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On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles

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The inverse scattering transform for the focusing non-linear Schrödinger (NLS) equation with non-zero boundary conditions at infinity and double zeros of the analytic scattering coefficients is presented. The direct problem is discussed, including a detailed analysis of the discrete spectrum in the presence of such double zeros. Such cases lead to double poles for the meromorphic matrices appearing in the inverse problem. Explicit formulae for the coefficients of the singular part of the Laurent expansions of the meromorphic functions at the discrete spectrum are also derived, and the inverse problem is reduced to a standard set of linear algebraic-integral equations. The general solution of the NLS equation in the case of an arbitrary finite number of double zeros is given, and an explicit formula for soliton solutions arising in the case of a single quartet of purely imaginary double zeros is presented. The long-time asymptotic behaviour of such a solution is also studied, and the solution is shown to describe the interaction of two solitons with same amplitude and velocity parameters, which diverge from each other logarithmically as in the case of zero boundary conditions.

Keywords: Solitons; inverse scattering; non-zero boundary conditions; non-linear Schrödinger equation.

1. Introduction

Non-linear Schrödinger (NLS) equations are fundamental physical models. They have applications in various fields such as acoustics, optics or the modelling of deep water waves. Recently, there have been many studies of solutions of the scalar and vector focusing NLS equations on a non-zero background. These solutions are believed to be relevant for modulational instability and the formation of rogue waves [see Zakharov & Shabat (1972), Satsuma & Yajima (1974), Sulem & Sulem (1999), Zakharov & Ostrovsky (2009)].

Recall that the NLS equation is a completely integrable system, for which the initial value problem can be solved via the inverse scattering transform (IST). On the other hand, the study of such solutions from a spectral point of view requires the use of the IST for the NLS equation with non-zero boundary conditions (NZBC). The IST for the scalar NLS equation and coupled NLS systems with zero boundary conditions (ZBC) is well known [see Zakharov & Shabat (1972), Ablowitz & Segur (1981), Novikov *et al.* (1984), Ablowitz *et al.* (2004)]. The IST for the scalar defocusing NLS equation with NZBC was also done a long time ago in Zakharov & Shabat (1973) [see also Faddeev & Takhtajan (1987)], but the IST for two-component NLS systems with NZBC remained open for a long time, and was only recently done in Prinari *et al* (2006) for the defocusing case [see also Biondini & Kraus (2015)]. The IST for the focusing two-component case with NZBC was also done in Kraus *et al.* (2015), building on the results of Biondini & Kovacic (2014), where the IST for the scalar focusing NLS equation with NZBC was formulated.

In particular, in Biondini & Kovacic (2014), expressions for general N-soliton solutions were obtained. As is common, however, those expressions are limited to solutions corresponding to the case in which all discrete eigenvalues are simple. On the other hand, for the focusing NLS equation with ZBC, it has been known since the original work of Zakharov & Shabat (1972) that more general solutions corresponding to double eigenvalues exist. More general 'multi-pole' solutions of the focusing NLS equation with ZBC were also studied in Aktosun *et al.* (2007), Schiebold (2010) and Schiebold (2014). A natural question is therefore whether such solutions exist in the focusing case with NZBC. (Note that these kinds of solutions do not exist in the defocusing scalar case with NZBC, cf. Faddeev & Takhtajan (1987), but are allowed in the defocusing two-component case with NZBC, see Biondini & Kraus (2015).) In this work, we answer this question affirmatively. More precisely, we characterize the behaviour of the eigenfunctions in the presence of a double zero of the analytic scattering coefficients; we formulate an appropriate enlarged Riemann–Hilbert problem (RHP); we obtain an explicit closed-form solution of the RHP; we recover the corresponding solution of the focusing NLS equation; we characterize the behaviour of soliton solutions in the particular case with a quartet of discrete eigenvalues.

Specifically, we study the focusing NLS equation

$$\mathrm{i}q_t + q_{xx} + 2(|q|^2 - q_o^2)q = 0 \tag{1.1}$$

with NZBC as $x \to \pm\infty$,

$$\lim_{x\to\pm\infty} q(x,t) = q_\pm. \tag{1.2}$$

Here, $|q_\pm| = q_o \neq 0$. The term $-2q_o^2 q$ in (1.1) is due to a rescaling chosen to make the boundary conditions independent of time for convenience. Namely, if $q(x,t)$ is a solution to (1.1) and (1.2), then $\tilde{q}(x,t) = \mathrm{e}^{2\mathrm{i}q_o^2 t} q(x,t)$ solves the classical NLS equation $\mathrm{i}\tilde{q}_t + \tilde{q}_{xx} + 2|\tilde{q}|^2\tilde{q} = 0$, with boundary conditions $\lim_{x\to\pm\infty} \tilde{q}(x,t) = \mathrm{e}^{2\mathrm{i}q_o^2 t} q_\pm$.

The structure of this work is the following: In Section 2, we give an outline of the direct scattering. In Section 3, we analyse the discrete spectrum and derive the residue conditions in the presence of double poles. In Section 4, we set up the inverse problem, we solve the RHP and give a formula for the solution of the focusing NLS equation as well as derive a trace formula for the analytic scattering coefficients and a 'theta condition' for the phase difference between the boundary values of the potential. Finally, in Section 5 we give an explicit formula for a soliton solution corresponding to a quartet of purely imaginary discrete eigenvalues and describe the asymptotic behaviour of such solutions.

2. Direct scattering with NZBC

Except for the treatment of the discrete spectrum, most of the direct scattering is unaffected by the presence of double poles as opposed to single poles. The direct scattering for the focusing NLS equation with NZBC and single poles has been presented in detail in Biondini & Kovacic (2014). Here we briefly present the results and refer the reader to Biondini & Kovacic (2014) for further details.

Recall that the focusing NLS equation (1.1) is the compatibility condition of the Lax pair

$$\phi_x = X\phi, \quad \phi_t = T\phi, \tag{2.1}$$

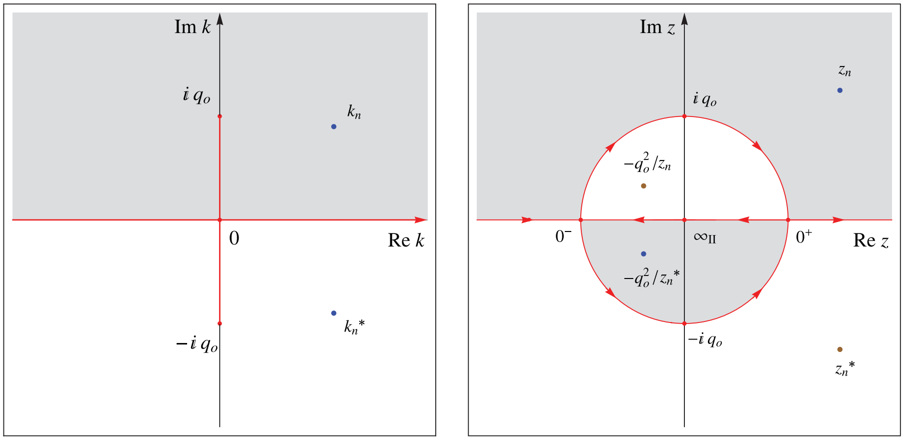

FIG. 1. Left: the first sheet of the Riemann surface, showing the branch cut (the real axis plus the segment of the imaginary axis between $\pm iq_o$) and the region where $\operatorname{Im}\lambda > 0$ (grey). Right: the complex z-plane, showing the regions $D^{\pm}$ where $\operatorname{Im}\lambda > 0$ (grey) and $\operatorname{Im}\lambda < 0$ (white), respectively. Also shown are the orientation of the contours for the RHP and the symmetries of the discrete spectrum of the scattering problem [namely, the zeros of $s_{2,2}(z)$ (in the grey domains) and those of $s_{1,1}(z)$ (in the white domains)]. See text for details.

where

$$X(x,t,k) = \mathrm{i}k\sigma_3 + Q, \quad T(x,t,k) = -2\mathrm{i}k^2\sigma_3 + \mathrm{i}\sigma_3(Q_x - Q^2 - q_o^2) - 2kQ, \tag{2.2a}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{2.2b}$$

That is, the condition $X_t - T_x + [X,T] = 0$ (where $[X,T]$ denotes matrix commutator, $[X,T] = XT - TX$) is satisfied precisely if q satisfies (1.1). The first of the equations (2.1) is called the scattering problem, and $q(x,t)$ the potential.

The asymptotic scattering problem is obtained by taking the limit $x \to \pm\infty$, obtaining $\phi_x = X_\pm \phi$, with $X_\pm = \mathrm{i}k\sigma_3 + Q_\pm$ and $Q_\pm = \lim_{x\to\pm\infty} Q(x,t)$. The eigenvalues of $X_\pm$ are $\pm\mathrm{i}\sqrt{k^2+q_o^2}$. These eigenvalues have two branch points at $k = \pm\mathrm{i}q_o$. We introduce the two-sheeted Riemann surface of equation $\lambda^2 = k^2 + q_o^2$, such that $\lambda(k)$ is single-valued on each sheet. We take the branch cut to be the interval $\mathrm{i}[-q_o, q_o]$. Note that λ is real-valued along the real axis, with $\lambda(k) = \operatorname{sign}(k)\sqrt{k^2+q_o^2}$ on Sheet I and $\lambda(k) = -\operatorname{sign}(k)\sqrt{k^2+q_o^2}$ on Sheet II. We also introduce the uniformization variable $z = \lambda(k) + k$. This defines a map from the Riemann surface onto the complex plane, which will allow us to work with a complex parameter instead of dealing with the more cumbersome two-sheeted Riemann surface. The inverse transformation is

$$\lambda = (z + q_o^2/z)/2, \quad k = (z - q_o^2/z)/2.$$

Under this transformation, the branch cut on either sheet of the Riemann surface is mapped to the circle of radius q_o, which will be denoted by C_o, in the complex plane. The first sheet of the Riemann surface is mapped onto the region in the complex plane outside C_o and the second sheet is mapped onto the region inside C_o, see Fig. 1. From now on, we will use the parameter z instead of k and rewrite all dependences in terms of z.

