Commun. Math. Phys. (2024) 405:192
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-024-05054-y

Communications in Mathematical Physics

On Maxwell-Bloch Systems with Inhomogeneous Broadening and One-sided Nonzero Background

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Received: 3 June 2023 / Accepted: 30 May 2024

Abstract: The inverse scattering transform is developed to solve the Maxwell-Bloch system of equations that describes two-level systems with inhomogeneous broadening, in the case of optical pulses that do not vanish at infinity in the future. The direct problem, which is formulated in terms of a suitably-defined uniformization variable, combines features of the formalism with decaying as well as non-decaying fields. The inverse problem is formulated in terms of a 2×2 matrix Riemann-Hilbert problem. A novel aspect of the problem is that no reflectionless solutions can exist, and solitons are always accompanied by radiation. At the same time, it is also shown that, when the medium is initially in the ground state, the radiative components of the solutions decay upon propagation into the medium, giving rise to an asymptotically reflectionless states. Like what happens when the optical pulse decays rapidly in the distant past and the distant future, a medium that is initially excited decays to the stable ground state as $t \to \infty$ and for sufficiently large propagation distances. Finally, the asymptotic state of the medium and certain features of the optical pulse inside the medium are considered, and the emergence of a transition region upon propagation in the medium is briefly discussed.

1. Introduction

Resonant interaction between light and optical media underlies several types of practical devices such as lasers and optical amplifiers [19,57,66,67]. Typically, only narrow ranges of light colors interact resonantly with electron transitions between a small number of specific pairs of working energy levels in the active atoms [6,19,20,38,59]. Frequently, there is only one resonant transition and light is monochromatic, yet even this simple case produces a host of important physical effects such as: electromagnetically induced/self-induced transparency [10,17,23,30,37,41,45,52,55,56,68,72], superradiance and superflourescence [18,25,36,39,61,69], chaos and instabilities [7–9,73], photon echo [34,35,46,60,75], and remarkably even the slowing down of light to a tiny fraction of its speed in vacuum [29,40,58,63,64].

Published online: 29 July 2024

For many experimental and practical setups, a sufficient theoretical description of the interaction between light and an active optical medium is semi-classical, with the light described classically and the medium quantum-mechanically [6]. In the case of a finite number of resonant electron transitions, the quantum description reduces to a finite number of ordinary differential equations for the elements of the corresponding density matrix [20]. When averaged over appropriate portions of the medium, this matrix renders a description of the macroscopic medium polarization as well as its average local level occupation [19,20]. The large separation between the period(s) of the electromagnetic field oscillations corresponding to the color(s) of the light and the scale of its pulse width(s) further simplifies the theoretical description. In particular, one can extract, and average over, the fast oscillations, and thus find the description only in terms of the slowly-varying envelopes corresponding to the evolution of the light intensity and phase [6]. Moreover, backscattering is neglected and thus only unidirectional propagation is assumed. The resulting equations are called the Maxwell-Bloch equations (MBEs), and are one of the fundamental models in modern nonlinear optics [2,47–49,65].

The Maxwell-Bloch equations for two- and certain three-level media are completely integrable in the sense of possessing a Lax Pair (zero-curvature) representation [2]. Integrability makes it possible to linearize exactly these equations via the *Inverse Scattering Transform* (IST) [3,21,22,31–33,53,54,70], and enables the use of various transformation methods to "dress" simple exact solutions into more complicated and physically relevant ones [71]. At first, only pulses whose intensity decays rapidly in both the distant past and the distant future were studied using IST. Recently, however, the IST formalism was extended to optical pulses with *nonzero background* (NZBG) in both the distant past and the distant future, and pulses riding on top of continuous light beams were also studied in [14,50]. In all these cases, the NZBG was assumed to be symmetric, i.e., approaching the same nonzero amplitude as $t \to \pm\infty$. The MBEs in the laboratory frame with periodic boundary conditions were also recently investigated in [27,28]. In this work we investigate Maxwell-Bloch systems with *one-sided NZBG*, corresponding to light pulses riding on continuous waves that are in the process of either turning on or off. Specifically, we consider one-sided boundary conditions with nonzero background in the distant future. The integrable Maxwell-Bloch equations are special among integrable equations, in that even the simplest problem involving them is an initial-boundary-value problem. In some situations, the medium can be assumed to be semi-infinite, and "prepared" in the distant past (mathematically, in the limit as $t \to -\infty$) in a (known) state characterized by assigned values for the distribution of atoms in the ground and excited states, and for the polarizations at every point. In the case of a 2-level system, macroscopically the medium can be in: (i) a pure ground state (with all atoms in the lowest energy level); (ii) a pure excited state (i.e., a medium with a complete "population inversion", with all the atoms in the excited state); (iii) a mixed state with an assigned fraction of atoms in each state (in this case, the medium exhibits nontrivial polarization fluctuations, encoded by the off-diagonal entries of the density matrix). A light pulse is then injected into the medium at the origin, and it propagates to the right. The Maxwell-Bloch equations determine the optical pulse in each point of the medium at any given time (and, in particular, the residual optical pulse along the medium), as well as the final state of the entire medium after a long time (i.e., the asymptotics of the density matrix as $t \to +\infty$). The state of the medium sample in the distant future cannot be assumed to be known, but it can be deduced from the state in the distant past (and the IST scattering data) by a rather sophisticated procedure first introduced in [2,33].

As in all signaling-type problems, the role of the spatial and temporal variables in Maxwell-Bloch systems is reversed as compared to pure initial-value problems. This is reflected in the IST treatment: the scattering and inverse scattering of the pulses takes place in time, and the "evolution" is actually propagation along the medium [2,49]. If all atoms in the medium are initially in the ground state, the propagation damps the "continuous-radiation" components of the solutions, and is thus not time-reversible [2]. In this case, the response of the medium to an incident electric field, to which the medium is totally transparent and which undergoes lossless propagation, is known as self-induced transparency. The properties of the system change drastically when atoms are initially all in the excited state, in which case the dynamics can give rise to the phenomenon called superfluorescence [32,34,70].

Two classes of solutions of the MBEs are naturally distinguished. If the medium is initially prepared in a pure state, and hence does not exhibit polarization fluctuations as $t \to -\infty$, then the solution is completely determined by the incident pulse (in this work, $q(t, 0)$). According to [33], such solutions are called "causal" solutions, in consideration of the fact that if $q(t, 0) = 0$ for all $t < t_o$ for some $t_o \in \mathbb{R}$, then one can show that $q(t, z) = 0$ for all $t < t_o + z$, which means that the causal solution for a potential of finite range has a front which propagates into the medium with the velocity of light, in agreement with the notion of causality. On the other hand, for a medium that is not initially in a pure state, the MBEs admit nontrivial solutions even if $q(t, 0) \equiv 0$, i.e., in the absence of an incident pulse. In this case, the solution is entirely determined by the polarization fluctuations as $t \to -\infty$, and, following the terminology introduced in [33,74], such solutions are called "spontaneous" solutions. The solution of the MBEs is in general a superposition of a causal and a spontaneous solution.

