is

$$q_{d2s}(x,t)=(A_2^2-A_1^2)\frac{A_1\mathrm{e}^{i\mathrm{Im}z_1}\mathrm{sech}L_1+A_2\mathrm{e}^{i\mathrm{Im}z_2}\mathrm{sech}L_2}{(A_1\tanh L_1-A_2\tanh L_2)^2+|A_1\mathrm{e}^{i\mathrm{Im}z_1}\mathrm{sech}L_1+A_2\mathrm{e}^{i\mathrm{Im}z_2}\mathrm{sech}L_2|^2}.$$

Finally, by rearranging terms in the above expression, one obtains the expression (3.1) for the solution, with $A(x, t)$, $P(x, t)$, and $Z(x, t)$ defined in Eq. (3.2).

3. Degenerate 2-soliton solutions

Proof of Theorem 7. First, let $q(x, t)$ be a degenerate solution with soliton parameters $A_1 < A_2$, $V_1 = V_2 = 0$, and ξ_j, ϕ_j given for $j = 1, 2$. Let $q'(x, t)$ be another, non-degenerate 2-soliton solution with soliton parameters $A'_j = A_j$, $\xi'_j = \xi_j$, $\phi'_j = \phi_j$, $V'_1 = -1/n$, and $V'_2 = A_1/(nA_2)$, for $j = 1, 2$ and $n \in \mathbb{N}$. In other words, all the parameters of solution q' are the same as those of q except for the soliton velocities, which are nonzero. (Primes do not denote differentiation here.)

From Eq. (2.19), both q and q' have zero momentum, so by Eq. (2.18) their CoMs ($\bar{\xi}$ and $\bar{\xi}'$, respectively) are constant in time. Therefore, one can evaluate their CoMs at special values of time: $t \to \infty$ and $t = 0$. We will first compute $\bar{\xi}'$, and then show that $\bar{\xi} = \lim_{n\to\infty} \bar{\xi}'$.

Since q' is a non-degenerate solution, the well-known result about the asymptotics of non-degenerate N-soliton solutions (see Theorem 15) applies

$$q'(x,t) = q'_1(x,t) + q'_2(x,t) + o(1), \qquad t \to \infty,$$

where, for $j = 1, 2$, $q'_j(x, t)$ is a 1-soliton solution with soliton parameters A_j, V'_j, $(\xi'_j)^+$, and $(\phi'_j)^+$. Therefore as $t \to \infty$, the solution q' can be regarded as a well-separated 2-body system with 2 solitons placed at $\bar{\xi}'_j$ with masses $2A_j$ for $j = 1, 2$, where $(\xi'_1)^+$ and $(\xi'_2)^+$ are given by

$$(\xi'_1)^+ = \xi_1 \qquad (\xi'_2)^+ = \xi_2 - \frac{1}{A_2} \ln \frac{(A_1 - A_2)^2 + (V'_1 - V'_2)^2}{(A_1 + A_2)^2 + (V'_1 - V'_2)^2}.$$

The centers of mass for q'_1 and q'_2, computed using definition (2.17), are $\bar{\xi}'_j = 2V'_j t + (\xi'_j)^+$, $j = 1, 2$. One concludes that CoM for this solution q' is

$$\bar{\xi}'(t) = \frac{2A_1\bar{\xi}'_1 + 2A_2\bar{\xi}'_2}{2A_1 + 2A_2} = \frac{A_1\xi_1 + A_2\xi_2}{A_1 + A_2} - \frac{1}{A_1 + A_2} \ln \frac{(A_1 - A_2)^2 + (1 + \frac{A_1}{A_2})^2/n^2}{(A_1 + A_2)^2 + (1 + \frac{A_1}{A_2})^2/n^2}.$$

(Alternatively, one can compute $\lim_{t\to\infty} \bar{\xi}'(t)$ directly from the definition (2.17), by using the fact that $q'_1(x, t)q'_2(x, t) = o(1)$ as $t \to \infty$ for all $x \in \mathbb{R}$.)

To obtain the CoM $\bar{\xi}(t)$ of $q(x, t)$, we consider the difference between the two centers of mass, which is proportional to

$$\left| \int_{\mathbb{R}} x(|q'(x,0)|^2 - |q(x,0)|^2)\, \mathrm{d}x \right| \le \left(\int_{-\infty}^{-W} + \int_{-W}^{W} + \int_{W}^{\infty} \right) |x| \cdot \left| |q'(x,0)|^2 - |q(x,0)|^2 \right| \mathrm{d}x.$$