We further introduce the eigenvector matrices $Y_\pm$ for the asymptotic scattering problem

$$Y_\pm = I + \frac{\mathrm{i}}{z}\sigma_3 Q_\pm$$

such that

$$X_\pm Y_\pm = Y_\pm \mathrm{i}\lambda\sigma_3.$$

Note that $\det Y_\pm = 1 + q_o^2/z^2 =: \gamma(z)$. The asymptotic problem as $x \to \pm\infty$ in the second part of the Lax pair is $\phi_t = T_\pm\phi$, with

$$T_\pm = -2\mathrm{i}k^2\sigma_3 - \mathrm{i}\sigma_3(Q_\pm^2 + q_o^2 I) - 2kQ_\pm.$$

Note that $Q_\pm^2 = -q_o^2 I$, therefore $T_\pm = -2\mathrm{i}k^2\sigma_3 - 2kQ_\pm = -2kX_\pm$. (This reflects the fact that the boundary conditions are independent of time: the compatibility condition for the asymptotic Lax pair simply becomes $[X_\pm, T_\pm] = 0$, hence the matrices are multiples of each other.) Consequently, we have $T_\pm Y_\pm = Y_\pm(-2\mathrm{i}k\lambda\sigma_3) = -\mathrm{i}/2(z^2 - q_o^4/z^2)Y_\pm\sigma_3$. This allows us to define the Jost eigenfunctions as simultaneous solutions of both parts of the Lax pair.

The continuous spectrum of $X_\pm$ is the set of values of k such that $\lambda(k)$ is real-valued. This is the case for $k \in \mathbb{R} \cup \mathrm{i}[-q_o, q_o]$. In the complex z-plane, this is the set $\mathbb{R} \cup C_o =: \Sigma$. For each $z \in \Sigma$, we define the Jost solutions $\phi_\pm(x,t,z)$ as those matrix solutions of both parts of the Lax pair which satisfy the asymptotic behaviour

$$\phi_\pm(x,t,z) = Y_\pm(z)\,\mathrm{e}^{\mathrm{i}\theta(x,t,z)\sigma_3} + o(1), \quad x \to \pm\infty, \tag{2.3}$$

with $\theta(x,t,z) = \lambda(z)(x - 2k(z)t)$. It is convenient to introduce the modified eigenfunctions $\mu_\pm(x,t,z)$ by removing the asymptotic oscillations,

$$\mu_\pm(x,t,z) = \phi_\pm(x,t,z)\,\mathrm{e}^{-\mathrm{i}\theta(x,t,z)\sigma_3},$$

such that the boundary conditions now are $\mu_\pm(x,t,z) \to Y_\pm(z)$ as $x \to \pm\infty$.

Let us introduce the domains $D^\pm = \{z \in \mathbb{C} : \operatorname{Im}\lambda(k) \gtrless 0\}$, denoted by the grey and white regions in Fig. 1 (right). It was shown in Biondini & Kovacic (2014) that the columns $\mu_{+,1}$ and $\mu_{-,2}$ are analytic in D^+, and $\mu_{+,2}$ and $\mu_{-,1}$ are analytic in D^-. This implies analyticity of the corresponding columns of $\phi_\pm$ in the respective regions. Note that throughout, superscripts $\pm$ are used to denote the analyticity regions, and subscripts refer to the asymptotic behaviour as $x \to \pm\infty$.

Since $\operatorname{tr} X = \operatorname{tr} T = 0$, it follows that $(\det\phi_\pm)_x = (\det\phi_\pm)_t = 0$. Therefore, $\det\phi_\pm(x,t,z) = \det Y_\pm(z) = \gamma(z)$ for $z \in \Sigma$, since $\det \mathrm{e}^{\mathrm{i}\theta(x,t,z)\sigma_3} = 1$. Note that $\gamma(\pm\mathrm{i}q_o) = 0$, but $\gamma(z) \neq 0$ for $z \in \Sigma_o := \Sigma \setminus \{\pm\mathrm{i}q_o\}$. This implies that for z in Σ_o, both ϕ_+ and ϕ_- are fundamental matrix solutions of the Lax pair, and there exists a regular matrix S that depends on z, but not on x and t, such that

$$\phi_+(x,t,z) = \phi_-(x,t,z)S(z), \quad z \in \Sigma_o. \tag{2.4}$$

(The behaviour of the eigenfunctions at the branch points $z = \pm\mathrm{i}q_o$ was studied in Biondini & Kovacic (2014).) The matrix $S(z)$ is called the scattering matrix, and its entries $s_{i,j}(z)$ are the scattering coefficients. Column-wise, (2.4) reads

$$\phi_{+,1} = s_{1,1}\phi_{-,1} + s_{2,1}\phi_{-,2}, \quad \phi_{+,2} = s_{1,2}\phi_{-,1} + s_{2,2}\phi_{-,2}. \tag{2.5}$$

The scattering coefficients can be expressed as Wronskians of columns of $\phi_\pm$ in the following way:

$$s_{1,1}(z) = \mathrm{Wr}(\phi_{+,1}, \phi_{-,2})/\gamma, \quad s_{1,2}(z) = \mathrm{Wr}(\phi_{+,2}, \phi_{-,2})/\gamma, \tag{2.6a}$$

$$s_{2,1}(z) = \mathrm{Wr}(\phi_{-,1}, \phi_{+,1})/\gamma, \quad s_{2,2}(z) = \mathrm{Wr}(\phi_{-,1}, \phi_{+,2})/\gamma. \tag{2.6b}$$

From these representations, we see that the scattering coefficient $s_{1,1}$ can be extended analytically to D^+, and similarly $s_{2,2}$ is analytic in D^-. But the coefficients $s_{1,2},\ s_{2,1}$ are not analytic in general.

We also define the reflection coefficients as

$$\rho(z) = s_{2,1}(z)/s_{1,1}(z), \quad \tilde{\rho}(z) = s_{1,2}(z)/s_{2,2}(z),\ z \in \Sigma. \tag{2.7}$$

The scattering problem admits two independent symmetries, which come from transformations that are compatible with the defining equation $\lambda^2 = k^2 + q_o^2$ of the Riemann surface. The first transformation is the map $(k, \lambda) \mapsto (k^*, \lambda^*)$, which maps points on a given sheet of the Riemann surface into the same sheet. (Recall that the transformation $k \mapsto k^*$ yields a symmetry in the case of ZBC.) In terms of z, this is the map $z \mapsto z^*$. Additionally, we have the map $(k, \lambda) \mapsto (k, -\lambda)$ between the two sheets of the Riemann surface (recall that $\lambda_I(k) = -\lambda_{II}(k)$), which in terms of z is $z \mapsto -q_o^2/z$, mapping points outside C_o into the region inside of C_o and vice versa. This transformation stems from the presence of the NZBC and the two-sheeted Riemann surface.

The Jost eigenfunctions satisfy the following relations for $z \in \Sigma$:

$$\phi_\pm(x,t,z) = -\sigma_* \phi_\pm^*(x,t,z^*)\sigma_*, \quad \sigma_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\phi_\pm(x,t,z) = \frac{\mathrm{i}}{z}\phi_\pm(x,t,-q_o^2/z)\sigma_3 Q_\pm.$$

The modified eigenfunctions satisfy similar symmetries. Since $\theta(x,t,z^*) = \theta^*(x,t,z)$ and $\mathrm{e}^{-\mathrm{i}\theta(x,t,z)\sigma_3}\sigma_* = \sigma_* \mathrm{e}^{\mathrm{i}\theta(x,t,z)\sigma_3}$, we have

$$\mu_\pm(x,t,z) = -\sigma_* \mu_\pm^*(x,t,z^*)\sigma_*, \quad z \in \Sigma.$$

Furthermore, since $\theta(x,t,-q_o^2/z) = -\theta(x,t,z)$ and $\mathrm{e}^{-\mathrm{i}\theta(x,t,z)\sigma_3}\sigma_3 Q_\pm = \sigma_3 Q_\pm \mathrm{e}^{\mathrm{i}\theta(x,t,z)\sigma_3}$, we also have

$$\mu_\pm(x,t,z) = \frac{\mathrm{i}}{z}\mu_\pm(x,t,-q_o^2/z)\sigma_3 Q_\pm, \quad z \in \Sigma.$$

Column-wise, these equations give the following relations between the individual solution functions:

$$\mu_{\pm,1}(x,t,z) = \sigma_* \mu_{\pm,2}^*(x,t,z^*), \quad \mu_{\pm,2}(x,t,z) = -\sigma_* \mu_{\pm,1}^*(x,t,z^*), \tag{2.8a}$$

$$\mu_{\pm,1}(x,t,z) = \frac{\mathrm{i}q_\pm^*}{z}\mu_{\pm,2}(x,t,-q_o^2/z), \quad \mu_{\pm,2}(x,t,z) = \frac{\mathrm{i}q_\pm}{z}\mu_{\pm,1}(x,t,-q_o^2/z). \tag{2.8b}$$

Using each of the symmetries for the matrices $\phi_\pm$ in the scattering relation (2.4) further yields the following symmetries for the scattering coefficients:

$$s_{2,2}(z) = s_{1,1}^*(z^*), \quad s_{1,2}(z) = -s_{2,1}^*(z^*), \tag{2.9a}$$

$$s_{1,1}(z) = (q_+^*/q_-^*)s_{2,2}\big(-q_o^2/z\big), \quad s_{1,2}(z) = (q_+/q_-^*)s_{2,1}\big(-q_o^2/z\big). \tag{2.9b}$$

We can also combine the two symmetries to obtain relations between values of the functions at z^* and $-q_o^2/z$. For the scattering coefficients, we then have

$$s_{1,1}^*(z^*) = (q_+/q_-)s_{1,1}\big(-q_o^2/z\big), \quad s_{1,2}^*(z^*) = -(q_+^*/q_-)s_{1,2}\big(-q_o^2/z\big),$$
$$s_{2,1}^*(z^*) = -(q_+/q_-^*)s_{2,1}\big(-q_o^2/z\big), \quad s_{2,2}^*(z^*) = (q_+^*/q_-^*)\, s_{2,2}\big(-q_o^2/z\big).$$

Finally, the symmetries of the scattering coefficients yield the following symmetries of the reflection coefficients:

$$\rho(z) = -\tilde{\rho}^*(z^*) = (q_-/q_-^*)\tilde{\rho}(-q_o^2/z) = -(q_-^*/q_-)\rho^*(-q_o^2/z^*), \quad z \in \Sigma. \tag{2.10}$$

Note that the above relations generically hold only for $z \in \Sigma$. However, whenever the involved functions are analytic in certain regions of the complex plane, the symmetries extend to these regions by the Schwarz reflection principle.