To avoid confusion, we emphasize that the definition of "causal" solutions introduced in [33], which we also adopt in this work, is different from the one recently introduced in [51], where the term "causal" was used to denote solutions for which the incident pulse $q(t, 0)$ is identically zero for all $t < 0$, and the optical field $q(t, z)$ vanishes for all $t < 0$ and $z \geq 0$. (The reason for the latter definition is that the MBEs are typically written in a comoving reference frame translating at the speed of light in vacuum—including in this work, as well as in [33,51]—and therefore solutions with $t < 0$ lie outside the light-cone frame and therefore are deemed to be unphysical.) While the two definitions of "causality" agree when $q(t, 0) = 0$, the first definition does not necessarily require the incident pulse to be of finite range. A related issue is the question of whether solitons are to be considered unphysical in the Maxwell-Bloch system because of the exponentially decaying tail, which extends to infinity in time. This is true, strictly speaking. However, it is still useful to include solitons in the description of two-level systems. Indeed, the situation is exactly the same as for the nonlinear Schrödinger (NLS) equation. The NLS equation is also written in a comoving frame like the Maxwell-Bloch system. Still, solitons have been enormously fruitful objects in order to understand the properties of optical systems governed by the NLS equation. In fact, a truncated soliton (i.e., a soliton with its exponential tail "chopped off") is described spectrally by a discrete eigenvalue plus a small radiative component. Such an initial system configuration is certainly physical. Moreover, since in the Maxwell-Bloch system in a stable system configuration (e.g., when atoms are initially in the ground state) the radiation decays upon propagation, and approximating the solution with solitons becomes increasingly more accurate upon propagation.

Another important feature of our work is that we do not limit ourselves to the sharp-line limit, but instead deal explicitly with the presence of inhomogeneous broadening.

(Recall that, in optical media, the density matrix depends on the detuning from the exact quantum transition frequency due to the Doppler shift caused by the thermal motion of the atoms in the medium, and the associated MBEs must account for the inhomogeneous broadening effect by averaging over the range of detuning with the atomic velocity distribution function. The sharp-line limit, sometimes also referred to as "infinitely narrow line", corresponds to the limiting case in which the broadening function is taken to be a Dirac delta.) As a byproduct, our treatment is not limited to media that are initially in a pure state. Indeed, accounting for inhomogeneous broadening also allows considering a medium initially in a mixed state without requiring a compatible non-vanishing optical pulse in the distant past, the latter becoming a constraint only in the sharp-line limit. Besides the obvious physical relevance, including inhomogeneous broadening is also crucial to circumvent one of the drawbacks highlighted in [51], namely the fact that no matter how fast the incident pulse $q(t, 0)$ decays as $t \to +\infty$, after an infinitesimal propagation distance the optical pulse $q(t, z)$ always decays at a fixed slow rate as $t \to +\infty$. This of course has consequences regarding the well-posedness of the IST, which in the case of zero boundary conditions requires that $q(\cdot, z) \in L^1(\mathbb{R})$ for all $z \geq 0$, while even if this condition is imposed at $z = 0$, it is generically violated for all $z > 0$ (see Corollary 3 in [51]). Such slow decay of the optical field was also noted in [33], but in both cases it can be attributed to the fact that neglecting inhomogeneous broadening results in the second operator of the Lax pair exhibiting an essential singularity at the origin in the spectral plane. With inhomogeneous broadening, there is no such essential singularity, and the behavior of the reflection coefficient at the origin cannot play any role in inducing a slow decay of the optical field as $t \to +\infty$.

The outline of this work is as follows. In Sect. 2 we briefly present some preliminary mathematical facts about the problem, to set up the framework for what follows. In Sect. 3 we formulate the direct scattering problem of the IST, including the Jost solutions, scattering matrix, symmetries, discrete eigenvalues, asymptotic behavior at the branch points, etc. In Sect. 4 we formulate the inverse problem both in terms of a Riemann-Hilbert problem from the left and one from the right, and we derive the trace formulae, which are needed to evaluate the propagation of the norming constants. In Sect. 5 we discuss the asymptotic values of the density matrix in the distant past and distant future, and in Sect. 6 we discuss the evolution (i.e., propagation) of the scattering data. In Sect. 7 we use the IST formalism to briefly discuss the asymptotic behavior of the medium and of the optical pulse. Finally, in Sect. 8 we offer some concluding remarks.

2. Maxwell–Bloch Equations, Lax Pair and Problem Formulation

Up to rescalings of dependent and independent variables, the MBE that describe the propagation of an electromagnetic pulse $q(t, z)$ in a two-level medium characterized by a (real) population density function $D(t, z, k)$ and (complex) polarization fluctuation $P(t, z, k)$ for the atoms can be written in dimensionless form [48] as

$$q_z(t, z) = -\int_{-\infty}^{\infty} P(t, z, k)\, g(k)\, dk, \tag{2.1a}$$

$$P_t(t, z, k) - 2ikP(t, z, k) = -2D(t, z, k)q(t, z), \tag{2.1b}$$

$$D_t(t, z, k) = 2\,\mathrm{Re}\left[q^*(t, z)P(t, z, k)\right], \tag{2.1c}$$

where $z = z_{\text{lab}}$ is the propagation distance, $t = t_{\text{lab}} - z_{\text{lab}}/c$ is a retarded time (c being the speed of light in vacuum), subscripts z and t denote partial differentiation, the parameter k is the deviation of the transition frequency of the atoms from its mean value, and the asterisk $*$ denotes complex conjugation. The function $g(k)$ is the so-called inhomogeneous broadening function, which describes the shape of the spectral line. A prototypical example is given by a Lorentzian distribution, i.e., $g(k) = \pi^{-1}\delta/(k^2+\delta^2)$, where $\delta > 0$ represents the width of the spectral line (the case $g(k) = \delta(k - k_o)$ corresponding to the so-called sharp-line limit, or infinitely narrow line). In the present work, we treat the case of a fairly arbitrary inhomogeneous broadening function $g(k)$, subject to the only physical constraints that $g(k) \geq 0$ for all $k \in \mathbb{R}$ is in $L^1(\mathbb{R})$, and normalized by the condition $\int_{-\infty}^{\infty} g(k)dk = 1$.

Introducing the matrix describing the optical field and the so-called density matrix of the medium,

$$Q(t,z) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \qquad \rho(t,z,k) = \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix}, \tag{2.2}$$

respectively, (2.1) can be written compactly as

$$\rho_t = [ik\sigma_3 + Q, \rho], \tag{2.3a}$$

$$Q_z = -\frac{1}{2}\int_{-\infty}^{+\infty} [\sigma_3, \rho]\, g(k)\, dk \tag{2.3b}$$

where $[A, B] = AB - BA$ is the matrix commutator, and σ_j for $j = 1, 2, 3$ are the standard Pauli matrices, with $\sigma_3 = \text{diag}(1, -1)$. It was then shown in [2,49] that (2.3) are integrable, with a Lax pair given by

$$v_t = Xv, \tag{2.4a}$$

$$v_z = Tv, \tag{2.4b}$$

with

$$X(t,z,k) = ik\sigma_3 + Q, \qquad T(t,z,k) = \frac{i\pi}{2}\mathscr{H}_k[\rho(t,z,\xi)g(\xi)], \tag{2.4c}$$

where $\mathscr{H}_k[f(\xi)]$ is the Hilbert transform,

$$\mathscr{H}_k[f(\xi)] = \frac{1}{\pi}⨍_{\mathbb{R}} \frac{f(\xi)}{\xi - k}\mathrm{d}\xi, \tag{2.5}$$

and the symbol $⨍$ denotes the principal value integral.

In the following, the medium is assumed to be semi-infinite, i.e., $z \geq 0$, and "prepared" in the distant past (i.e., as $t \to -\infty$) in a (known) state characterized by assigned values for the distribution of atoms in the ground and excited states via $D(t, z, k)$, and for the polarization $P(t, z, k)$ at every point.