It is easy to show using Eq. (2.5) that $x|q'(x, 0)|^2$ and $x|q(x, 0)|^2$ are both $O(x \exp(-2A_1|x|))$ as $x \to \pm\infty$ (recall that $A_1 < A_2$). Notice that this estimate is independent of both soliton velocities V_j and V'_j. Thus, for any given $\epsilon > 0$, there exists large enough $W > 0$ such that the first and third integrals on the right hand side are less than ϵ. Moreover, for any finite fixed value of W, the second integral on the right hand side can also be made less than ϵ for large enough n because $q'(x, 0)$ converges uniformly to $q(x, 0)$ as $n \to \infty$ on $[-W, W]$. In summary, the above difference tends to zero as $n \to \infty$, yielding the CoM of a 2-soliton solution with degenerate velocities $V_1 = V_2 = 0$ as

$$\bar{\xi}(t) = \bar{\xi}(0) = \int_{-\infty}^{\infty} x|q(x,0)|^2 \mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} x|q'(x,0)|^2 \mathrm{d}x = \lim_{n\to\infty} \bar{\xi}'(0).$$

Now, let $q''(x, t)$ be a degenerate 2-soliton solution with the same soliton parameters as q except for an arbitrary velocity V. Let its CoM be $\bar{\xi}''(t)$. By using Galilean transformation (cf. Section II C), the relation $q''(x, t) = \mathrm{e}^{i(Vx - V2t)} q(x - 2Vt, t)$ holds. By using the definition of CoM (2.17), the following relation between $\bar{\xi}(t)$ and $\bar{\xi}''(t)$ is obtained:

$$\bar{\xi}''(t) = \bar{\xi}(t) + 2Vt.$$

Therefore Eq. (3.4) is obtained, which completes the proof for Theorem 7. □

Proof of Lemma 9. The only time-dependent part in the modulus of the solution (3.9) is $|\Xi(x, t; w, \phi)|$. Also, L_1 and L_2 are independent of t. Notice that from the assumption $V = 0$, the

only part of $|q_{d2s}(x,t)|$ depending on t is $\cos\left[(A_1^2 - A_2^2)t + \phi\right]$. As a result, $|q_{d2s}(x,t)|$ is periodic with period T given by Eq. (3.10).

After differentiating $|q_{d2s}(x,t)|$ with respect to t, we get $\partial_t|q_{d2s}(x+2Vt,t)| = C(x,t)\sin[(A_1^2 - A_2^2)t + \phi]$, where $C(x,t) = A_1A_2B(x,t)\tanh P(x,t)\mathrm{sech}P(x,t)\mathrm{sech}L_1(x)\mathrm{sech}L_2(x)/|\Xi(x,t)|^2$. In general, $C(x,t)$ has no roots in time variable t for all spatial variable x. Therefore there always exist 2 critical points in one period for any fixed x,

$$t_1 = -\frac{\phi}{A_1^2 - A_2^2} + mT, \qquad t_2 = \frac{\pi - \phi}{A_1^2 - A_2^2} + mT,$$

with $m \in \mathbb{Z}$. The lemma is thus proved. □

Proof of Theorem 10. By using the same notations as before and the calculation in Subsection 2 of the Appendix, the degenerate 2-soliton solution can be rewritten as

$$q_{d2s}(x,t) = \frac{(A_2^2 - A_1^2)(A_1 e^{i\mathrm{Im}z_1}\mathrm{sech}L_1 + A_2 e^{i\mathrm{Im}z_2}\mathrm{sech}L_2)}{(A_1\tanh L_1 - A_2\tanh L_2)^2 + |A_1 e^{i\mathrm{Im}z_1}\mathrm{sech}L_1 + A_2 e^{i\mathrm{Im}z_2}\mathrm{sech}L_2|^2}, \tag{A2}$$

with

$$L_1 = A_1 x - \frac{A_1A_2}{A_1 + A_2}w, \qquad L_2 = A_2 x + \frac{A_1A_2}{A_1 + A_2}w, \qquad z_n = A_n^2 t + \phi_n\,.$$

Letting $x = A_2 w/(A_1 + A_2) + y$ we have $L_1 = A_1 y$ and $L_2 = A_2 y + A_2 w$. We then look at the asymptotics of the solution as $w \to \infty$ with $y = O(1)$. It is easy to show that $L_1 = O(1)$ and $L_2 = O(w)$. As $w \to \infty$ the solution (A2) simplifies to

$$q_{d2s}(y,t) = \frac{A_1(A_2^2 - A_1^2)}{A_1^2 + A_2^2 - 2A_1A_2\tanh(A_1 y)}\mathrm{sech}\,(A_1 y)\,e^{i(A_1^2 t + \phi_1)} + o(1)\,.$$

Simplifying the above expression further results in

$$q_{d2s}(y,t) = A_1\mathrm{sech}\left(A_1 y - \ln\frac{A_1 + A_2}{A_2 - A_1}\right)e^{i(A_1^2 t + \phi_1)} + o(1)\,.$$