To properly set up the inverse problem, one needs the asymptotic behaviour of the Jost eigenfunctions and scattering coefficients as $k \to \infty$. This corresponds to two different limits in terms of the uniformization variable $z = k + \lambda$. Note that on the first sheet of the Riemann surface, $z \to \infty$ as $k \to \infty$. On the second sheet, however, $z \to 0$ as $k \to \infty$. Therefore, we need to consider the asymptotic behaviour both as $z \to \infty$ and $z \to 0$. To this purpose, the eigenfunctions $\mu_\pm$ can be expanded in a Neumann series:

$$\mu_-(x,t,z) = \sum_{n=0}^{\infty} \mu^{(n)}(x,t,z)$$

with

$$\mu^{(0)}(x,t,z) = Y_-, \quad \mu^{(n+1)}(x,t,z) = \int_{-\infty}^{x} Y_- \mathrm{e}^{\mathrm{i}\lambda(z)(x-y)\sigma_3}\big(Y_-^{-1}\Delta Q_-(y,t)\mu^{(n)}(y,t,z)\big)\mathrm{e}^{-\mathrm{i}\lambda(z)(x-y)\sigma_3}\,\mathrm{d}y,$$

where $\Delta Q_- = Q - Q_-$. Recall that the individual columns of $\mu_\pm$ are analytic in different regions of the complex plane. The following relations are written in terms of the matrices $\mu_\pm$ rather than column-wise, so they are to be understood as taken in the appropriate region for each column. For example, for the first column of μ_-, $z \to \infty$ in the lower half plane and for the second column $z \to \infty$ in the upper half plane. Then for each $m \in \mathbb{N}$,

$$\mu_d^{(2m)} = O\big(1/z^m\big), \quad \mu_d^{(2m+1)} = O\Big(1/z^{m+1}\Big), \quad \mu_o^{(2m)} = O\Big(1/z^{m+1}\Big), \quad \mu_o^{(2m+1)} = O\Big(1/z^{m+1}\Big),$$

as $z \to \infty$ in the appropriate region of the complex plane, where M_o and M_d denote, respectively, the off-diagonal and diagonal parts of a matrix M. Similarly,

$$\mu_d^{(2m)} = O\big(z^m\big), \quad \mu_d^{(2m+1)} = O\big(z^m\big), \quad \mu_o^{(2m)} = O\Big(z^{m-1}\Big), \quad \mu_o^{(2m+1)} = O\big(z^m\big),$$

as $z \to 0$ in the appropriate region. Explicitly, computing the first terms of the expansion yields

$$\mu_-(x,t,z) = I + \frac{\mathrm{i}}{z}\sigma_3 Q(x,t) + \frac{\mathrm{i}}{z}\int_{-\infty}^{x}([\sigma_3 Q_-, \Delta Q_-(y,t)] + \Delta Q_-(y,t)\sigma_3\Delta Q_-(y,t))\,\mathrm{d}y + O(z^{-2}) \tag{2.11}$$

as $z \to \infty$. This equation will be used to reconstruct the potential $q(x,t)$. Indeed, we see that

$$q(x,t) = \lim_{z\to\infty} -\mathrm{i}z\mu_{-,1,2}(x,t,z). \tag{2.12}$$

Similarly, one sees that

$$\mu_+(x,t,z) = I + O(1/z), \quad z \to \infty, \tag{2.13}$$

$$\mu_+(x,t,z) = \frac{\mathrm{i}}{z}\sigma_3 Q_+(x,t) + O(1), \quad z \to 0. \tag{2.14}$$

The asymptotic behaviour of the Jost eigenfunctions further implies the following asymptotics of the scattering matrix, by using the relations (2.6): in the appropriate regions of the complex plane (that is, in the upper half plane for $s_{1,1}$, in the lower half plane for $s_{2,2}$, and along the real axis for $s_{1,2}$ and $s_{2,1}$),

$$S(z) = I + O(1/z), \quad z \to \infty, \tag{2.15}$$

$$S(z) = \mathrm{diag}(q_-/q_+, q_+/q_-) + O(z), \quad z \to 0. \tag{2.16}$$

3. Discrete spectrum with double poles

Those values of $z \in \mathbb{C} \setminus \Sigma$ for which there exist eigenfunctions in $L^2(\mathbb{R})$ constitute the discrete spectrum of the scattering problem. As is shown in Biondini & Kovacic (2014), these are precisely the values of z in D^+ for which $s_{1,1}(z) = 0$ and those values in D^- for which $s_{2,2}(z) = 0$. (Note that zeros of the scattering coefficients along the continuous spectrum Σ are possible, but we do not consider this case here.) We need to take these values into account as they yield poles of the functions $\mu_{+,1}/s_{1,1}$ and $\mu_{+,2}/s_{2,2}$ occurring in the inverse problem, so in order to regularize the RHP, we need to subtract the principal part of the Laurent expansion of the functions at each pole.

In Biondini & Kovacic (2014), the case of single zeros of the analytic scattering coefficients was studied, and up to this point, our treatment was not different from that work. We will now deviate from that path, however, and consider the case that z_o is a double zero of the scattering coefficient $s_{1,1}$, that is, we have $s_{1,1}(z_o) = s'_{1,1}(z_o) = 0$, and $s''_{1,1}(z_o) \neq 0$. (Here and in what follows, we let $'$ denote the derivative with respect to z.) Consequently, $\mu_{+,1}/s_{1,1}$ has a double pole at z_o, and in addition to the residue we need to compute the coefficient of the term of order -2.

First note that from the first of each of the symmetries (2.9a) and (2.9b) it follows that if $s_{1,1}(z_o) = 0$, then also

$$s_{2,2}\left(z_o^*\right) = s_{2,2}\left(-q_o^2/z_o\right) = s_{1,1}\left(-q_o^2/z_o^*\right) = 0.$$

Therefore, the discrete eigenvalues occur in quartets of complex numbers. We suppose now that $s_{1,1}$ has N double zeros in the part of D^+ in the upper half plane (that is, $|z| > q_o$ and $\mathrm{Im}\, z > 0$), and denote them by $z_n,\ n = 1, \ldots, N$; then the discrete spectrum of the scattering problem is the set

$$Z = \left\{z_n,\ z_n^*,\ -q_o^2/z_n,\ -q_o^2/z_n^*\right\}_{n=1,\ldots,N}.$$

For brevity, we will henceforth use the shorthand notations $\hat{z}_n := -q_o^2/z_n$ and $\check{z}_n := -q_o^2/z_n^*$. We first show that if $z_n \in \mathbb{C}$ with $|z_n| > q_o$ and $\mathrm{Im}\, z_n > 0$ is a double zero of $s_{1,1}$, then there are *norming constants*

b_n and d_n that are independent of x and t such that

$$\mu_{+,1}(x,t,z_n) = \mathrm{e}^{-2\mathrm{i}\theta_n(x,t)} b_n \mu_{-,2}(x,t,z_n) \tag{3.1}$$

and

$$\mu'_{+,1}(x,t,z_n) = \mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}((d_n - 2i\theta'_n(x,t)b_n)\mu_{-,2}(x,t,z_n) + b_n\mu'_{-,2}(x,t,z_n)). \tag{3.2}$$

Here, $\theta_n(x,t) := \theta(x,t,z_n)$ and $\theta'_n(x,t) := \theta'(x,t,z_n)$. To see this, note that from the Wronskian representation (2.6) it follows that if $s_{1,1}(z_n) = 0$, then

$$\mathrm{Wr}(\phi_{+,1}(x,t,z_n), \phi_{-,2}(x,t,z_n)) = 0, \tag{3.3}$$

and therefore there exists a constant b_n such that

$$\phi_{+,1}(x,t,z_n) = b_n \phi_{-,2}(x,t,z_n). \tag{3.4}$$

Note that b_n is independent of both x and t, since the Jost solutions are simultaneous solutions of both parts of the Lax pair. We use the relations $\phi_{+,1} = \mu_{+,1}\mathrm{e}^{\mathrm{i}\theta}$ and $\phi_{-,2} = \mu_{-,2}\mathrm{e}^{-\mathrm{i}\theta}$ between the Jost eigenfunctions and the modified eigenfunctions to find that in terms of $\mu_\pm$,

$$\mu_{+,1}(x,t,z_n) = \mathrm{e}^{-2\mathrm{i}\theta_n(x,t)} b_n \mu_{-,2}(x,t,z_n).$$

Since z_n is a double zero of $s_{1,1}$, from (2.9) we also get

$$0 = s'_{1,1}(z_n) = \left[\frac{1}{\gamma}\left(\mathrm{Wr}\big(\phi'_{+,1}, \phi_{-,2}\big) + \mathrm{Wr}\big(\phi_{+,1}, \phi'_{-,2}\big)\right) - \frac{\gamma'}{\gamma^2}\mathrm{Wr}(\phi_{+,1}, \phi_{-,2})\right]_{z=z_n}.$$

The last term on the right-hand side vanishes because of (3.3), and using (3.4) we have

$$0 = \mathrm{Wr}(\phi'_{+,1}(x,t,z_n) - b_n\phi'_{-,2}(x,t,z_n), \phi_{-,2}(x,t,z_n)),$$

Thus, there exists another constant d_n such that

$$\phi'_{+,1}(x,t,z_n) = d_n\phi_{-,2}(x,t,z_n) + b_n\phi'_{-,2}(x,t,z_n).$$

Expressing $\phi_\pm$ in terms of $\mu_\pm$ yields that at $z = z_n$,

$$\mu'_{+,1}\,\mathrm{e}^{\mathrm{i}\theta_n} + \mathrm{i}\theta'_n\mu_{+,1}\,\mathrm{e}^{\mathrm{i}\theta_n} = d_n\mu_{-,2}\,\mathrm{e}^{-\mathrm{i}\theta_n} + b_n\mu'_{-,2}\,\mathrm{e}^{-\mathrm{i}\theta_n} - b_n\mathrm{i}\theta'_n\mu_{-,2}\,\mathrm{e}^{-\mathrm{i}\theta_n},$$

and hence, using (3.1),

$$\mu'_{+,1}(x,t,z_n) = \mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}((d_n - 2i\theta'_n(x,t)b_n)\mu_{-,2}(x,t,z_n) + b_n\mu'_{-,2}(x,t,z_n)).$$

Since z_n is a double zero of $s_{1,1}$, it follows from (2.9) that $\check{z}_n$ is a double zero of $s_{1,1}$ and z_n^* and $\hat{z}_n$ are double zeros of $s_{2,2}$. Thus, in the same way as above it can be shown that furthermore there are constants