Equations (2.3) imply that the determinant of the density matrix, $\det \rho(t, z, k) = -(D^2(t,z,k)+|P(t,z,k)|^2)$, is independent of t. Moreover, (2.3) are invariant under the transformation $\rho(t, z, k) \mapsto \rho(t, k, k) + cI$, with c an arbitrary constant and I the 2×2 identity matrix. Thus, without loss of generality, the density matrix can be redefined to be traceless, and with determinant equal to -1 for all $z \geq 0$, so $D^2(t,z,k)+|P(t,z,k)|^2 = 1$. An electromagnetic pulse $q(t, 0)$ is then injected into the medium at the origin and it propagates into it ($z > 0$). As mentioned above, the inhomogeneous broadening function is assumed to be integrable, and the entries of the density matrix $\rho(t, z, k)$ are bounded functions for all $k, t \in \mathbb{R}$ and all $z \geq 0$, ensuring that the integrals in Eqs. (2.3) and the Hilbert transform (2.4c) are well-defined.

The IST to solve the initial-value problem for the above MBEs with localized fields [i.e., with $q(t,z) \to 0$ as $t \to \pm\infty$] was developed in [2] in the case of an initially stable medium $\lim_{t\to-\infty} D(t,z,k) = -1$) and in [33] in the case of an arbitrary initial state of the medium. The IST with a symmetric NZBG [i.e., $q(t,z) \to q_\pm(z)$ with $|q_+(z)| = |q_-(z)| = q_o$ as $t \to \pm\infty$] was carried out in [14]. Here, we develop the IST for one-sided NZBG, namely:

$$q(t,z) \to \begin{cases} 0 & t \to -\infty, \\ q_+(z) & t \to +\infty, \end{cases} \tag{2.6}$$

with $|q_+(z)| = A > 0$ for all $z \geq 0$. This type of boundary conditions describes optical fields that start from zero and then never return back to zero, remaining as a continuous wave (CW). We point out that the same methodology can be used to study the situation when $q_+(z) \equiv 0$ and $q(t,z) \to q_-(z) \neq 0$ as $t \to -\infty$, corresponding to optical fields that start as a CW and then get extinguished. For brevity, however, we omit the details. Note that, as in all earlier works and like in the study of optical fibers [5], in the formulation of the IST the set up is essentially that of a signaling problem, in which the propagation distance z is the evolution variable, and t is treated as a transverse variable.

The first half of the Lax pair, namely (2.4a), which is referred to as the scattering problem, coincides with the scattering problem for the focusing NLS equation [4,76]. As a result, the formulation of the direct problem is similar to that for the IST for the focusing NLS equation with one-sided nonzero background [62]. On the other hand, the "evolution" of the scattering data for the MBEs with a one-sided NZBG is substantially different and more complicated than for the NLS equation, and also from the case of the MBEs with a zero background [2], and a symmetric NZBG [14]. Moreover, the formulation of the inverse problem in the present work is also substantially more involved than in the focusing NLS case [62], or in the MBEs with either rapidly decaying or symmetric NZBG.

The asymptotic scattering problem as $t \to -\infty$ reduces to the one with zero background, while when $t \to +\infty$ it becomes $v_t = X_+ v$ where $X_+ = ik\sigma_3 + Q_+$, $Q_+(z)$ being $Q(t,z)$ with $q(t,z)$ replaced by its boundary value $q_+(z)$. The eigenvalues of X_+ are $\pm i\lambda$ with $\lambda^2 = k^2 + A^2$, where, as mentioned before, $A = |q_+|$. As with the focusing NLS equation [15], to deal with the branching of the scattering parameter we consider the two-sheeted Riemann surface defined by $\lambda(k) = (k^2 + A^2)^{1/2}$ with a branch cut along the segment of the imaginary axis $[-iA, iA]$, and then introduce a uniformization variable defined by the conformal mapping [26].

$$\zeta = k + \lambda(k), \tag{2.7a}$$

which is inverted by the identities

$$k = \tfrac{1}{2}\left(\zeta - A^2/\zeta\right), \qquad \lambda = \zeta - k = \tfrac{1}{2}\left(\zeta + A^2/\zeta\right). \tag{2.7b}$$

The above transformation maps the two-sheeted Riemann surface defined by $\lambda(k)$ onto the complex ζ-plane. Specifically, one has the following results (cf. Fig. 1): (i) Sheet I and sheet II of the Riemann surface are mapped, respectively, onto the exterior and the interior of the circle $\mathscr{C}$ of radius A. (ii) The branch cut Σ on either sheet is mapped onto $\mathscr{C}$. (iii) The real k-axis on sheet I and sheet II is mapped, respectively, onto $(-\infty, -A) \cup (A, +\infty)$

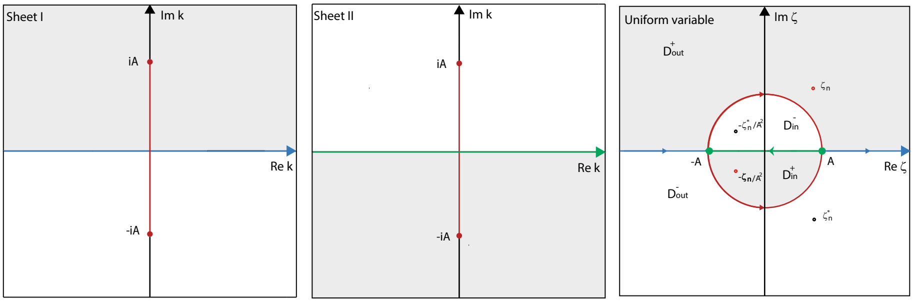

Fig. 1. Left and center: the two sheets of the Riemann surface associated with $\lambda^2 = k^2 + A^2$. Right: the complex plane for the uniformization variable $\zeta = k + \lambda$. The grey regions (D_{in}^+ and D_{out}^+) correspond to values of ζ for which $\operatorname{Im}\lambda > 0$, while the white regions (D_{in}^- and D_{out}^-) correspond to $\operatorname{Im}\lambda < 0$. The circle $\mathscr{C}$ (in red) corresponds to the cut Σ on either sheet, while $(-\infty, -A) \cup (A, +\infty)$ (blue) and $(-A, A)$ (green) correspond to the real k-axis on sheets I and II, respectively

and $(-A, A)$. (iv) $\zeta(\pm iA) = \pm iA$ from either sheet, while $\zeta(0_{\text{I}}^{\pm}) = \pm A$ and $\zeta(0_{\text{II}}^{\pm}) = \mp A$. (v) The upper half-planes $\operatorname{Im} k > 0$ on sheet I and sheet II are mapped, respectively, into the exterior and the interior of the circle $\mathscr{C}$ in the upper half-plane $\operatorname{Im}\zeta > 0$, whereas the lower half-planes $\operatorname{Im} k < 0$ on sheet I and sheet II are mapped, respectively, into the exterior and the interior of the circle $\mathscr{C}$ in the lower half-plane $\operatorname{Im}\zeta < 0$. It is therefore convenient to introduce the following regions in the complex ζ-plane:

$$D^+ = \{\zeta \in \mathbb{C} : (|\zeta|^2 - A^2)\, \operatorname{Im}\zeta > 0\}, \quad D^- = \{\zeta \in \mathbb{C} : (|\zeta|^2 - A^2)\, \operatorname{Im}\zeta < 0\},$$

which correspond to the regions where $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$, respectively, on either sheet. The complex ζ-plane is then partitioned into four regions: the upper/lower half ζ-plane outside the circle $\mathscr{C}$, denoted as $D_{\text{out}}^{\pm}$, respectively, and the lower/upper half ζ-plane inside the circle $\mathscr{C}$ denoted as $D_{\text{in}}^{\pm}$, respectively. In the following, we will also denote with $\mathscr{C}^{\pm}$ the upper and lower semicircles of radius A, respectively.