Notice that the right hand side of the above expression is a 1-soliton solution (cf. Eq. (2.10)) with amplitude A_1, zero velocity, displacement $(1/A_1)\ln(A_1 + A_2)/(A_2 - A_1)$, and initial phase ϕ_1. A similar result is obtained by taking $x = -A_1 w/(A_1 + A_2) + y$ and again looking at the asymptotic behavior of the solution as $w \to \infty$ with $y = O(1)$. Finally, by looking at the asymptotic behavior as $w \to \infty$ away from $x = (-1)^{n+1}A_{3-n}w/(A_1 + A_2) + y$ with $y = O(1)$, it is easy to show that $q_{d2s}(x,t) = o(1)$ there. Combining these results, one finally obtains Eq. (3.12). □

4. Fully degenerate soliton solutions

Proof of Theorem 11. Since in this proof we are only interested in the time dependence of the N-soliton solution, after substituting $x = 2Vt + y$ into Eq. (2.6b) we will only consider the parts of the solution that are dependent on t. With the above substitution, the function $l_j(x,t)$ becomes

$$l_j(2Vt + y, t) = \exp\left[i(A_j^2 + V^2)t + A_j y + iVy + \beta_j\right]\,. \tag{A3}$$

Next, let us look at the component $\tilde{p}(\cdot)$ in the solution (2.5). We separate the real and imaginary parts and write it as

$$\tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_\lambda} = C\binom{i_1,\ldots,i_k}{j_1,\ldots,j_\lambda}\exp i\left[\left(\sum_{\nu=1}^{\lambda}A_{j_\nu}^2 - \sum_{\mu=1}^{k}A_{i_\mu}^2 + (\lambda - k)V^2\right)t + (\lambda - k)Vy + D\binom{i_1,\ldots,i_k}{j_1,\ldots,j_\lambda}\right], \tag{A4}$$

where $C(\cdot)$ and $D(\cdot)$ are real. Their explicit expressions are omitted for brevity since they are rather complicated. Notice, however, that $C(\cdot)$ is independent of t, and $D(\cdot)$ is independent of both y and t. Then, using the representation (2.5) of the solution from the operator formalism, we notice that every term in the solution expression (2.5) is either in the form (A3) or (A4). Thus the solution $q(x,t)$ is periodic if all the periods of all the exponential functions appearing in the solution are commensurate. In that case, the period $T \in \mathbb{R}^+$ of $q(x,t)$ is the least common multiple of all the periods. It is obvious from Eqs. (A3) and (A4) that if V and all A_j satisfy Eq. (3.13), such number T exists, which implies that the solution $q(x,t)$ is periodic along the line $x = 2Vt + y$. □

Proof of Theorem 12. Here we will use the same idea as the one used in the proof of Theorem 7. The only difference is the start settings.

Let $q(x,t)$ be a fully degenerate N-soliton solution with zero velocity $V=0$ and other parameters A_j, ξ_j and ϕ_j given for $j=1,\ldots,N$. Let n be an integer and denote that $q'(x,t)$ is an N-soliton solution with non-degenerate soliton velocities $V'_j = C_j/n$, where C_j satisfies the following properties: (i) $C_j \neq 0$; (ii) C_j are distinct and are in the increasing order; (iii) the relation $\sum_{j=1}^N A_j C_j = 0$ holds. The other parameters of $q'(x,t)$ are the same as those appeared in $q(x,t)$. Immediately, we get that for fixed integer n, $\{V'_j\}_{j=1}^N$ are distinct and in increasing order, and satisfy the relation $\sum_{j=1}^N A_j V'_j = 0$. So the momentum of both q and q' are zeros and the CoMs for both solutions are constants.

Then by using a similar approach to the one used in Subsection 3 of the Appendix and by using Theorem 15 (which is the well-known result of the asymptotic behaviors of non-degenerate N-soliton solutions), it is easy to prove the desired results. Moreover, one can generalize the result to non-zero velocities cases by using Galilean transformation (cf. Section II C) and the definition of CoM (2.17). This completes our proof of Theorem 12. □

5. Long-time asymptotics of non-degenerate soliton solutions

Proof of Theorem 14. We first show that along the line $x = 2Vt + y$ with $V \neq V_1, V_2$ the solution (2.8) satisfies $\lim_{t\to\pm\infty} q(x,t) = 0$. There are three different ranges for V:

(i) $V < V_1$. By Eq. (4.1), $\lim_{t\to\infty} l_j(x,t) = 0$ for $j = 1, 2$. Therefore $\lim_{t\to\infty} q(x,t) = 0$. On the other hand, $\lim_{t\to-\infty} l_j(x,t) = \infty$ for $j = 1, 2$. Dividing numerator and denominator of Eq. (2.8) by $|l_1 l_2|^2$, we again obtain $\lim_{t\to\infty} q(x,t) = 0$.