$\tilde{b}_n$, $\tilde{d}_n$, $\hat{b}_n$, $\hat{d}_n$, $\check{b}_n$ and $\check{d}_n$ such that

$$\mu_{+,2}(x,t,z_n^*) = \mathrm{e}^{2\mathrm{i}\theta_n^*(x,t)}\tilde{b}_n\mu_{-,1}(x,t,z_n^*), \tag{3.5a}$$

$$\mu'_{+,2}(x,t,z_n^*) = \mathrm{e}^{2\mathrm{i}\theta_n^*(x,t)}\Big(\big(\tilde{d}_n + 2\mathrm{i}\theta_n'^*(x,t)\tilde{b}_n\big)\mu_{-,1}(x,t,z_n^*) + \tilde{b}_n\mu'_{-,1}(x,t,z_n^*)\Big), \tag{3.5b}$$

$$\mu_{+,2}(x,t,\hat{z}_n) = \mathrm{e}^{2\mathrm{i}\hat{\theta}_n(x,t)}\hat{b}_n\mu_{-,1}(x,t,\hat{z}_n), \tag{3.5c}$$

$$\mu'_{+,2}(x,t,\hat{z}_n) = \mathrm{e}^{2\mathrm{i}\hat{\theta}_n(x,t)}\Big(\big(\hat{d}_n + 2\mathrm{i}\hat{\theta}'_n(x,t)\hat{b}_n\big)\mu_{-,1}(x,t,\hat{z}_n) + \hat{b}_n\mu'_{-,1}(x,t,\hat{z}_n)\Big), \tag{3.5d}$$

$$\mu_{+,1}(x,t,\check{z}_n) = \mathrm{e}^{-2\mathrm{i}\check{\theta}_n(x,t)}\check{b}_n\mu_{-,2}(x,t,\check{z}_n), \tag{3.5e}$$

$$\mu'_{+,1}(x,t,\check{z}_n) = \mathrm{e}^{-2\mathrm{i}\check{\theta}_n(x,t)}\Big(\big(\check{d}_n - 2\mathrm{i}\check{\theta}'_n(x,t)\check{b}_n\big)\mu_{-,2}(x,t,\check{z}_n) + \check{b}_n\mu'_{-,2}(x,t,\check{z}_n)\Big), \tag{3.5f}$$

with

$$\theta_n^*(x,t) = \theta(x,t,z_n^*), \quad \hat{\theta}_n(x,t) = \theta(x,t,\hat{z}_n), \quad \check{\theta}_n(x,t) = \theta(x,t,\check{z}_n),$$
$$\theta_n'^*(x,t) = \theta'(x,t,z_n^*), \quad \hat{\theta}'_n(x,t) = \theta'(x,t,\hat{z}_n), \quad \check{\theta}'_n(x,t) = \theta'(x,t,\check{z}_n).$$

The symmetries of the eigenfunctions yield corresponding relations between the norming constants:

$$\tilde{b}_n = -b_n^*, \quad \hat{b}_n = b_n\frac{q_-}{q_+^*}, \quad \check{b}_n = -\hat{b}_n^*, \quad \tilde{d}_n = -d_n^*, \quad \hat{d}_n = -d_n\frac{q_-z_n^2}{q_+^*q_o^2}, \quad \check{d}_n = -\hat{d}_n^*. \tag{3.6}$$

To see this, consider equation (3.5a) and use the symmetries (2.8a). Then

$$-\sigma_*\mu_{+,1}^*(x,t,z_n) = \tilde{b}_n\mathrm{e}^{2\mathrm{i}\theta_n^*(x,t)}\sigma_*\mu_{-,2}^*(x,t,z_n).$$

Multiplying by $\sigma_*^{-1} = -\sigma_*$, taking the complex conjugate and using $\theta(x,t,z^*) = \theta^*(x,t,z)$, we obtain

$$\mu_{+,1}(x,t,z_n) = -\tilde{b}_n^*\mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}\mu_{-,2}(x,t,z_n).$$

Comparing this with (3.1), we find that $\tilde{b}_n = -b_n^*$. The other relations are obtained in the same manner.

Next we obtain the residue conditions that will be needed for the inverse problem. We denote, respectively, by

$$\operatorname*{Res}_{z=z_o}[f(z)] = F_{-1}, \quad \operatorname*{P_{-2}}_{z=z_o}[f(z)] = F_{-2}, \tag{3.7a}$$

the residue and the coefficient of $(z-z_o)^{-2}$ in the Laurent expansion of a meromorphic function $F(z)$ around z_o,

$$F(z) = \sum_{n=-\infty}^{\infty}\frac{F_n}{(z-z_o)^n}, \quad 0 < |z-z_o| < R, \tag{3.7b}$$

where R is the associated radius of convergence.

Recall that, if f and g are analytic in a region $\Omega \subset \mathbb{C}$, and g has a double zero at $z_o \in \Omega$ and $f(z_o) \neq 0$, the coefficients of the principal part of the Laurent expansion of f/g at $z = z_o$ are

$$\operatorname*{Res}_{z=z_o}\left[\frac{f}{g}\right] = \frac{2f'(z_o)}{g''(z_o)} - \frac{2f(z_o)g'''(z_o)}{3(g''(z_o))^2}, \quad \underset{z=z_n}{P_{-2}}\left[\frac{f}{g}\right] = \frac{2f(z_o)}{g''(z_o)}.$$

Using these formulae, we obtain the following expressions for the coefficients of $\mu_{+,1}/s_{1,1}$, using (3.1) and (3.2):

$$\underset{z=z_n}{P_{-2}}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = \frac{2\mu_{+,1}(x,t,z_n)}{s''_{1,1}(z_n)} = \frac{2b_n}{s''_{1,1}(z_n)}\mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}\mu_{-,2}(x,t,z_n),$$

$$\operatorname*{Res}_{z=z_n}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = \frac{2b_n}{s''_{1,1}(z_n)}\mathrm{e}^{-2i\theta_n(x,t)}\left[\mu'_{-,2}(x,t,z_n) + \mu_{-,2}(x,t,z_n)\left(\frac{d_n}{b_n} - 2i\theta'_n(x,t) - \frac{s'''_{1,1}(z_n)}{3s''_{1,1}(z_n)}\right)\right].$$

Denoting $A_n := 2b_n/s''_{1,1}(z_n)$ and $B_n := (d_n/b_n) - (s'''_{1,1}(z_n)/3s''_{1,1}(z_n))$, the expressions take the more compact form

$$\underset{z=z_n}{P_{-2}}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = A_n\mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}\mu_{-,2}(x,t,z_n), \tag{3.8a}$$

$$\operatorname*{Res}_{z=z_n}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = A_n\mathrm{e}^{-2\mathrm{i}\theta_n(x,t)}(\mu'_{-,2}(x,t,z_n) + \mu_{-,2}(x,t,z_n)(B_n - 2i\theta'_n(x,t))). \tag{3.8b}$$

Since $\mu_{+,1}/s_{1,1}$ also has a double pole at $z = \check{z}_n$, and $\mu_{-,2}/s_{2,2}$ has double poles at $z = z_n^*$ and $z = \hat{z}_n$, we analogously obtain the expressions

$$\underset{z=z_n^*}{P_{-2}}\left[\frac{\mu_{+,2}}{s_{2,2}}\right] = \tilde{A}_n\,\mathrm{e}^{2\mathrm{i}\theta_n^*(x,t)}\mu_{-,1}(x,t,z_n^*), \tag{3.9a}$$

$$\operatorname*{Res}_{z=z_n^*}\left[\frac{\mu_{+,2}}{s_{2,2}}\right] = \tilde{A}_n\,\mathrm{e}^{2\mathrm{i}\theta_n^*(x,t)}(\mu'_{-,1}(x,t,z_n^*) + \mu_{-,1}(x,t,z_n^*)(\tilde{B}_n + 2\mathrm{i}\theta_n'^*(x,t))), \tag{3.9b}$$

$$\underset{z=\hat{z}_n}{P_{-2}}\left[\frac{\mu_{+,2}}{s_{2,2}}\right] = \hat{A}_n\,\mathrm{e}^{2\mathrm{i}\hat{\theta}_n(x,t)}\mu_{-,1}(x,t,\hat{z}_n), \tag{3.9c}$$

$$\operatorname*{Res}_{z=\hat{z}_n}\left[\frac{\mu_{+,2}}{s_{2,2}}\right] = \hat{A}_n\,\mathrm{e}^{2\mathrm{i}\hat{\theta}_n(x,t)}(\mu'_{-,1}(x,t,\hat{z}_n) + \mu_{-,1}(x,t,\hat{z}_n)(\hat{B}_n + 2\mathrm{i}\hat{\theta}'_n(x,t))), \tag{3.9d}$$

$$\underset{z=\check{z}_n}{P_{-2}}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = \check{A}_n\,\mathrm{e}^{-2\mathrm{i}\check{\theta}_n(x,t)}\mu_{-,2}(x,t,\check{z}_n), \tag{3.9e}$$

$$\operatorname*{Res}_{z=\check{z}_n}\left[\frac{\mu_{+,1}}{s_{1,1}}\right] = \check{A}_n\,\mathrm{e}^{-2\mathrm{i}\check{\theta}_n(x,t)}(\mu'_{-,2}(x,t,\check{z}_n) + \mu_{-,2}(x,t,\check{z}_n)(\check{B}_n - 2\mathrm{i}\check{\theta}'_n(x,t))). \tag{3.9f}$$

The constants occurring are related through the symmetries (3.6) as follows:

$$\tilde{A}_n = \frac{2\tilde{b}_n}{s''_{2,2}(z_n^*)} = -A_n^*, \quad \tilde{B}_n = \frac{\tilde{d}_n}{\tilde{b}_n} - \frac{s'''_{2,2}(z_n^*)}{3s''_{2,2}(z_n^*)} = B_n^*,$$
$$\hat{A}_n = \frac{2\hat{b}_n}{s''_{2,2}(\hat{z}_n)} = A_n \frac{q_- q_o^4}{q_-^* z_n^4}, \quad \hat{B}_n = \frac{\hat{d}_n}{\hat{b}_n} - \frac{s'''_{2,2}(\hat{z}_n)}{3s''_{2,2}(\hat{z}_n)} = B_n \frac{q_o^2}{\hat{z}_n^2} + \frac{2}{\hat{z}_n},$$
$$\check{A}_n = \frac{2\check{b}_n}{s''_{1,1}(\check{z}_n)} = -\hat{A}_n^*, \quad \check{B}_n = \frac{\check{d}_n}{\check{b}_n} - \frac{s'''_{2,2}(\check{z}_n)}{3s''_{2,2}(\check{z}_n)} = \hat{B}_n^*.$$