Note that in general the density matrix $\rho(t, z, k)$ is only defined for $k \in \mathbb{R}$. In terms of the uniformization variable ζ, we can evaluate it for all $\zeta \in \mathbb{R}$, specifically for $|\zeta| > A$ on sheet I, and for $-A < \zeta < A$ on sheet II, where obviously $\rho(t, z, -A^2/\zeta) = \rho(t, z, \zeta)$ since ρ is a single-valued function of k. But we cannot assume ρ is defined off the real k axis.

Note also that, in analogy with what happens for the symmetric NZBG, the background solution should reduce to $\rho = h_3\sigma_3$ with $h_3 = \pm 1$ in the limit $A \to 0$, and to maintain consistency one needs to choose the opposite sign of h_3 on sheet II.

3. Direct Scattering Problem

3.1. Jost solutions, analyticity and scattering matrix. As usual, the direct problem in the IST consists in mapping the solution of the MBEs into a suitable set of scattering data. And, as usual, this is done by introducing Jost eigenfunctions, which are solutions of the scattering problem with prescribed exponential asymptotic behavior at infinity. Specifically, in light of the asymptotic behaviors of the scattering problem as $t \to \pm\infty$ discussed above, we define the Jost eigenfunctions as

$$\Phi(t, z, \zeta) = \left(\bar{\phi}(t, z, \zeta)\,, \phi(t, z, \zeta)\right) = I_2 \mathrm{e}^{ik(\zeta)t\sigma_3}\,(1 + o(1)), \qquad t \to -\infty, \qquad (3.1\text{a})$$

$$\Psi(t,z,\zeta) = \big(\psi(t,z,\zeta),\ \bar\psi(t,z,\zeta)\big) = Y_+(\zeta,z)\,\mathrm{e}^{i\lambda(\zeta)t\sigma_3}\,(1+o(1)), \qquad t\to+\infty, \tag{3.1b}$$

where I_2 is the 2×2 identity matrix,

$$Y_+(\zeta,z) = I_2 + \frac{i}{\zeta}\sigma_3 Q_+(z), \tag{3.2}$$

and $\bar\phi$, ϕ, ψ and $\bar\psi$ denote the first and second columns of Φ and Ψ, respectively. (Note that in this work the overbar is not used to denote Schwarz conjugation.) Then $\Phi(t,z,\zeta)$ is defined for $k(\zeta)\in\mathbb{R}$ [i.e., for $\zeta\in\mathbb{R}$], while $\Psi(t,z,\zeta)$ is defined for $\lambda(\zeta)\in\mathbb{R}$ [i.e., for $\zeta\in\mathbb{R}\cup\mathscr{C}$].

One can rigorously define the Jost solutions as solutions of the following integral equations:

$$\Phi(t,z,\zeta) = \mathrm{e}^{ik(\zeta)t\sigma_3} + \int_{-\infty}^{t} \mathrm{e}^{ik(\zeta)(t-\tau)\sigma_3} Q(\tau,z)\Phi(\tau,z,\zeta)\mathrm{d}\tau, \tag{3.3a}$$

$$\Psi(t,z,\zeta) = Y_+(\zeta,z)\,\mathrm{e}^{i\lambda(\zeta)t\sigma_3} - \int_{t}^{\infty} K_+(t-\tau,z,\zeta)\Delta Q_+(\tau,z)\Psi(\tau,z,\zeta)\mathrm{d}\tau, \tag{3.3b}$$

where $\Delta Q_+(t,z) = Q(t,z) - Q_+(z)$ and $K_+(t,z,\zeta) := Y_+(z,\zeta)\mathrm{e}^{i\lambda(\zeta)t\sigma_3}Y_+^{-1}(z,\zeta)$. Standard Neumann series iterations on these Volterra integral equations allow one to prove the following (e.g., see Ref. [4] for details):

Theorem 3.1. *If the potential $q(t,z)$ is such that $q(t,z) - q_+(z)\,H(t) \in L_t^{1,1}(\mathbb{R})$, where $H(t)$ is the Heaviside step function, the following columns of the Jost eigenfunctions can be analytically extended onto the corresponding regions of the complex ζ-plane:*

$$\begin{aligned} &\phi(t,z,\zeta):\ \zeta\in\mathbb{C}^+, \qquad &&\bar\phi(t,z,\zeta):\ \zeta\in\mathbb{C}^-,\\ &\psi(t,z,\zeta):\ \zeta\in D^+, \qquad &&\bar\psi(t,z,\zeta):\ \zeta\in D^-. \end{aligned} \tag{3.4}$$

As usual, we define the continuous spectrum of the scattering problem as the set of values of k where all four eigenfunctions are simultaneously defined. Unlike what happens with symmetric NZBG (for which the continuous spectrum includes $\mathscr{C}$), here the continuous spectrum is limited to the real ζ-axis. Abel's theorem implies that, for any matrix solution v of the scattering problem, $\partial_t(\det v) = 0$. In addition, for all $\zeta\in\mathbb{R}$, $\lim_{t\to-\infty}\Phi(t,z,\zeta)\,\mathrm{e}^{-ik(\zeta)t\sigma_3} = I_2$ and $\lim_{t\to+\infty}\Psi(t,z,\zeta)\,\mathrm{e}^{-i\lambda(\zeta)t\sigma_3} = Y_+(\zeta,z)$. Hence, $\forall t,\zeta\in\mathbb{R}$ we have $\det\Phi(t,z,\zeta) = 1$ and $\det\Psi(t,z,\zeta) = \det Y_+(\zeta,z)$. Thus, for all $\zeta\in\mathbb{R}$, both Φ and Ψ are two fundamental matrix solutions of the scattering problem, and one can express one set of eigenfunctions in terms of the other one:

$$\Psi(t,z,\zeta) = \Phi(t,z,\zeta)\,S(\zeta,z)\,, \qquad S(\zeta,z) = \begin{pmatrix} a(\zeta,z) & \bar b(\zeta,z)\\ b(\zeta,z) & \bar a(\zeta,z)\end{pmatrix}, \quad \zeta\in\mathbb{R}, \tag{3.5a}$$

$$\Phi(t,z,\zeta) = \Psi(t,z,\zeta)\,S^{-1}(\zeta,z)\,, \qquad S^{-1}(\zeta,z) = \begin{pmatrix} \bar c(\zeta,z) & d(\zeta,z)\\ \bar d(\zeta,z) & c(\zeta,z)\end{pmatrix}, \quad \zeta\in\mathbb{R}, \tag{3.5b}$$

where $S(\zeta, z)$ is the scattering matrix, whose entries are referred to as the scattering coefficients. Note that unlike the case of symmetric NZBG, here the scattering matrix is not unimodular, since (3.5) implies $\det S = \det \Psi$, i.e., explicitly:

$$\det S(\zeta, z) = 2\lambda(\zeta)/\zeta = 1 + A^2/\zeta^2. \tag{3.6}$$

As usual, one can express the scattering coefficients as Wronskians of the Jost solutions:

$$a(\zeta, z) = \mathrm{Wr}(\psi(t, z, \zeta), \phi(t, z, \zeta)), \qquad \bar{a}(\zeta, z) = \mathrm{Wr}(\bar{\phi}(t, z, \zeta), \bar{\psi}(t, z, \zeta)), \tag{3.7a}$$
$$b(\zeta, z) = \mathrm{Wr}(\bar{\phi}(t, z, \zeta), \psi(t, z, \zeta)), \qquad \bar{b}(\zeta, z) = \mathrm{Wr}(\bar{\psi}(t, z, \zeta), \phi(t, z, \zeta)). \tag{3.7b}$$