(ii) $V_1 < V < V_2$. In the limit $t \to \infty$, we have $\lim_{t\to\infty} l_1(x,t) = \infty$ and $\lim_{t\to\infty} l_2(x,t) = 0$. Thus, proceeding as before, we obtain $\lim_{t\to\infty} q(x,t) = 0$. By similar arguments, one can achieve the same result when $t \to -\infty$.

(iii) $V_2 < V$. The result follows via similar arguments from the case (ii).

Now we look at the limits as $t \to \pm\infty$ along the line $x = 2V_1 t + y$. After using Eq. (4.1), we immediately have $\lim_{t\to\infty} |l_2| = 0$. Thus solution (2.8) becomes

$$q(x,t) = [l_1(x,t) + o(1)] \Big/ \left[1 + \frac{|l_1(x,t)|^2}{(\alpha_1^* + \alpha_1)^2} + o(1)\right], \qquad t \to \infty,$$

which when simplified yields Eq. (4.2a). Also, by Eq. (4.1) we have $\lim_{t\to-\infty} |l_2| = +\infty$. After dividing numerator and denominator of Eq. (2.8) by $|l_2|^2$, we have,

$$\begin{aligned} q(x,t) &= \frac{\alpha_{1,2} l_1(x,t)}{1 + \frac{1}{(\alpha_1^*+\alpha_1)^2}|\alpha_{1,2}|^2 |l_1(x,t))|^2} + o(1) \\ &= A_1 e^{i[V_1 x + (A_1^2 - V_1^2)t + \phi_1^-]} \operatorname{sech}\left[A_1(x - 2V_1 t - \xi_1^-)\right] + o(1), \qquad t \to -\infty\,. \end{aligned}$$

The asymptotics for $q(x,t)$ along the line $x = 2V_2 t + y$ is obtained in a similar way. □

To prove the Theorem 15, we will need the following two results, both of which are verified by direct calculation:

Proposition 21. Let $1 \le n \le N$ and $x = 2Vt + y$ for any $V, y \in \mathbb{R}$. Then, as $t \to \pm\infty$,

$$|l_n(2Vt + y, t)| = \begin{cases} O(e^{2A_n(V-V_n)t}), & V \neq V_n, \\ \exp(A_n y + \ln(2A_n) - A_n \xi_n) + o(1), & V = V_n\,. \end{cases}$$

Lemma 22. Let $1 \le k, \lambda \le N$ and $x = 2Vt + y$ for any $V, y \in \mathbb{R}$. Let $1 \le i_1 < i_2 < \cdots < i_k \le N$ and $1 \le j_1 < j_2 < \cdots < j_\lambda \le N$. Denoting $F = l_n^{t_1} (l_n^*)^{t_2}$ where $t_1, t_2 \in \{0, 1\}$.

(i) *If $V = V_n$ for $1 \le n < N$ and $t \to -\infty$, then the following identity holds:*

$$\prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=n+1}^{N} |l_s(x,t)|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=n+1}^{N} |l_s(x,t)|^2, \\ o(1), & \text{otherwise}\,. \end{cases}$$

(ii) *If $V = V_n$ for $1 < n \le N$ and $t \to \infty$, then the following identity holds:*

$$\prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=1}^{n-1} |l_s(x,t)|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=1}^{n-1} |l_s(x,t)|^2, \\ o(1), & \text{otherwise.} \end{cases}$$

Note that, above and below for $n = 1$ and $n = N$, we define $\prod_{s=1}^{n-1} |l_s(x,t)|^2 = 1$ and $\prod_{s=n+1}^{N} |l_s(x,t)|^2 = 1$, respectively.

Proof of Theorem 15. Using Proposition 21 and Lemma 22, and similar ideas as in the proof of Theorem 14, it is easy to show that $\lim_{t\to\pm\infty} q(x,t) = 0$ along the line $x = 2Vt + y$ for $\forall V, y \in \mathbb{R}$ with $V \ne V_n$ for $n = 1, 2, \ldots, N$.

To prove the second part of Theorem 15, i.e., the asymptotics along the line $x = 2V_n t + y$ with $n = 1, \ldots, N$, let $x = 2V_n t + y$ for $n = 1, \ldots, N$ and $y \in \mathbb{R}$, as before. For all $n = 1, \ldots, N$, dividing numerator and denominator of solution (2.5) simultaneously by $\prod_{s=n+1}^{N} |l_s(x,t)|^2$, by using Lemma 22 we get

$$\begin{aligned} q(x,t) &= \frac{\tilde{p}\binom{n+1,n+2,\ldots,N}{n,n+1,\ldots,N} \Big/ \prod_{s=n+1}^{N} |l_s(x,t)|^2 + o(1)}{\frac{\tilde{p}\binom{n+1,\ldots,N}{n+1,\ldots,N}}{\prod_{s=n+1}^{N} |l_s(x,t)|^2} + \frac{\tilde{p}\binom{n,\ldots,N}{n,\ldots,N}}{\prod_{s=n+1}^{N} |l_s(x,t)|^2} + o(1)} \\ &= \frac{\prod_{s=n+1}^{N} \alpha_{n,s} l_n}{1 + \frac{1}{(\alpha_n^* + \alpha_n)^2} \left| \prod_{s=n+1}^{N} \alpha_{n,s} l_n \right|^2} + o(1), \qquad t \to -\infty. \end{aligned}$$