4. Inverse scattering with double poles

4.1 *Formulation of the RHP*

We now have all the necessary information to formulate the inverse problem. As usual, we start from the scattering relations (2.5) and rearrange terms to obtain relations between the functions $\phi_{+,1}$, $\phi_{-,2}$ and $s_{1,1}$ (which are analytic in the region D^+) and $\phi_{+,2}$, $\phi_{-,1}$ and $s_{2,2}$ (which are analytic in D^-) on the boundary Σ of these regions. Recalling the definition of the reflection coefficients (2.7), and replacing $\phi_\pm$ by the modified eigenfunctions, we have

$$\mu_{-,1}(x,t,z) = \frac{\mu_{+,1}(x,t,z)}{s_{1,1}(z)} - \rho(z)\,\mathrm{e}^{-2\mathrm{i}\theta(x,t,z)}\mu_{-,2}(x,t,z), \tag{4.1a}$$

$$\frac{\mu_{+,2}(x,t,z)}{s_{2,2}(z)} = \tilde{\rho}(z)\,\mathrm{e}^{2\mathrm{i}\theta(x,t,z)}\frac{\mu_{+,1}(x,t,z)}{s_{1,1}(z)} + (1-\rho(z)\tilde{\rho}(z))\mu_{-,2}(x,t,z). \tag{4.1b}$$

We introduce the meromorphic matrices

$$M^+(x,t,z) = \left(\frac{\mu_{+,1}(x,t,z)}{s_{1,1}(z)}, \mu_{-,2}(x,t,z)\right), \quad z \in D^+, \tag{4.2a}$$

$$M^-(x,t,z) = \left(\mu_{-,1}(x,t,z), \frac{\mu_{+,2}(x,t,z)}{s_{2,2}(z)}\right), \quad z \in D^-, \tag{4.2b}$$

as well as the jump matrix

$$J(x,t,z) = \begin{pmatrix} 0 & -\mathrm{e}^{2\mathrm{i}\theta(x,t,z)}\tilde{\rho}(z) \\ \mathrm{e}^{-2\mathrm{i}\theta(x,t,z)}\rho(z) & \rho(z)\tilde{\rho}(z) \end{pmatrix}.$$

With these definitions, (4.1) yields the jump condition for a multiplicative matrix RHP:

$$M^-(x,t,z) = M^+(x,t,z)(I - J(x,t,z)), \quad z \in \Sigma. \tag{4.3}$$

Note that $M^\pm$ possess poles at the discrete eigenvalues. In order to regularize this RHP, we need to take care of the behaviour of $M^\pm$ as $z \to \infty$ and $z \to 0$ (the eigenfunctions have a pole at $z = 0$), and the poles of $\mu_{+,1}/s_{1,1}$ in D^+ as well as those of $\mu_{+,2}/s_{2,2}$ in D^-.

The asymptotic behaviour as $z \to \infty$ and as $z \to 0$ is determined from (2.11), (2.13) and (2.15), which imply

$$M^{\pm}(x,t,z) = I + O(1/z), \quad z \to \infty. \tag{4.4}$$

$$M^{\pm}(x,t,z) = \frac{\mathrm{i}}{z}\sigma_3 Q_-(x,t) + O(1), \quad z \to 0. \tag{4.5}$$

Also, for the jump matrix, (2.15) and (2.16) yield

$$J(x,t,z) = O(1/z), \quad z \to \infty, \quad \text{and} \quad J(x,t,z) = O(z), \quad z \to 0,$$

where both limits are taken along the real axis, since J is only defined for $z \in \Sigma$.

To regularize the RHP, we subtract the asymptotic values as $z \to \infty$ and $z \to 0$, given by (4.4) and (4.5) and the singular contributions from the poles $z_n, z_n^*, \hat{z}_n, \check{z}_n$, $n = 1, \ldots, N$. We let $\eta_n := z_n$ and $\eta_{n+N} := \check{z}_n$, then η_n, $n = 1, \ldots, 2N$, are the poles in D^+ and $\hat{\eta}_n := -q_o^2/\eta_n$, $n = 1, \ldots, 2N$, are the poles in D^- (note that $\hat{\eta}_n = \hat{z}_n$ and $\hat{\eta}_{N+n} = z_n^*$, $n = 1, \ldots, N$). Thus, (4.3) becomes

$$\begin{aligned} M^- - I - \frac{\mathrm{i}}{z}\sigma_3 Q_- - \sum_{n=1}^{2N}\left\{ \frac{\operatorname*{Res}_{z=\eta_n}[M^+]}{z-\eta_n} + \frac{\operatorname*{P_{-2}}_{z=\eta_n}[M^+]}{(z-\eta_n)^2} + \frac{\operatorname*{Res}_{z=\hat{\eta}_n}[M^-]}{z-\hat{\eta}_n} + \frac{\operatorname*{P_{-2}}_{z=\hat{\eta}_n}[M^-]}{(z-\hat{\eta}_n)^2} \right\} \\ = M^+ - I - \frac{\mathrm{i}}{z}\sigma_3 Q_- - \sum_{n=1}^{2N}\left\{ \frac{\operatorname*{Res}_{z=\eta_n}[M^+]}{z-\eta_n} + \frac{\operatorname*{P_{-2}}_{z=\eta_n}[M^+]}{(z-\eta_n)^2} + \frac{\operatorname*{Res}_{z=\hat{\eta}_n}[M^-]}{z-\hat{\eta}_n} + \frac{\operatorname*{P_{-2}}_{z=\hat{\eta}_n}[M^-]}{(z-\hat{\eta}_n)^2} \right\} - M^+ J, \end{aligned} \tag{4.6}$$

where, recalling the notation introduced in (3.7),

$$\operatorname*{Res}_{z=\eta_n}[M^+] = \left(\operatorname*{Res}_{z=\eta_n}\left[\frac{\mu_{+,1}}{s_{1,1}}\right], 0\right), \quad \operatorname*{P_{-2}}_{z=\eta_n}[M^+] = \left(\operatorname*{P_{-2}}_{z=\eta_n}\left[\frac{\mu_{+,1}}{s_{1,1}}\right], 0\right), \tag{4.7a}$$

$$\operatorname*{Res}_{z=\hat{\eta}_n}[M^-] = \left(0, \operatorname*{Res}_{z=\hat{\eta}_n}\left[\frac{\mu_{+,2}}{s_{2,2}}\right]\right), \quad \operatorname*{P_{-2}}_{z=\hat{\eta}_n}[M^-] = \left(0, \operatorname*{P_{-2}}_{z=\hat{\eta}_n}\left[\frac{\mu_{+,2}}{s_{2,2}}\right]\right). \tag{4.7b}$$

The left-hand side now is analytic in the region D^-, and the right-hand side, except for the last term M^+J, is analytic in D^+ (recall that J is defined only on Σ), and both are of order $1/z$ as $z \to \infty$ in the respective regions. We can thus apply the Cauchy projectors $P_\pm$ on Σ, which are defined as usual by

$$P_\pm[f](z) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{f(\zeta)}{\zeta - (z \pm \mathrm{i}0)}\,\mathrm{d}\zeta.$$

Applying the above projectors to (4.6) and using Plemelj's formulae, we arrive at the following integral representation for the solution of the RHP:

$$M(x,t,z) = I + \frac{\mathrm{i}}{z}\sigma_3 Q_- + \sum_{n=1}^{2N}\left\{\frac{\operatorname*{Res}_{z=\eta_n}[M^+]}{z-\eta_n} + \frac{\operatorname*{P_{-2}}_{z=\eta_n}[M^+]}{(z-\eta_n)^2} + \frac{\operatorname*{Res}_{z=\hat\eta_n}[M^-]}{z-\hat\eta_n} + \frac{\operatorname*{P_{-2}}_{z=\hat\eta_n}[M^-]}{(z-\hat\eta_n)^2}\right\}$$
$$+ \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{M^+(x,t,\zeta)J(x,t,\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad z\in\mathbb{C}\setminus\Sigma. \tag{4.8}$$

Note that, as usual, the equations for M^+ and M^- are identical except for the integral being the projector P^+ and P^-, respectively.

4.2 *Residue conditions*

In order to obtain a closed system of equations for the solution of the RHP, we need to supplement (4.8) by obtaining expressions for the values of the eigenfunctions and their derivatives at the discrete eigenvalues. This is done decalling (3.8) and (3.9). We see from (4.7) that only the first column of M has poles at the points η_n, and only the second column has poles at $\hat\eta_n$. Therefore, we can use (4.8), evaluating the non-singular columns of M at the discrete eigenvalues.

Note that it is sufficient to find expressions for the values at the eigenvalues $\hat\eta_n$, $n=1,\ldots,2N$, since the values of the eigenfunctions at η_n are then determined by the symmetries. We introduce the constants $\hat A_{N+n} := \tilde A_n$ and $\hat B_{N+n} = \tilde B_n$, and we define the functions

$$C_n(x,t,z) = \frac{\hat A_n \mathrm{e}^{2\mathrm{i}\theta\left(x,t,\hat\eta_n\right)}}{z-\hat\eta_n}, \quad D_n(x,t) = \hat B_n + 2\mathrm{i}\theta'\left(x,t,\hat\eta_n\right).$$

Evaluating the second column of (4.8) at $z=\eta_k$, we get (letting $(M^+J)_j$ denote the jth column of M^+J)

$$\mu_{-,2}(x,t,\eta_k) = \begin{pmatrix}\mathrm{i}q_-/\eta_k\\ 1\end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{(M^+J)_2}{\zeta-\eta_k}\,\mathrm{d}\zeta$$
$$+ \sum_{n=1}^{2N}\left\{\frac{C_n(x,t,\eta_k)}{\eta_k-\hat\eta_n}\mu_{-,1}\left(x,t,\hat\eta_n\right) + C_n(x,t,\eta_k)[\mu'_{-,1}\left(x,t,\hat\eta_n\right) + \mu_{-,1}\left(x,t,\hat\eta_n\right)D_n(x,t)]\right\}.$$