In turn, (3.7) imply the following:

Theorem 3.2. *Under the same hypotheses as in Theorem 3.1, the following scattering coefficients can be analytically extend off the real ζ-axis in the following regions:*

$$\begin{aligned} a(\zeta, z):\quad \zeta \in D^+_{out}, \qquad \bar{a}(\zeta, z):\quad \zeta \in D^-_{out}, \\ b(\zeta, z):\quad \zeta \in D^+_{in}, \qquad \bar{b}(\zeta, z):\quad \zeta \in D^-_{in}. \end{aligned} \tag{3.8}$$

Moreover, $a(\zeta, z)$ is continuous for $\zeta \in \mathbb{R} \cup \mathscr{C}^+ \setminus \{iA\}$; $\bar{a}(\zeta, z)$ is continuous for $\zeta \in \mathbb{R} \cup \mathscr{C}^- \setminus \{-iA\}$; $b(\zeta, z)$ is continuous for $\zeta \in \mathbb{R} \cup \mathscr{C}^- \setminus \{-iA\}$ and $\bar{b}(\zeta, z)$ is continuous for $\zeta \in \mathbb{R} \cup \mathscr{C}^+ \setminus \{iA\}$.

The first set of reflection coefficients that will be needed in the inverse problem, which we refer to as the reflection coefficients from the left, is given by:

$$r_-(\zeta, z) = \frac{b(\zeta, z)}{a(\zeta, z)}, \qquad \bar{r}_-(\zeta, z) = \frac{\bar{b}(\zeta, z)}{\bar{a}(\zeta, z)}, \qquad \zeta \in \mathbb{R}. \tag{3.9a}$$

Similarly, one can define reflection coefficients from the right in terms of the entries of $S^{-1}(\zeta, z)$:

$$r_+(\zeta, z) = \frac{d(\zeta, z)}{c(\zeta, z)} \equiv -\frac{\bar{b}(\zeta, z)}{a(\zeta, z)}, \qquad \bar{r}_+(\zeta, z) = \frac{\bar{d}(\zeta, z)}{\bar{c}(\zeta, z)} \equiv -\frac{b(\zeta, z)}{\bar{a}(\zeta, z)}, \qquad \zeta \in \mathbb{R}. \tag{3.9b}$$

Note that, unlike $r_-(\zeta, z)$ and $\bar{r}_-(\zeta, z)$, the reflection coefficients from the right are such that $r_+(\zeta, z)$ is also defined on $\mathscr{C}^+$ (except, possibly, at $\zeta = iA$, where $\bar{b}(\zeta, z)$ and $a(\zeta, z)$ might have a pole), and $\bar{r}_+(\zeta, z)$ is also defined on $\mathscr{C}^-$ (except, possibly, at $\zeta = -iA$, where $b(\zeta, z)$ and $\bar{a}(\zeta, z)$ might have a simple pole, see [15,62]). One can also write an integral representation of the scattering matrix $S(\zeta, z)$, which is analogous to the one with symmetric NZBG with $Y_- \equiv I_2$. Since such a representation will not be used, however, it is omitted for brevity.

3.2. Symmetries of eigenfunctions and scattering coefficients. The scattering problem admits two nontrivial symmetries: $(k,\lambda)\mapsto(k^*,\lambda^*)$ (i.e., switching between the upper and lower half k-planes) and $(k,\lambda)\mapsto(k,-\lambda)$ (i.e., switching between opposite sheets). In terms of the uniformization variable ζ, these correspond to the maps $\zeta\mapsto\zeta^*$ (i.e., upper/lower half ζ-plane) and $\zeta\mapsto -A^2/\zeta$ (outside/inside the circle $\mathscr{C}$).

Regarding the first involution, the boundary conditions (3.1) yield the following symmetries:

Lemma 3.3. *The Jost solutions satisfy the following symmetry:*

$$\Psi(t,z,\zeta)=-i\sigma_2\Psi^*(t,z,\zeta^*)i\sigma_2,\qquad \Phi(t,z,\zeta)=-i\sigma_2\Phi^*(t,z,\zeta^*)i\sigma_2,\quad \zeta\in\mathbb{R}. \tag{3.10a}$$

Componentwise, we have

$$\bar\psi^*(t,z,\zeta^*)=-i\sigma_2\psi(t,z,\zeta),\qquad \zeta\in D^+\cup\mathscr{C}\cup\mathbb{R}, \tag{3.10b}$$
$$\psi^*(t,z,\zeta^*)=i\sigma_2\bar\psi(t,z,\zeta),\qquad \zeta\in D^-\cup\mathscr{C}\cup\mathbb{R}, \tag{3.10c}$$
$$\phi^*(t,z,\zeta^*)=-i\sigma_2\bar\phi(t,z,\zeta),\qquad \zeta\in\mathbb{C}^-\cup\mathbb{R}, \tag{3.10d}$$
$$\bar\phi^*(t,z,\zeta^*)=i\sigma_2\phi(t,z,\zeta),\qquad \zeta\in\mathbb{C}^+\cup\mathbb{R}. \tag{3.10e}$$

Lemma 3.4. *The scattering matrix and reflection coefficients obey the following symmetry relations:*

$$S^*(\zeta^*,z)=-i\sigma_2S(\zeta,z)i\sigma_2,\quad \zeta\in\mathbb{R}, \tag{3.11a}$$
$$\bar r_-^*(\zeta^*,z)=-r_-(\zeta,z),\quad \zeta\in\mathbb{R}, \tag{3.11b}$$
$$\bar r_+^*(\zeta^*,z)=-r_+(\zeta,z),\quad \zeta\in\mathscr{C}^+\cup\mathbb{R}\setminus\{iA\}. \tag{3.11c}$$

Componentwise, we have

$$\bar a^*(\zeta^*,z)=a(\zeta,z):\qquad \zeta\in D^+_{out}\cup\mathscr{C}^+\cup\mathbb{R}, \tag{3.11d}$$
$$\bar b^*(\zeta^*,z)=-b(\zeta,z):\qquad \zeta\in D^-_{in}\cup\mathscr{C}^-\cup\mathbb{R}.$$

Using the above symmetries, (3.6) can then be written as

$$|a(\zeta,z)|^2+|b(\zeta,z)|^2=2\lambda/\zeta\qquad \zeta\in\mathbb{R},\ z\geq 0. \tag{3.12}$$

For the second involution, since $\lambda(-A^2/\zeta)=-\lambda(\zeta)$ and $k(-A^2/\zeta)=k(\zeta)$, taking into account the boundary conditions (3.1), one can easily establish the following additional symmetry relations:

Lemma 3.5. *The Jost solutions satisfy the following symmetry relations:*

$$\Psi(t,z,\zeta)=\frac{i}{\zeta}\Psi(t,z,-A^2/\zeta)\sigma_3Q_+(z),\qquad \Phi(t,z,\zeta)=\Phi(t,z,-A^2/\zeta),\quad \zeta\in\mathbb{R}. \tag{3.13a}$$

Componentwise, we have

$$\bar\psi(t,z,\zeta)=\frac{iq_+(z)}{\zeta}\psi(t,z,-A^2/\zeta),\qquad \zeta\in D^-\cup\mathscr{C}, \tag{3.13b}$$
$$\psi(t,z,\zeta)=\frac{iq_+^*(z)}{\zeta}\bar\psi(t,z,-A^2/\zeta),\qquad \zeta\in D^+\cup\mathscr{C}, \tag{3.13c}$$
$$\phi(t,z,\zeta)=\phi(t,z,-A^2/\zeta),\quad \zeta\in\mathbb{R}\cup\mathscr{C}^+, \tag{3.13d}$$
$$\bar\phi(t,z,\zeta)=\bar\phi(t,z,-A^2/\zeta),\qquad \zeta\in\mathbb{R}\cup\mathscr{C}^-. \tag{3.13e}$$