As a result, asymptotic soliton parameters are $\beta_N^- = \beta_N$ as well as $\beta_n^- = \beta_n + \sum_{s=n+1}^{N} \ln \alpha_{n,s}$ for all $n = 1, \ldots, N-1$, implying $\xi_N^- = \xi_N$ and $\phi_N^- = \phi_N$ as well as

$$\xi_n^- = \xi_n - \frac{1}{A_n} \sum_{s=n+1}^{N} \ln \frac{(A_n - A_s)^2 + (V_n - V_s)^2}{(A_n + A_s)^2 + (V_n - V_s)^2},$$

$$\phi_n^- = \phi_n + 2 \sum_{s=n+1}^{N} \arctan \frac{2A_s(V_n - V_s)}{A_n^2 - A_s^2 + (V_n - V_s)^2},$$

for all $n = 1, \ldots, N-1$. The asymptotics as $t \to \infty$ is calculated in a similar way. One then obtains $\beta_1^+ = \beta_1$ as well as $\beta_n^+ = \beta_n + \sum_{s=1}^{n-1} \ln \alpha_{n,s}$ for all $n = 2, \ldots, N$, implying $\xi_1^+ = \xi_1$ and $\phi_1^+ = \phi_1$ as well as

$$\xi_n^+ = \xi_n - \frac{1}{A_n} \sum_{s=1}^{n-1} \ln \frac{(A_n - A_s)^2 + (V_n - V_s)^2}{(A_n + A_s)^2 + (V_n - V_s)^2},$$

$$\phi_n^+ = \phi_n + 2 \sum_{s=1}^{n-1} \arctan \frac{2A_s(V_n - V_s)}{A_n^2 - A_s^2 + (V_n - V_s)^2},$$

for all $n = 2, \ldots, N$. This completes the calculation of the asymptotics. □

6. Long-time asymptotics of doubly degenerate soliton solutions

The goal of this section is to prove Theorem 16. First, the following two lemmas are needed, which are obtained by direct calculation:

Lemma 23. Let $m = 1, \ldots, M$, $s = 0, \ldots, d_m - 1$, and $x = 2Vt + y$ with $V, y \in \mathbb{R}$. As $t \to \pm\infty$,

$$|l_{n_m+s}(2Vt + y, t)| = \begin{cases} O(\mathrm{e}^{2A_{n_m+s}(V - V_{n_m+s})t}), & V \ne V_{n_m+s}, \\ \exp\left(A_{n_m+s} y + \ln(2A_{n_m+s}) - A_{n_m+s}\xi_{n_m+s}\right) + o(1), & V = V_{n_m+s}. \end{cases}$$

Lemma 24. Let $1 \le k, \lambda \le N$ and $x = 2Vt + y$ for $V, y \in \mathbb{R}$. Let $1 \le i_1 < i_2 < \cdots < i_k \le N$ and $1 \le j_1 < j_2 < \cdots < j_\lambda \le N$. Let $s = 0, \ldots, d_m - 1$, $t_{s,1}, t_{s,2} \in \{0, 1\}$ and define

$$F = \prod_{s=0}^{d_m-1} l_{n_m+s}^{t_{s,1}} (l_{n_m+s}^*)^{t_{s,2}}. \tag{A5}$$

(i) *If $V = V_{n_m}$ for $m = 1, \ldots, M$ and $t \to -\infty$, the following asymptotics holds:*

$$\prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=n_m+d_m}^{N} |l_s|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=n_m+d_m}^{N} |l_s|^2, \\ o(1), & \text{otherwise.} \end{cases}$$

(ii) *If $V = V_{n_m}$ for $m = 1, \ldots, M$ and $t \to \infty$, the following asymptotics holds:*

$$\prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=1}^{n_m-1} |l_s|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l_{i_\mu}^* \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=1}^{n_m-1} |l_s|^2, \\ o(1), & \text{otherwise.} \end{cases}$$

Note that, above and below for $m = 1$ and $m = M$, we denote $\prod_{s=1}^{n_m-1} |l_s|^2 = 1$ and $\prod_{s=n_m+d_m}^{N} |l_s|^2 = 1$, respectively.