Now we make use of the symmetry (2.8b), which gives the relation $\mu_{-,2}(x,t,\eta_k) = (\mathrm{i}q_-/\eta_k)\mu_{-,1}(x,t,\hat\eta_k)$, to substitute the left-hand term by an expression involving only $\mu_{-,1}$. This yields

$$0 = \begin{pmatrix}\mathrm{i}q_-/\eta_k\\ 1\end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{(M^+J)_2}{\zeta-\eta_k}\,\mathrm{d}\zeta + \sum_{n=1}^{2N}\left\{C_n(x,t,\eta_k)\mu'_{-,1}\left(x,t,\hat\eta_n\right)\right.$$
$$\left. + \left[C_n(x,t,\eta_k)\left(\frac{1}{\eta_k-\hat\eta_n} + D_n(x,t)\right) - \frac{\mathrm{i}q_-\delta_{kn}}{\eta_k}\right]\mu_{-,1}\left(x,t,\hat\eta_n\right)\right\}, \tag{4.9}$$

where δ_{kn} denotes the Kronecker delta. To obtain further equations for the unknown values at the derivatives of the eigenfunctions, we take the derivative with respect to z of the second column of (4.8) and evaluate at $z = \eta_k$. This yields

$$\mu'_{-,2}(x,t,\eta_k) = \begin{pmatrix} -\mathrm{i}q_-/\eta_k^2 \\ 0 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{(M^+J)_2}{(\zeta-\eta_k)^2}\,\mathrm{d}\zeta - \sum_{n=1}^{2N}\left\{\frac{C_n(x,t,\eta_k)}{\eta_k-\hat\eta_n}\mu'_{-,1}\left(x,t,\hat\eta_n\right) + \frac{C_n(x,t,\eta_k)}{\eta_k-\hat\eta_n}\left[D_n(x,t)+\frac{2}{\eta_k-\hat\eta_n}\right]\mu_{-,1}\left(x,t,\hat\eta_n\right)\right\}. \tag{4.10}$$

Taking the derivative with respect to z in (2.8b) and evaluating at $z = \hat z_n$, we obtain

$$\mu'_{-,2}(x,t,\eta_k) = -\frac{\mathrm{i}q_-}{\eta_k^2}\mu_{-,1}(x,t,\hat\eta_k) + \frac{\mathrm{i}q_-q_o^2}{\eta_k^3}\mu'_{-,1}(x,t,\hat\eta_k),$$

which we use to express the left-hand side of (4.10) in terms of μ_{-1} and its derivatives, yielding

$$0 = \begin{pmatrix} -\mathrm{i}q_-/\eta_k^2 \\ 0 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{(M^+J)_2}{(\zeta-\eta_k)^2}\,\mathrm{d}\zeta - \sum_{n=1}^{2N}\left\{\left[\frac{C_n(x,t,\eta_k)}{\eta_k-\hat\eta_n} + \frac{\mathrm{i}q_-q_o^2\delta_{kn}}{\eta_k^3}\right]\mu'_{-,1}\left(x,t,\hat\eta_n\right) + \left[\frac{C_n(x,t,\eta_k)}{\eta_k-\hat\eta_n}\left(D_n(x,t)+\frac{2}{\eta_k-\hat\eta_n}\right) - \frac{\mathrm{i}q_-\delta_{kn}}{\eta_k^2}\right]\mu_{-,1}\left(x,t,\hat\eta_n\right)\right\}. \tag{4.11}$$

Equations (4.9) and (4.11) for $k = 1,\dots,2N$ comprise a system of $4N$ equations with the $4N$ unknowns $\mu_{-,1}\left(x,t,\hat\eta_n\right)$, $\mu'_{-,1}\left(x,t,\hat\eta_n\right)$, $n = 1,\dots,2N$. These equations, together with (4.8), give a closed system of equations for $M^\pm(x,t,z)$ in terms of the scattering data.

4.3 *Reconstruction formula for the potential*

In order to recover the potential from the solution of the RHP, we recall (2.12). Therefore, we can use (4.8), with the projector P^+, to find an asymptotic expression for $(\mu_-)_{1,2}$ as $z \to \infty$:

$$\mu_{-,1,2}(x,t,z) = \frac{1}{z}\left(\mathrm{i}q_- + \sum_{n=1}^{2N}\frac{(\operatorname{Res}_{z=\hat\eta_n}[M^-])_{1,2}}{1-\hat\eta_n/z} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{(M^+J)_{1,2}}{\zeta/z-1}\,\mathrm{d}\zeta\right) + O\left(\frac{1}{z^2}\right),$$

Hence, by (2.12) we obtain the solution of the focusing NLS equation with NZBC in the presence of double poles of the scattering coefficients as

$$q(x,t) = q_- + \frac{1}{2\pi}\int_\Sigma (M^+J)_{1,2}\,\mathrm{d}\zeta - \mathrm{i}\sum_{n=1}^{2N}\hat A_n \mathrm{e}^{2\mathrm{i}\theta\left(x,t,\hat\eta_n\right)}\left(\mu'_{-,1,1}\left(x,t,\hat\eta_n\right) + \mu_{-,1,1}\left(x,t,\hat\eta_n\right)D_n(x,t)\right). \tag{4.12}$$

4.4 *Trace formula and asymptotic phase difference*

Now we derive a formula for the scattering coefficients in terms of the discrete eigenvalues and reflection coefficients by formulating an RHP for $s_{1,1}$ and $s_{2,2}$. First, we factor out the zeros of the scattering coefficients. As $s_{1,1}$ has double zeros at the values z_n and $-q_o^2/z_n^*,\ n=1,\dots,N$, the function

$$\bar{s}_{1,1}(z)=s_{1,1}(z)\prod_{n=1}^{N}\frac{(z-z_n^*)^2\left(z+q_o^2/z_n\right)^2}{(z-z_n)^2\left(z+q_o^2/z_n^*\right)^2}$$

is analytic in D^+ and has no zeros in the region, and it has the same asymptotic behaviour as $s_{1,1}$ as $z\to\infty$. Likewise,

$$\bar{s}_{2,2}(z)=s_{2,2}(z)\prod_{n=1}^{N}\frac{(z-z_n)^2\left(z+q_o^2/z_n^*\right)^2}{(z-z_n^*)^2\left(z+q_o^2/z_n\right)^2}$$

is analytic and has no zeros in D^-, and exhibits the same asymptotic behaviour as $s_{2,2}$ as $z\to\infty$. Furthermore, $\bar{s}_{1,1}\bar{s}_{2,2}=s_{1,1}s_{2,2}$ on Σ. We now use the fact that $\det S=s_{1,1}s_{2,2}-s_{1,2}s_{2,1}=1$ on Σ. This implies that

$$\frac{1}{s_{1,1}(z)s_{2,2}(z)}=1-\frac{s_{2,1}(z)}{s_{1,1}(z)}\frac{s_{1,2}(z)}{s_{2,2}(z)}=1-\rho(z)\tilde{\rho}(z)=1+\rho(z)\rho^*(z^*),$$

having used the first of the symmetries (2.10) for the reflection coefficients. Consequently, we have $\bar{s}_{1,1}(z)\bar{s}_{2,2}(z)=1/(1+\rho(z)\rho^*(z^*))$ on Σ. After taking logarithms, we have

$$\log\bar{s}_{1,1}(z)+\log\bar{s}_{2,2}(z)=-\log(1+\rho(z)\rho^*(z^*)),\quad z\in\Sigma,$$

which is the jump condition for a scalar additive RHP. Note that (2.15) implies that both $\log\bar{s}_{1,1}$ and $\log\bar{s}_{2,2}$ tend to 0 as $z\to\infty$. Applying the Cauchy projectors $P^{\pm}$, we obtain

$$\log\bar{s}_{1,1}(z)=-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log(1+\rho(\zeta)\rho^*(\zeta^*))}{\zeta-(z+\mathrm{i}0)}\,\mathrm{d}\zeta,\quad \log\bar{s}_{2,2}(z)=-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log(1+\rho(\zeta)\rho^*(\zeta^*))}{\zeta-(z-\mathrm{i}0)}\,\mathrm{d}\zeta.$$

Thus, we obtain the following formulae for the scattering coefficients:

$$s_{1,1}(z)=\exp\left(-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log(1+\rho(\zeta)\rho^*(\zeta^*))}{\zeta-z}\,\mathrm{d}\zeta\right)\prod_{n=1}^{N}\frac{(z-z_n)^2\left(z+q_o^2/z_n^*\right)^2}{(z-z_n^*)^2\left(z+q_o^2/z_n\right)^2},\tag{4.13a}$$

$$s_{2,2}(z)=\exp\left(-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log(1+\rho(\zeta)\rho^*(\zeta^*))}{\zeta-z}\,\mathrm{d}\zeta\right)\prod_{n=1}^{N}\frac{(z-z_n^*)^2\left(z+q_o^2/z_n\right)^2}{(z-z_n)^2\left(z+q_o^2/z_n^*\right)^2}.\tag{4.13b}$$

Finally, we derive a condition on the asymptotic phase difference of the boundary values q_- and q_+ (i.e. the analogue of the so-called 'theta' condition in Faddeev & Takhtajan (1987)). To this end, we take the limit $z\to 0$ in D^+ in (4.13a) and recall the asymptotic behaviour of the scattering coefficients (2.16). We then obtain after taking logarithms,

$$\arg\frac{q_-}{q_+}=\arg q_- -\arg q_+=\frac{1}{2\pi}\int_\Sigma\frac{\log(1+\rho(\zeta)\rho^*(\zeta^*))}{\zeta}\,\mathrm{d}\zeta+\sum_{n=1}^{N}8\arg z_n.\tag{4.14}$$

5. Double-pole soliton solutions

We now discuss a special kind of solutions; namely, we consider the case in which no jump occurs along the continuous spectrum, which happens if the reflection coefficients ρ and $\tilde{\rho}$ vanish identically.

The potential q is then said to be reflectionless. In this case, the RHP reduces to a system of algebraic equations, and the solutions obtained this way are the soliton solutions. Since the reconstruction formula of q requires only the first component of the eigenfunctions, we obtain a system of $4N$ equations. Using (4.9) and (4.11), we can write this as

$$GX = V,$$

where $X_n = \mu_{-,1,1}\big(x,t,\hat{\eta}_n\big)$, $X_{N+n} = \mu'_{-,1,1}\big(x,t,\hat{\eta}_n\big)$, $V_n = -\mathrm{i}q_-/\eta_n$, $V_{N+n} = -\mathrm{i}q_-/\eta_n^2$, for $n = 1,\ldots, 2N$, and

$$G_{k,n} = C_n(x,t,\eta_k)\left(D_n(x,t) + \frac{1}{\eta_k - \hat{\eta}_n}\right) - \frac{\mathrm{i}q_-\delta_{kn}}{\eta_k}, \quad G_{k,2N+n} = C_n(x,t,\eta_k),$$
$$G_{2N+k,n} = \frac{C_n(x,t,\eta_k)}{\eta_k - \hat{\eta}_n}\left(D_n(x,t) + \frac{2}{\eta_k - \hat{\eta}_n}\right) - \frac{\mathrm{i}q_-\delta_{kn}}{\eta_k^2}, \quad G_{2N+k,2N+n} = \frac{C_n(x,t,\eta_k)}{\eta_k - \hat{\eta}_n} + \frac{\mathrm{i}q_-q_o^2\delta_{kn}}{\eta_k^3}.$$

The reconstruction formula for the potential then becomes

$$q(x,t) = q_- - \mathrm{i}\sum_{n=1}^{2N} \hat{A}_n\, \mathrm{e}^{2\mathrm{i}\theta\left(x,t,\hat{\eta}_n\right)}\left(\mu'_{-,1,1}\big(x,t,\hat{\eta}_n\big) + \mu_{-,1,1}\big(x,t,\hat{\eta}_n\big)D_n(x,t)\right), \tag{5.1}$$

which provides the pure double-pole solutions of the focusing NLS equation with NZBC.