Lemma 3.6. *The scattering matrix satisfy the following symmetry relations:*

$$S(\zeta, z) = \tfrac{i}{\zeta} S(-A^2/\zeta, z)\sigma_3 Q_+(z), \quad \zeta \in \mathbb{R}. \tag{3.14a}$$

Componentwise, we have

$$a(\zeta, z) = \frac{iq_+^*(z)}{\zeta}\bar{b}(-A^2/\zeta, z), \qquad \zeta \in D_{\text{out}}^+ \cup \mathbb{R} \cup \mathscr{C}^+, \tag{3.14b}$$

$$\bar{a}(\zeta, z) = \frac{iq_+(z)}{\zeta} b(-A^2/\zeta, z), \qquad \zeta \in D_{\text{out}}^- \cup \mathbb{R} \cup \mathscr{C}^-. \tag{3.14c}$$

$$c(\zeta, z) = -\frac{i\zeta}{q_+(z)} d(-A^2/\zeta, z), \qquad \zeta \in D_{\text{out}}^+ \cup \mathbb{R} \cup \mathscr{C}^+, \tag{3.14d}$$

$$\bar{c}(\zeta, z) = -\frac{i\zeta}{q_+^*(z)} \bar{d}(-A^2/\zeta, z), \qquad \zeta \in D_{\text{out}}^- \cup \mathbb{R} \cup \mathscr{C}^-. \tag{3.14e}$$

Note that, unlike what happens for the focusing NLS equation, the above symmetries depend explicitly on z. The explicit z-dependence will be determined in Sect. 6, where we discuss the propagation of the background, as well as that of all relevant quantities in the inverse scattering formalism.

Combining the first and second involutions above also yields the following symmetry for the scattering matrix:

$$S^*(\zeta^*, z) = \frac{i}{\zeta}\sigma_2 S(-A^2/\zeta, z)\sigma_3 Q_+(z)\sigma_2. \tag{3.15}$$

As a direct consequence we have the following symmetry for the reflection coefficients from the left:

$$r_-^*(\zeta^*, z)\, r_-(-A^2/\zeta, z) = \bar{r}_-(\zeta, z)\, \bar{r}_-^*(-A^2/\zeta^*, z) = -1, \quad \zeta \in \mathbb{R} \tag{3.16}$$

On the other hand, the second symmetry implies that the reflection coefficients from the right satisfy the symmetry:

$$r_+(\zeta, z)\, r_+(-A^2/\zeta, z) = \mathrm{e}^{2i \arg q_+(z)}, \quad \zeta \in \mathbb{R}. \tag{3.17}$$

Importantly, (3.16) and (3.17) imply that $r_\pm(\zeta, z) \neq 0$ for all $\zeta \in \mathbb{R}$. This is significant for two reasons: (i) It means that no reflectionless solutions exist for the problem with one-sided NZBG. This situation is similar to what happens for the focusing and defocusing NLS equation with asymmetric NZBG [13,24], as well for the defocusing Manakov system with non-parallel boundary conditions [1]. (ii) Since $r_\pm(\zeta, z)$ appear in the denominator of the jump matrices in the inverse problem (cf. Sect. 4), it ensures that the jump condition does not introduce singularities in the Riemann-Hilbert problem.

3.3. Discrete eigenvalues and residue conditions. A discrete eigenvalue of the scattering problem is a value $\zeta \in D^+ \cup D^-$ for which there exists a nontrivial solution $v(t, z, \zeta)$ to the scattering problem in (2.4) with entries in $L^2(\mathbb{R}, \mathrm{d}t)$. These eigenvalues occur for $\zeta \in D_{\text{out}}^+$ iff the functions $\phi(t, z, \zeta)$ and $\psi(t, z, \zeta)$ are linearly dependent (i.e., iff $a(\zeta, z) = 0$); for $\zeta \in D_{\text{out}}^-$ iff the functions $\bar{\psi}(t, z, \zeta)$ and $\bar{\phi}(t, z, \zeta)$ are linearly dependent (i.e., iff $\bar{a}(\zeta, z) = 0$); for $\zeta \in D_{\text{in}}^-$ iff the functions $\phi(t, z, \zeta)$ and $\bar{\psi}(t, z, \zeta)$ are linearly dependent (i.e., iff $\bar{b}(\zeta, z) = 0$); finally, for $\zeta \in D_{\text{in}}^+$ iff the functions $\psi(t, z, \zeta)$

and $\bar{\phi}(t, z, \zeta)$ are linearly dependent (i.e., iff $b(\zeta, z) = 0$). The conjugation symmetry (3.11a) and the second symmetry (3.14b) then imply that the discrete eigenvalues occur in quartets: $\{\zeta_n, \zeta_n^*, -A^2/\zeta_n, -A^2/\zeta_n^*\}_{n=1}^N$.

Next, we derive the residue conditions at each of the discrete eigenvalues. Here we assume that discrete eigenvalues are simple and finite in number, and that there are no spectral singularities, i.e., no real zeros of the scattering coefficients. In the case of rapidly decaying optical pulses, sufficient conditions that guarantee the absence of spectral singularities and a finite number (or absence) of discrete eigenvalues were established in [42–44,77]. On the other hand, when a non-zero background is considered, the characterization of incident pulses without spectral singularities is an open problem, even for the focusing NLS equation in the case of a symmetric NZBG, and it therefore is beyond the scope of the present work. On the other hand, it is not difficult to construct explicit examples of incident optical pulses (e.g., see [12,16]) for which no spectral singularities are present. And most importantly, we emphasize that potentials with spectral singularities are non-generic, and therefore the IST formulation in the present work covers the generic case. Similarly, the assumption that the discrete eigenvalues are simple has only the purpose of simplifying the notation/presentation.

Theorem 3.7. *Consider a discrete eigenvalue $\zeta_n \in D_{\text{out}}^+$, i.e., a simple zero of $a(\zeta, z)$. The Wronskian relation (3.7a) imply that there exist $b_n, \bar{b}_n \neq 0$ independent of t such that*

$$\psi(t, z, \zeta_n) = b_n(z)\phi(t, z, \zeta_n), \qquad \bar{\psi}(t, z, \zeta_n^*) = \bar{b}_n(z)\bar{\phi}(t, z, \zeta_n^*). \tag{3.18}$$

In terms of the modified eigenfunctions $\mu = (\mu^- \ \mu^+) = \Phi\, \mathrm{e}^{-ikt\sigma_3}$ and $\nu = (\nu^+ \ \nu^-) = \Psi\, \mathrm{e}^{-i\lambda t\sigma_3}$ we obtain

$$Res_{\zeta=\zeta_n}[\nu^+(t, z, \zeta)/a(\zeta, z)] = C_n(z)\, \mathrm{e}^{-i\zeta_n t}\mu^+(t, z, \zeta_n), \tag{3.19a}$$

$$Res_{\zeta=\zeta_n^*}[\nu^-(t, z, \zeta)/\bar{a}(\zeta, z)] = \bar{C}_n(z)\, \mathrm{e}^{i\zeta_n^* t}\mu^-(t, z, \zeta_n^*), \tag{3.19b}$$

where the norming constants are $C_n(z) = b_n(z)/a'(\zeta_n, z)$ and $\bar{C}_n(z) = \bar{b}_n(z)/\bar{a}'(\zeta_n^*, z)$.