Proof of Theorem 16. Let $x = 2Vt + y$ for $V, y \in \mathbb{R}$. If $V \neq V_{n_m}$ for all $m = 1, 2, \ldots, M$, using Lemma 23 it is easy to show $\lim_{t\to\pm\infty} q(x,t) = 0$.

Next, we compute the asymptotics for the N-soliton solutions along the line $x = 2V_{n_m} t + y$ for $m = 1, \ldots, M$.

First, let us consider the $t \to -\infty$ case. There are two subcases depending on the soliton groups:

(i) If $d_m = 1$, by dividing every term by $\prod_{s=n_m+1}^{N} |l_s|^2$ (if $m = M$, defining $\prod_{s=n_m+1}^{N} |l_s|^2 = 1$) and using Lemma 24, the solution (2.5) reduces to

$$q(x,t) = \prod_{s=n_m+1}^{N} \alpha_{n_m,s} l_{n_m} \Big/ \left[1 + \frac{1}{(\alpha_{n_m}^* + \alpha_{n_m})^2} \prod_{s=n_m+1}^{N} |\alpha_{n_m,s} l_{n_m}|^2 \right] + o(1), \qquad t \to -\infty .$$

As a result, the parameter is $\beta_{n_m}^- = \beta_{n_m} + \sum_{s=n_m+1}^{N} \ln \alpha_{n_m,s}$. In other words,

$$\xi_{n_m}^- = \xi_{n_m} - \frac{1}{A_{n_m}} \sum_{s=n_m+1}^{N} \ln \frac{(A_{n_m} - A_s)^2 + (V_{n_m} - V_s)^2}{(A_{n_m} + A_s)^2 + (V_{n_m} - V_s)^2},$$

$$\phi_{n_m}^- = \phi_{n_m} + 2 \sum_{s=n_m+1}^{N} \arctan \frac{2A_s(V_{n_m} - V_s)}{A_{n_m}^2 - A_s^2 + (V_{n_m} - V_s)^2} .$$

(ii) If $d_m = 2$, the corresponding soliton group contains two eigenvalues with indices n_m and $n_m + 1$. Let us define

$$\tilde{l}_{n_m}^- = \left(\prod_{s=n_m+2}^{N} \alpha_{n_m,s} \right) l_{n_m}, \qquad \tilde{l}_{n_m+1}^- = \left(\prod_{s=n_m+2}^{N} \alpha_{n_m+1,s} \right) l_{n_m+1} . \tag{A6}$$

(If $m = M$, we define $\prod_{s=n_m+2}^{N} \alpha_{n_m,s} = 1$ and $\prod_{s=n_m+2}^{N} \alpha_{n_m+1,s} = 1$.) By dividing every term in the solution (2.5) by $\prod_{s=n_m+2}^{N} |l_s|^2$ and using the Lemma 24 and definition (A6), as $t \to -\infty$ the solution (2.5) rewrites to

$$q(x,t) = q_{\text{num}}(x,t)/q_{\text{denom}}(x,t) + o(1) , \tag{A7}$$

where

$$q_{\text{num}}(x,t) = \tilde{l}_{n_m}^- + \tilde{l}_{n_m+1}^- + \frac{(\alpha_{n_m} - \alpha_{n_m+1})^2 |\tilde{l}_{n_m}^-|^2 \tilde{l}_{n_m+1}^-}{(\alpha_{n_m}^* + \alpha_{n_m})^2 (\alpha_{n_m}^* + \alpha_{n_m+1})^2} + \frac{(\alpha_{n_m+1} - \alpha_{n_m})^2 \tilde{l}_{n_m}^- |\tilde{l}_{n_m+1}^-|^2}{(\alpha_{n_m+1}^* + \alpha_{n_m})^2 (\alpha_{n_m+1}^* + \alpha_{n_m+1})^2} ,$$

$$\begin{aligned} q_{\text{denom}}(x,t) = 1 &+ \frac{|\tilde{l}_{n_m}^-|^2}{(\alpha_{n_m}^* + \alpha_{n_m})^2} + \frac{\tilde{l}_{n_m}^{-,*} \tilde{l}_{n_m+1}^-}{(\alpha_{n_m}^* + \alpha_{n_m+1})^2} + \frac{\tilde{l}_{n_m+1}^{-,*} \tilde{l}_{n_m}^-}{(\alpha_{n_m+1}^* + \alpha_{n_m})^2} \\ &+ \frac{|\tilde{l}_{n_m+1}^-|^2}{(\alpha_{n_m+1}^* + \alpha_{n_m+1})^2} + \frac{(\alpha_{n_m}^* - \alpha_{n_m+1}^*)^2 (\alpha_{n_m} - \alpha_{n_m+1})^2 |\tilde{l}_{n_m}^- \tilde{l}_{n_m+1}^-|^2}{(\alpha_{n_m}^* + \alpha_{n_m})^2 (\alpha_{n_m+1}^* + \alpha_{n_m+1})^2 |\alpha_{n_m}^* + \alpha_{n_m+1}|^4} . \end{aligned}$$