5.1 *One-soliton solutions with an imaginary eigenvalue*

Next we present in detail the simplest case of a one-soliton solution and we discuss its asymptotic behaviour. We consider the case that there is one double zero z_1 of $s_{1,1}$ in the portion of D^+ in the upper half plane, yielding a single quartet of double eigenvalues.

(i) Recall that the NLS equation is invariant under scaling: If $q(x,t)$ solves the equation, so does $cq(cx, c^2t)$, for $c \in \mathbb{R}$. This allows us to restrict ourselves to the case $q_o = 1$ without loss of generality.

(ii) We further consider the special case that the discrete eigenvalue is purely imaginary, that is, $z_1 = Z\mathrm{e}^{\mathrm{i}\pi/2}$ with $Z > 1$. Consequently, from the theta condition (4.14) it follows that the phase difference between q_- and q_+ is 4π, hence $q_- = q_+$. We let $q_- = \mathrm{e}^{i\alpha}$.

(iii) Due to the symmetries, there are just two independent norming constants for each quartet of eigenvalues. We set $b_1 = \mathrm{e}^{\xi+\mathrm{i}\beta}$ and $d_1 = \mathrm{e}^{\psi+\mathrm{i}\delta}$. The other constants are then determined by (3.6).

With these assumptions, the reconstruction formula for q becomes

$$q(x,t) = \mathrm{e}^{\mathrm{i}\alpha} - \mathrm{i}\tilde{A}_1\,\mathrm{e}^{2\mathrm{i}\theta_1^*(x,t)}(\mu'_{-,1,1}(x,t,z_1^*) + \mu_{-,1,1}(x,t,z_1^*)(\tilde{B}_1 + 2\mathrm{i}\theta_1'^*(x,t)))$$
$$- \mathrm{i}\hat{A}_1\,\mathrm{e}^{2\mathrm{i}\hat{\theta}_1(x,t)}(\mu'_{-,1,1}(x,t,\hat{z}_1) + \mu_{-,1,1}(x,t,\hat{z}_1)(\hat{B}_1 + 2\mathrm{i}\hat{\theta}_1'(x,t))).$$

The explicit formula for the potential is

$$q(x,t) = \mathrm{e}^{\mathrm{i}\alpha}q_N(x,t)/q_D(x,t), \tag{5.2}$$

where

$$q_N(x,t) = g(x,t) + (8Z^4\Delta_1\Delta_2^2\cos\tau + 4\Delta_1 f_2(\tau)\mathrm{i}S + 4Z^4\Delta_1^3\Delta_2 f_4(\tau))\cosh\chi \\ - 4\Delta_1 f_3(\tau)Y\sinh\chi - 8\mathrm{i}Z^2\Delta_1\Delta_2 S - \Delta_2^2 Z^2(7+10Z^2+7Z^4) - 8f_1(\tau), \quad (5.3a)$$

$$q_D(x,t) = g(x,t) - (8Z^2\Delta_1 S\sin\tau + 8Z^4\Delta_1\Delta_2^2\cos\tau)\cosh\chi \\ - 8Z^2\Delta_1 Y\cos\tau\sinh\chi - 16Z^4\cos(2\tau) + \Delta_2^2 Z^4(\Delta_1^2+4), \quad (5.3b)$$

with

$$\Delta_1 = Z + \frac{1}{Z}, \quad \Delta_2 = Z - \frac{1}{Z}, \quad \omega = \delta - \beta, \quad \Omega = Z^3\Delta_1\Delta_2\,\mathrm{e}^{\psi-\xi},$$
$$\chi(x) = \Delta_2 x + \xi, \quad \tau(t) = \Delta_1\Delta_2 t - \alpha - \beta,$$
$$Y(x) = Z^2\Delta_1^2\Delta_2 x - \Omega\sin\omega, \quad S(t) = 2\Delta_1\Delta_2(Z^4+1)t - \Omega\cos\omega,$$
$$f_1(\tau) = (Z^8+1)\cos(2\tau) + (Z^8-1)\mathrm{i}\sin(2\tau), \quad f_2(\tau) = (Z^4-1)\cos\tau + (Z^4+1)\mathrm{i}\sin\tau,$$
$$f_3(\tau) = (Z^4+1)\cos\tau + (Z^4-1)\mathrm{i}\sin\tau, \quad f_4(\tau) = \Delta_2\cos\tau + \Delta_1\,\mathrm{i}\sin\tau,$$
$$g(x,t) = Z^4\Delta_1^4\cosh(2\chi) + 2S^2 + 2Y^2.$$

A double-pole solution of the focusing NLS equation with NZBC is illustrated in Fig. 2a, which shows an example with $q_- = 1$, $z_1 = 2\mathrm{i}$, and the choice of norming constants $b_1 = d_1 = \mathrm{e}^{1+\mathrm{i}}$.

5.2 *Long-time asymptotics*

We now study the long-time asymptotic behaviour of these solutions. To do so, it is convenient to rewrite the numerator and denominator in terms of exponentials in x and t and compare the resulting powers:

$$q_N(x,t) = \frac{Z^4\Delta_1^4}{2}\left(\mathrm{e}^{2\chi}+\mathrm{e}^{-2\chi}\right) + [4Z^4\Delta_1\Delta_2^2\cos\tau + 2Z^4\Delta_1^3\Delta_2 f_4(\tau)\left(\mathrm{e}^{\chi}+\mathrm{e}^{-\chi}\right) \\ - 2\Delta_1 f_3(\tau)Y(\mathrm{e}^{\chi}-\mathrm{e}^{-\chi}) + 2\Delta_1 f_2(\tau)\,\mathrm{i}\,\mathrm{sign}(S)\left(\mathrm{e}^{\chi+\log|S|}+\mathrm{e}^{-\chi+\log|S|}\right) + 2\mathrm{e}^{2\log|S|} + 2Y^2 \\ - 8f_1(\tau) - \Delta_2^2 Z^4\left(7+10Z^2+7Z^4\right)] - 8\mathrm{i}Z^4\Delta_1\Delta_2\,\mathrm{sign}(S)\,\mathrm{e}^{\log|S|}, \quad (5.4a)$$

$$q_D(x,t) = \frac{Z^4\Delta_1^4}{2}\left(\mathrm{e}^{2\chi}+\mathrm{e}^{-2\chi}\right) + 2\,\mathrm{e}^{2\log|S|} + 2Y^2 - 4Z^2\Delta_1\sin\tau\,\mathrm{sign}(S)\left(\mathrm{e}^{\log|S|+\chi}+\mathrm{e}^{\log|S|-\chi}\right) \\ - 4Z^4\Delta_1\Delta_2^2\cos\tau\left(\mathrm{e}^{\chi}+\mathrm{e}^{-\chi}\right) - 4Z^2\Delta_1 Y\cos\tau\left(\mathrm{e}^{\chi}-\mathrm{e}^{-\chi}\right) - 16Z^4\cos(2\tau) + \Delta_2^2 Z^4\left(\Delta_1^2+4\right). \quad (5.4b)$$

Note that the highest power in terms of x in the exponential is $\pm 2\chi$, and the highest power in terms of t is $2\log|S|$. Therefore, we study the behaviour of the solution along curves of the form $\chi = K\log|S|$ for different values of K. Without loss of generality, we can assume that $|t| \gg 0$ is sufficiently large such that also $|S| > 1$.

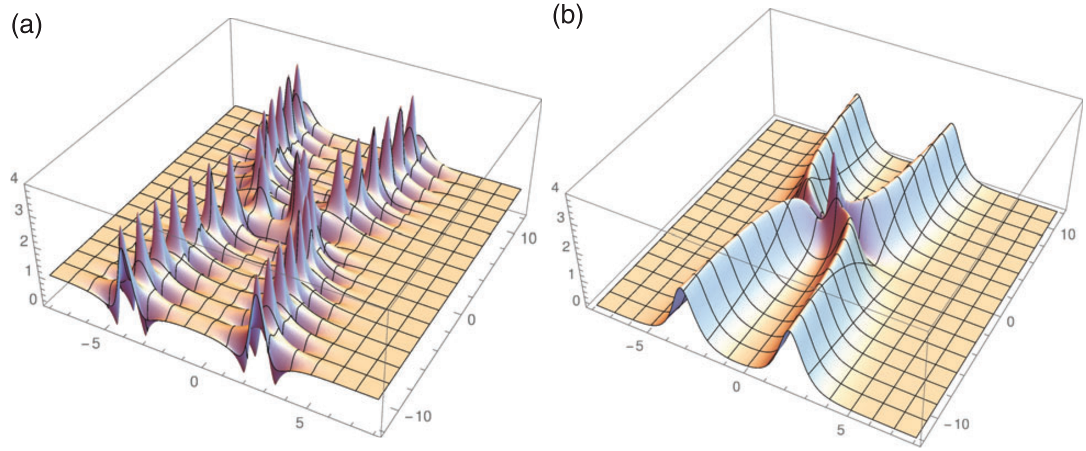

FIG. 2. (a) One-soliton solution of the focusing NLS equation with NZBC and double poles, with $q_- = 1$, $z_1 = 2\mathrm{i}$, $b_1 = d_1 = \mathrm{e}^{1+\mathrm{i}}$. (b) One-soliton solution of the focusing NLS equation with ZBC and double poles, obtained through the limit $q_o \to 0$, with a pair of eigenvalues $z_1 = 2\mathrm{i}$, $z_2 = -2\mathrm{i}$ and norming constants $b_1 = d_1 = \mathrm{e}^{1+\mathrm{i}}$. The plots show $|q(x,t)|$.