Hereafter, primes will denote differentiation with respect to the spectral parameter ζ. Note that, since we assumed the eigenvalues to be simple, we have $a(\zeta_n, z) = 0$ and $a'(\zeta_n, z) \neq 0$. The first symmetry implies that $\bar{b}_n(z) = -b_n^*(z)$ and $a'(\zeta_n, z) = (\bar{a}'(\zeta_n^*, z))^*$, which yields the following:

Lemma 3.8. *Assume that $a(\zeta, z)$ has simple zeros $\{\zeta_n\}_{n=1}^{N_1} \in D_{\text{out}}^+$. Then the norming constants defined in the Theorem 3.7 obey the following symmetry relation:*

$$\bar{C}_n(z) = -C_n^*(z), \qquad n = 1, \ldots, N_1. \tag{3.20}$$

In turn, $\hat{\zeta}_n = -A^2/\zeta_n^* \in D_{\text{in}}^+$ and $\hat{\zeta}_n^* \equiv -A^2/\zeta_n \in D_{\text{in}}^-$, and as a result of the second symmetry $b(-A^2/\zeta_n^*, z) = 0$ and $\bar{b}(z, -A^2/\zeta_n) = 0$. Then, from the Wronskian representations it follows:

Theorem 3.9. *Let ζ_n be a simple zero of $a(\zeta, z)$. Then there exist $\hat{b}_n, \bar{\hat{b}}_n \neq 0$ independent of t such that*

$$\begin{aligned} \psi(t, z, -A^2/\zeta_n^*) &= \hat{b}_n(z)\bar{\phi}(t, z, -A^2/\zeta_n^*), \\ \psi(t, z, -A^2/\zeta_n) &= \bar{\hat{b}}_n(z)\phi(t, z, -A^2/\zeta_n^*). \end{aligned} \tag{3.21}$$

Similar symmetries exist for the norming constants associated to eigenvalues inside $\mathscr{C}$, namely:

$$Res_{\zeta=\hat{\zeta}_n}[\nu^+(t,z,\zeta)/b(\zeta,z)] = \hat{C}_n(z)\,\mathrm{e}^{i\zeta_n^* t}\mu^-(t,z,\hat{\zeta}_n), \tag{3.22a}$$

$$Res_{\zeta=\hat{\zeta}_n^*}[\nu^-(t,z,\zeta)/\bar{b}(\zeta,z)] = \hat{\bar{C}}_n(z)\,\mathrm{e}^{-i\zeta_n t}\mu^+(t,z,\hat{\zeta}_n^*), \tag{3.22b}$$

with

$$\hat{C}_n(z) = \frac{\hat{b}_n(z)}{b'(\hat{\zeta}_n,z)}, \qquad \hat{\bar{C}}_n(z) = \frac{\hat{\bar{b}}_n(z)}{\bar{b}'(\hat{\zeta}_n^*,z)}. \tag{3.22c}$$

Note that the proportionality constants satisfy different symmetries for discrete eigenvalues inside and outside $\mathscr{C}$: for discrete eigenvalues outside $\mathscr{C}$ one has $\bar{b}_n(z) = -b_n^*(z)$, while inside $\mathscr{C}$ one has $\hat{\bar{b}}_n(z) = \hat{b}_n^*(z)$. This is a result of the problem having asymmetric boundary conditions as $t \to \pm\infty$. Inside $\mathscr{C}$ the first symmetry implies $\bar{b}'(\hat{\zeta}_n^*,z) = -b'(\hat{\zeta}_n,z)$, which yields the following:

Lemma 3.10. *Under the same hypotheses as in Lemma 3.8, norming constants defined in the Theorem 3.9 satisfy the following symmetry:*

$$\hat{\bar{C}}_n(z) = -\hat{C}_n^*(z), \qquad n = 1, \ldots, N_1. \tag{3.23}$$

From the second symmetry for the eigenfunctions we have $(-i\zeta_n^*/q_+(z))\bar{\psi}^*(t,z,\zeta_n^*) = \hat{b}_n(z)\bar{\phi}(t,z,\zeta_n^*)$, and on the other hand $(-i\zeta_n^*/q_+(z))\bar{\psi}(t,z,\zeta_n^*) \equiv (-i\zeta_n^*/q_+(z))\,\bar{b}_n(z)\bar{\phi}(t,z,\zeta_n^*)$. Therefore,

$$\hat{b}_n(z) = -\frac{i\zeta_n^*}{q_+(z)}\bar{b}_n(z). \tag{3.24}$$

Using again the second symmetry one can obtain

$$\hat{\bar{b}}_n(z) = -\frac{i\zeta_n}{q_+^*(z)}b_n(z). \tag{3.25}$$

Finally, note that the definition of $\hat{C}_n(z)$ and the second symmetry imply, respectively,

$$\hat{C}_n(z) = -\frac{i\zeta_n^*}{q_+(z)}\frac{\bar{b}_n(z)}{b'(\hat{\zeta}_n,z)}, \qquad b'(\hat{\zeta}_n,z) = -i\frac{(\zeta_n^*)^3}{q_+(z)A^2}\bar{a}'(\zeta_n^*,z).$$

We then obtain the following relationship between the norming constants inside and outside $\mathscr{C}$:

$$\hat{C}_n(z) = (A/\zeta_n^*)^2\bar{C}_n(z) \equiv -(A/\zeta_n^*)^2 C_n^*(z). \tag{3.26}$$

3.4. Asymptotic behavior as $\zeta \to \infty$ and $\zeta \to 0$. To normalize the inverse problem in Sect. 4, we will need the asymptotic behavior of the eigenfunctions and the scattering coefficients. Note that $k \to \infty$ corresponds to $\zeta \to \infty$ in Sheet I, and $\zeta \to 0$ in Sheet II. Standard Wentzel-Kramers-Brillouin (WKB) expansions in the scattering problem rewritten in terms of ζ yield the following asymptotic behaviors for the eigenfunctions:

Proposition 3.11. *As $\zeta \to \infty$ in the appropriate regions of the complex plane,*

$$\Psi_d(t,z,\zeta)\,e^{-i\lambda(\zeta)t\sigma_3} = I_2 + o(1),$$
$$\Psi_o(t,z,\zeta)\,e^{-i\lambda(\zeta)t\sigma_3} = \tfrac{i}{\zeta}\sigma_3 Q(t,z) + o(1/\zeta), \tag{3.27a}$$
$$\Phi_d(t,z,\zeta)\,e^{-ik(\zeta)t\sigma_3} = I_2 + o(1),$$
$$\Phi_o(t,z,\zeta)\,e^{-ik(\zeta)t\sigma_3} = \tfrac{i}{\zeta}\sigma_3 Q(t,z) + o(1/\zeta), \tag{3.27b}$$

where subscripts "d" and "o" denote diagonal and off-diagonal part of a matrix. Similarly, as $\zeta \to 0$ in the appropriate regions of the complex plan,

$$\Psi_o(t,z,\zeta)\,e^{-i\lambda(\zeta)t\sigma_3} = \tfrac{i}{\zeta}\sigma_3 Q_+(z) + O(1),$$
$$\Psi_d(t,z,\zeta)\,e^{-i\lambda(\zeta)t\sigma_3} = Q(t,z)Q_+^{-1}(z) + o(1), \tag{3.27c}$$
$$\Phi_d(t,z,\zeta)\,e^{-ik(\zeta)t\sigma_3} = I_2 + O(\zeta), \tag{3.27d}$$
$$\Phi_o(t,z,\zeta)\,e^{-ik(\zeta)t\sigma_3} = -\tfrac{i\zeta}{A^2}\sigma_3 Q(t,z) + o(\zeta).$$