This is similar to Eq. (2.8). Let us define

$$\beta^-_{n_m} = \beta_{n_m} + \sum_{s=n_m+2}^{N} \ln \alpha_{n_m,s}, \qquad \beta^-_{n_m+1} = \beta_{n_m+1} + \sum_{s=n_m+2}^{N} \ln \alpha_{n_m+1,s}\,.$$

We rewrite the solution (A7) in the polar solution form [cf. Eq. (3.1)] as $t \to -\infty$ along $x = 2V_{n_\mathrm{m}} + y$,

$$q(x,t) = A^-_{n_m}(x,t)\mathrm{e}^{iZ^-_{n_m}(x,t)} + o(1),$$

where $A^-_{n_m}(x,t), Z^-_{n_m}(x,t) \in \mathbb{R}$ are given in Theorem 16. Thus, the asymptotic period and CoM of the soliton group are

$$T^-_{n_m} = T_{n_m}, \qquad \bar{\xi}^-_{n_m} = \bar{\xi}_{n_m} - \frac{1}{A_{n_m} + A_{n_m+1}} \sum_{s=n_m+2}^{N} \sum_{l=0}^{1} \ln \frac{(A_{n_m+l} - A_s)^2 + (V_{n_m+l} - V_s)^2}{(A_{n_m+l} + A_s)^2 + (V_{n_m+l} - V_s)^2}\,. \tag{A8a}$$

The following results with $t \to \infty$ are obtained similarly

$$T^+_{n_m} = T_{n_m}, \qquad \bar{\xi}^+_{n_m} = \bar{\xi}_{n_m} - \frac{1}{A_{n_m} + A_{n_m+1}} \sum_{s=1}^{n_m-1} \sum_{l=0}^{1} \ln \frac{(A_{n_m+l} - A_s)^2 + (V_{n_m+l} - V_s)^2}{(A_{n_m+l} + A_s)^2 + (V_{n_m+l} - V_s)^2}. \tag{A8b}$$

This completes the calculation of the long-time asymptotics. □

7. Long-time asymptotics of arbitrarily degenerate soliton solutions

The goal of this section is to prove Theorem 18. We will need the following result which is obtained by direct calculation.

Lemma 25. Let us consider the m-th soliton group in an N-soliton solution. This soliton group has degree of degeneracy d_m. Let $1 \le k, \lambda \le N$ and $x = 2Vt + y$ for $V, y \in \mathbb{R}$. Let $1 \le i_1 < i_2 < \cdots < i_k \le N$ and $1 \le j_1 < j_2 < \cdots < j_\lambda \le N$. Let F be given by Eq. (A5) as before.

(i) *If $V = V_{n_m}$ for $m = 1, \ldots, M$. Letting $t \to -\infty$, the following asymptotics holds:*

$$\prod_{\mu=1}^{k} l^*_{i_\mu} \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=n_m+d_m}^{N} |l_s|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l^*_{i_\mu} \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=n_m+d_m}^{N} |l_s|^2, \\ o(1), & \text{otherwise.} \end{cases}$$

(ii) *If $V = V_{n_m}$ for $m = 1, \ldots, M$, letting $t \to \infty$, the following asymptotics holds:*

$$\prod_{\mu=1}^{k} l^*_{i_\mu} \prod_{\nu=1}^{\lambda} l_{j_\nu} \Big/ \prod_{s=1}^{n_m-1} |l_s|^2 = \begin{cases} F, & \text{if } \prod_{\mu=1}^{k} l^*_{i_\mu} \prod_{\nu=1}^{\lambda} l_{j_\nu} = F \prod_{s=1}^{n_m-1} |l_s|^2, \\ o(1), & \text{otherwise.} \end{cases}$$

Above and below, for $m = 1$ and $m = M$ we denote $\prod_{s=1}^{n_m-1} |l_s|^2 = 1$ and $\prod_{s=n_m+d_m}^{N} |l_s|^2 = 1$, respectively, similarly to Subsection 6 of the Appendix.

Proof of Theorem 18. Using Lemma 25, it is easy to show that along the line $x = 2Vt + y$ with $V \ne V_{n_m}$ for $m = 1, \ldots, M$ and $y \in \mathbb{R}$, the solution satisfies $\lim_{t\to\pm\infty} q(x,t) = 0$.