(i) $K > 1$. The leading order exponent is $2\chi = 2K \log |S| > 0$, and factoring out $\mathrm{e}^{2\chi}$ in both the numerator (5.4a) and the denominator (5.4b), we obtain

$$
\begin{aligned}
q_N(x,t) = \mathrm{e}^{2\chi} \bigg[& \frac{Z^4 \Delta_1^4}{2}\left(1 + |S|^{-4K}\right) + \left(4Z^4 \Delta_1 \Delta_2^2 \cos\tau + 2Z^4 \Delta_1^3 \Delta_2 f_4(\tau)\right)\left(|S|^{-K} + |S|^{-3K}\right) \\
& - 2\Delta_1 f_3(\tau) Y \left(|S|^{-K} - |S|^{-3K}\right) + 2\Delta_1 f_2(\tau) \mathrm{i}\, \mathrm{sign}(S) \left(|S|^{1-K} + |S|^{1-3K}\right) + 2|S|^{2-2K} \\
& + \left(2Y^2 - 8f_1(\tau) - \Delta_2^2 Z^4\left(7 + 10Z^2 + 7Z^4\right)\right)|S|^{-2K} - 8\mathrm{i} Z^4 \Delta_1 \Delta_2\, \mathrm{sign}(S) |S|^{1-2K} \bigg], \\
q_D(x,t) = \mathrm{e}^{2\chi} \bigg[& \frac{Z^4 \Delta_1^4}{2}\left(1 + |S|^{-4K}\right) + 2|S|^{2-2K} + 2Y^2 |S|^{-2K} \\
& - 4Z^2 \Delta_1 Y \cos\tau \left(|S|^{-K} - |S|^{-3K}\right) - 4Z^2 \Delta_1 \sin\tau\, \mathrm{sign}(S) \left(|S|^{1-K} + |S|^{1-3K}\right) \\
& - 4Z^4 \Delta_1 \Delta_2^2 \cos\tau \left(|S|^{-K} + |S|^{-3K}\right) + \left(\Delta_2^2 Z^4 \left(\Delta_1^2 + 4\right) - 16 Z^4 \cos(2\tau)\right)|S|^{-2K} \bigg].
\end{aligned}
$$

As $t \to \infty$, all terms except for the first one in both the numerator and denominator decay to zero. Thus, $q_N(x,t) = \mathrm{e}^{2\chi}\left[\frac{1}{2} Z^4 \Delta_1^4 + O\left(|S|^{1-K}\right)\right]$ and $q_D(x,t) = \mathrm{e}^{2\chi}\left[\frac{1}{2} Z^4 \Delta_1^4 + O\left(|S|^{1-K}\right)\right]$. We conclude that along these curves,

$$
q(x,t) = \mathrm{e}^{\mathrm{i}\alpha}\left(1 + O\left(|S|^{1-K}\right)\right), \qquad t \to \infty.
$$

(ii) $K=1$. In this case, we consider the branch $\chi=\log|S|$. We have that $2\chi=2\log|S|=\chi+\log|S|$ is the leading exponent in the numerator and denominator. Factoring out the term $\mathrm{e}^{2\chi}=S^2$, we obtain

$$\begin{aligned} q_N(x,t)=S^2\Big[&Z^4\Delta_1^4\tfrac{1}{2}\left(1+|S|^{-4}\right)+\Big(4Z^4\Delta_1\Delta_2^2\cos\tau+2Z^4\Delta_1^3\Delta_2 f_4(\tau)\Big)\Big(|S|^{-1}+|S|^{-3}\Big)\\ &-2\Delta_1 f_3(\tau)Y\Big(|S|^{-1}-|S|^{-3}\Big)+2\Delta_1 f_2(\tau)i\,\mathrm{sign}(S)\Big(1+|S|^{-2}\Big)+2\\ &+\Big(2Y^2-8f_1(\tau)-\Delta_2^2Z^4(7+10Z^2+7Z^4)\Big)|S|^{-2}-8\mathrm{i}Z^4\Delta_1\Delta_2\,\mathrm{sign}(S)|S|^{-1}\Big], \end{aligned}$$

and since $Y=O(\log|S|)$, we conclude that

$$q_N(x,t)=S^2\left[\frac{Z^4\Delta_1^4}{2}+2+2\Delta_1 f_2(\tau)\mathrm{i}\,\mathrm{sign}(S)+O\left(\frac{\log|S|}{|S|}\right)\right],\quad t\to\infty.$$

For the denominator, we obtain

$$q_D(x,t)=S^2\left[\frac{Z^4\Delta_1^4}{2}+2-4Z^2\Delta_1\sin\tau\,\mathrm{sign}(S)+O\left(\frac{\log|S|}{|S|}\right)\right],\quad t\to\infty.$$

Simplifying common factors in the numerator and denominator and setting $F=(Z^4\Delta_1^4+4)/(4\Delta_1)$, we then see that

$$q(x,t)=\mathrm{e}^{\mathrm{i}\alpha}\frac{1+\mathrm{sign}(S)((\mathrm{i}(Z^4-1)/F)\cos\tau-((1+Z^4)/F)\sin\tau)+O(\log|S|/|S|)}{1-\mathrm{sign}(S)(2Z^2/F)\sin\tau+O(\log|S|/|S|)},\quad t\to\infty.$$

Note that $2Z^2/F\leqslant\frac{1}{2}$ for $Z>1$, so the function remains bounded. Thus, along the curve $\chi=\log|S|$—that is, along the logarithmic trajectory $x=(\log[|2\Delta_1\Delta_2(Z^4+1)t-\Omega\cos\omega|]-\xi)/\Delta_2$—the solution oscillates as $t\to\infty$.

(iii) $-1<K<1,\ K\neq0$. In this case, the dominating exponent is $2\log|S|$, and after factoring $\mathrm{e}^{2\log|S|}=S^2$, we have

$$q_N(x,t)=S^2\Big[2+O\big(|S|^{K-1}\big)\Big],\quad q_D(x,t)=S^2\Big[2+O\big(|S|^{K-1}\big)\Big],\quad t\to\infty,$$

and therefore

$$q(x,t)=\mathrm{e}^{\mathrm{i}\alpha}(1+O\big(|S|^{K-1}\big)),\quad t\to\infty.$$

(iv) $K=-1$. In this case, $-2\chi=2\log|S|=-\chi+\log|S|$ is the dominating exponent and we have

$$\begin{aligned} q_N(x,t)&=S^2\left[\frac{Z^4\Delta_1^4}{2}+2+2\Delta_1 f_2(\tau)\,\mathrm{i}\,\mathrm{sign}(S)+O\left(\frac{\log|S|}{|S|}\right)\right],\\ q_D(x,t)&=S^2\left[\frac{Z^4\Delta_1^4}{2}+2-4Z^2\Delta_1\sin\tau\,\mathrm{sign}(S)+O\left(\frac{\log|S|}{|S|}\right)\right] \end{aligned}$$

as $t \to \infty$, and therefore

$$q(x,t) = \mathrm{e}^{\mathrm{i}\alpha} \frac{1 + \operatorname{sign}(S)((\mathrm{i}(Z^4-1)/F)\cos\tau - ((1+Z^4)/F)\sin\tau) + O(\log|S|/|S|)}{1 - \operatorname{sign}(S)(2Z^2/F)\sin\tau + O(\log|S|/|S|)}, \quad t \to \infty,$$

with $F = (Z^4\Delta_1^4 + 4)/(4\Delta_1)$ as in (ii).

(v) $K < -1$: The dominant exponent is -2χ, and in the same manner as in (i), we obtain

$$q(x,t) = \mathrm{e}^{\mathrm{i}\alpha}\big(1 + O\big(|S|^{K+1}\big)\big), \quad t \to \infty.$$

In summary, we have shown that as $|t| \to \infty$ the solution is localized around the logarithmic curves $\chi = \pm\log|S|$, along which it oscillates.

5.3 *Limit of zero background*

Recall that the focusing NLS with ZBC also admits solutions arising from double zeros of the analytic scattering coefficient, see Zakharov & Shabat (1972). Next we show that the double-pole solution of the NLS equation with NZBC does indeed reduce to that of the case with ZBC in the limit $q_o \to 0$. To do so, one must first write a double-pole solution of the focusing NLS equation with a generic value of q_o. (Note that such a solution can be obtained without having to solve the algebraic system from scratch, since one can simply apply the scaling invariance of the NLS equation to write $q(x,t) = \mathrm{e}^{\mathrm{i}\alpha} q_o\, q_N(q_o x, q_o^2 t)/q_D(q_o x, q_o^2 t)$, with $q_N(x,t)$ and $q_D(x,t)$ as in (5.3).) Then one takes the limit of such solution as $q_o \to 0$ with Z fixed, to obtain the double-pole solution of focusing NLS with ZBC, which was first reported in Zakharov & Shabat (1972). An example of such a solution is shown in Fig. 2b in the case of a purely imaginary eigenvalue $z_1 = 2\mathrm{i}$.

6. Concluding remarks

We end this work with a few general remarks.

The localization along logarithmic curves also occurs in the case of double poles and ZBC, see Zakharov & Shabat (1972), and for the two-component defocusing NLS equation with NZBC studied in Biondini & Kraus (2015). The oscillatory behaviour, on the other hand, is also observed in the case of non-stationary soliton solutions to the NLS equation with single poles and NZBC, cf. Biondini & Kovacic (2014). In this case, however, the solitons propagate along straight lines.

We should also note that one can in the same manner study multi-soliton solutions, as well as solutions arising from combinations of double and single poles [see Aktosun *et al.* (2007), Schiebold (2010), Schiebold (2014)].

On the other hand, studying the possible limit to an infinite number of discrete eigenvalues it is a non-trivial problem. In this respect, we should remark that, unlike what happens for the Korteweg–deVries equation, even for the focusing NLS equation with ZBC one can construct potentials with an infinite number of discrete eigenvalues accumulating on the real axis, cf. Zhou (1989). The study of the corresponding solutions is a problem that is still open even in the case of ZBC, and one would expect the problem to be even more challenging in the case of NZBC.

Finally, we should remark that the limit $Z \to 1$ of the soliton solutions of the focusing NLS equation with NZBC and simple zeros, see Biondini & Kovacic (2014), yields the well-known Peregrine soliton Peregrine (1983), namely a rational solution of the focusing NLS equation which is homoclinic to 1 both in the large x and the large t asymptotics. An interesting question is whether the limit $Z \to 1$ of the double-pole solution (5.2) yields higher-order rational solutions of the NLS equation on a non-zero background (e.g. see Ankiewicz *et al.*, 2010).

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