Next, we compute the asymptotics of this N-soliton solution along the line $x = 2Vt + y$ with $V = V_{n_\mathrm{m}}$. Let $x = 2V_{n_\mathrm{m}}t + y$ where $m = 1, \ldots, M$ and $y \in \mathbb{R}$. By dividing numerator and denominator in the solution (2.5) by $\prod_{s=n_m+d_m}^{N} |l_s|^2$, applying Lemma 25 and performing simple calculation, as $t \to -\infty$, the solution rewrites to

$$q(x,t) = \frac{\sum_{k=1}^{d_m} \tilde{p}\binom{n_m+d_m,\ldots,N}{n_{m+k-1},n_m+d_m,\ldots,N} \Big/ \tilde{p}\binom{n_m+d_m,\ldots,N}{n_m+d_m,\ldots,N} + \sum_{k=1}^{d_m-1} \sum_{\substack{i_1,\ldots,i_k=n_m \\ i_1<\cdots<i_k}} \sum_{\substack{j_1,\ldots,j_{k+1}=n_m \\ j_1<\cdots<j_{k+1}}}^{n_m+d_m-1} \tilde{p}\binom{i_1,\ldots,i_k,n_m+d_m,\ldots,N}{j_1,\ldots,j_{k+1},n_m+d_m,\ldots,N} \Big/ \tilde{p}\binom{n_m+d_m,\ldots,N}{n_m+d_m,\ldots,N}}{1 + \sum_{k=1}^{d_m} \sum_{\substack{i_1,\ldots,i_k=n_m \\ i_1<\cdots<i_k}} \sum_{\substack{j_1,\ldots,j_k=n_m \\ j_1<\cdots<j_k}}^{n_m+d_m-1} \tilde{p}\binom{i_1,\ldots,i_k,n_m+d_m,\ldots,N}{j_1,\ldots,j_k,n_m+d_m,\ldots,N} \Big/ \tilde{p}\binom{n_m+d_m,\ldots,N}{n_m+d_m,\ldots,N}} + o(1),$$

where the expression $\tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_\lambda}$ is defined similarly to Eq. (2.6a) with $\tilde{l}$ instead of l with $\tilde{l}_i = l_i \prod_{n_m+d_m \le \nu \le N} \alpha_{i,\nu}$. Recall that the parameter $\alpha_{i,j}$ is defined in Eq. (2.13d).

Next, let us examine the quotient term $\tilde{p}\binom{i_1,\ldots,i_k,n_m+d_m,\ldots,N}{j_1,\ldots,j_\lambda,n_m+d_m,\ldots,N} \Big/ \tilde{p}\binom{n_m+d_m,\ldots,N}{n_m+d_m,\ldots,N}$ where $n_m \le i_1 < \cdots < i_k \le n_m + d_m - 1$ and $n_m \le j_1 < \cdots < j_\lambda \le n_m + d_m - 1$ with $1 \le k, \lambda \le d_m - 1$. By the definition (2.6a), it is easy to show that $\tilde{p}\binom{i_1,\ldots,i_k,n_m+d_m,\ldots,N}{j_1,\ldots,j_\lambda,n_m+d_m,\ldots,N} \Big/ \tilde{p}\binom{n_m+d_m,\ldots,N}{n_m+d_m,\ldots,N} = \tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_\lambda}$. Thus the solution is (with $x = 2V_{n_\mathrm{m}}t + y$)

$$q(x,t) = \frac{\sum_{j=1}^{d_m} \tilde{l}_j + \sum_{k=1}^{d_m-1} \sum_{\substack{i_1,\ldots,i_k=n_m \\ i_1<\cdots<i_k}} \sum_{\substack{j_1,\ldots,j_{k+1}=n_m \\ j_1<\cdots<j_{k+1}}}^{n_m+d_m-1} \tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_{k+1}}}{1 + \sum_{k=1}^{d_m} \sum_{\substack{i_1,\ldots,i_k=n_m \\ i_1<\cdots<i_k}} \sum_{\substack{j_1,\ldots,j_k=n_m \\ j_1<\cdots<j_k}}^{n_m+d_m-1} \tilde{p}\binom{i_1,\ldots,i_k}{j_1,\ldots,j_k}} + o(1) = q_m^-(x,t) + o(1), \qquad t \to -\infty,$$

where $q_m{}^-(x,t)$ is a d_m-soliton solution with soliton amplitudes $A_{n_m}, \ldots, A_{n_m+d_m-1}$, velocity V_{n_m} and $\beta^-_{n_m}, \ldots, \beta^-_{n_m+d_m-1}$,

$$\beta^-_{n_m+s} = \beta_{n_m+s} + \sum_{s'=n_m+d_m}^{N} \ln \alpha_{n_m+s,s'} .$$

The argument is similar when $t \to \infty$. Notice that $n_m + d_m = n_{m+1}$, thus the formulas for the soliton parameters stated in Theorem 18 are obtained. □

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