If the potential satisfies the condition in Theorem 3.1, the Wronskian representations (3.7) and the above asymptotic behavior of the eigenfunctions then yield the following asymptotic behavior for the scattering coefficients:

Proposition 3.12. *As $\zeta \to \infty$ in the appropriate regions of the complex plane,*

$$\lim_{\zeta\to\infty} a(\zeta,z) = 1, \quad \zeta \in D_{\text{out}}^+ \cup \mathbb{R}, \tag{3.28a}$$
$$\lim_{\zeta\to\infty} \bar a(\zeta,z) = 1, \quad \zeta \in D_{\text{out}}^- \cup \mathbb{R}, \tag{3.28b}$$
$$\lim_{\zeta\to\infty} b(\zeta,z) = \lim_{\zeta\to\infty} \bar b(\zeta,z) = 0, \quad \zeta \in \mathbb{R}. \tag{3.28c}$$

Similarly, as $\zeta \to 0$ in the appropriate regions of the complex plane,

$$b(\zeta,z) = \frac{iq_+^*(z)}{\zeta} + O(1), \quad \zeta \in D_{\text{in}}^+ \cup (-A,A), \tag{3.28d}$$
$$\bar b(\zeta,z) = \frac{iq_+(z)}{\zeta} + O(1), \quad \zeta \in D_{\text{in}}^- \cup (-A,A), \tag{3.28e}$$
$$\lim_{\zeta\to 0} a(\zeta,z) = \lim_{\zeta\to 0} \bar a(\zeta,z) = 0, \quad \zeta \in (-A,A), \tag{3.28f}$$
$$r_\pm(\zeta,z) = O(1/\zeta^2), \qquad \zeta \in (-A,A). \tag{3.28g}$$

Using (3.9a) and (3.9b) one then obtains following:

Proposition 3.13. *As $\zeta \to \infty$ on the real ζ- axis,*

$$\lim_{\zeta\to\infty} r_\pm(\zeta,z) = 0, \qquad \zeta \in \mathbb{R}. \tag{3.29a}$$

Similarly, as $\zeta \to 0$ on the real ζ- axis,

$$r_\pm(\zeta,z) = O(1/\zeta^2), \qquad \zeta \in (-A,A). \tag{3.29b}$$

Note that b, $\bar b$ and $r_\pm$ all have poles at $\zeta = 0$. We will see that this is not an obstacle to the formulation of the inverse problem, however.

3.5. Behavior at the branch points. Next we discuss the behavior of the Jost eigenfunctions and the scattering coefficients at the points $\zeta = \pm iA$, which correspond to the branch points $k = \pm iA$ of $\lambda(k)$ in the k-plane, and are therefore still referred to as branch points even if there is no branching in the ζ-plane.

Since $\lambda(\pm iA) = 0$, at $\zeta = \pm iA$ the two columns of $Y_+(\zeta, z)$, which specify the asymptotic behavior of the Jost eigenfunctions $\Psi(t, z, \zeta)$, become linearly dependent, and the two exponentials $\mathrm{e}^{\pm i\lambda(\zeta)t}$ reduce to the identity. It is convenient to introduce the weighted spaces $L^{1,j}(\mathbb{R}_t^{\pm}) := \{f : \mathbb{R} \to \mathbb{C} \,|\, (1+|t|)^j f(t) \in L^1(\mathbb{R}_t^{\pm})\}$, and consider the integral equations (3.3). Notice that even though $Y_+(z, \pm iA)$ is not invertible, the term $K_+(t-\tau, z, \zeta)$ appearing in the (3.3b) remains finite as $\zeta \to \pm iA$:

$$\lim_{\zeta\to\pm iA} K_+(t-\tau, z, \zeta) = I_2 + (t-\tau)(Q_+(z) \mp A\sigma_3), \qquad (3.30)$$
$$\lim_{\zeta\to\pm iA} \frac{\partial K_+(t-\tau, z, \zeta)}{\partial\zeta} = O_{2\times 2}.$$

Thus, if $q(t, z) - q_+(z) \in L^{1,1}(\mathbb{R}_t^+)$, the integral in (3.3b) is convergent at $\zeta = \pm iA$. Moreover, $\Psi(t, z, \zeta)$ is well-defined and continuous at the branch points $\zeta = \pm iA$, with

$$\Psi(t, z, \zeta) = \Psi_{\pm,1}(t, z) + o(1), \quad \zeta \to \pm iA, \qquad (3.31)$$

where $\Psi_{\pm,1}(t, z) \equiv \big(\psi_{\pm,1}(t, z), \bar\psi_{\pm,1}(t, z)\big) := \Psi(t, z, \pm iA)$.

Furthermore, if $q(t, z) - q_+(z) \in L^{1,2}(\mathbb{R}_t^+)$, it follows that $\partial\Psi_+(t, z, \zeta)/\partial\zeta$ is well-defined and continuous as $\zeta \to \pm iA$. Therefore one obtains

$$\Psi(t, z, \zeta) = \Psi_{\pm,1}(t, z) + \Psi_{\pm,2}(t, z)(\zeta \mp iA) + o(\zeta \mp iA), \quad \zeta \to \pm iA, \qquad (3.32a)$$

with $\Psi_{\pm,1}(t, z)$ given above, and

$$\Psi_{\pm,2}(t, z) \equiv \big(\psi_{\pm,2}(t, z), \bar\psi_{\pm,2}(t, z)\big) := \left.\frac{\partial\Psi(t, z, \zeta)}{\partial\zeta}\right|_{\zeta=\pm iA}. \qquad (3.32b)$$

Higher order expansions can be found similarly by placing further restrictions on the potential and looking at higher order derivatives in ζ.

Recall that $\det\Psi(t, z, \pm iA) = 0$, and therefore the columns of $\Psi(t, z, \pm iA)$ are proportional to each other. Using the asymptotic behavior of $\Psi(t, z, \zeta)$ when $t \to \infty$ and the fact that $\lambda(\zeta) = 0$ when $\zeta \to \pm iA$, one can show that

$$\psi(t, z, \pm iA) = \pm e^{-i\arg q_+(z)}\bar\psi(t, \pm iA). \qquad (3.33)$$

On the other hand, the Jost eigenfunction $\Phi(t, z, \zeta)$ is not defined at $\zeta = \pm iA$, but the individual columns are continuous at the appropriate branch point (cf. 3.3a). Namely, $\phi(t, z, iA)$ and $\bar\phi(t, z, -iA)$ are well-defined and continuous. Using the Wronskian representations (3.7a) and (3.7b), one can obtain the behavior of the scattering coefficients at the points $\pm iA$.

If $q(t, z) - q_+(z) \in L^{1,1}(\mathbb{R}_t^+)$ then using (3.31) one obtains

$$a(\zeta, z) = \mathrm{Wr}(\psi_{+,1}(t, z), \phi(t, z, \zeta)) + o(1), \quad \zeta \to iA, \qquad (3.34a)$$
$$\bar a(\zeta, z) = \mathrm{Wr}(\bar\phi(t, z, \zeta), \bar\psi_{+,1}(t, z)) + o(1), \quad \zeta \to -iA, \qquad (3.34b)$$
$$b(\zeta, z) = \mathrm{Wr}(\bar\phi(t, z, \zeta), \psi_{+,1}(t, z)) + o(1), \quad \zeta \to -iA, \qquad (3.34c)$$
$$\bar b(\zeta, z) = \mathrm{Wr}(\bar\psi_{+,1}(t, z), \phi(t, z, \zeta)) + o(1), \quad \zeta \to iA. \qquad (3.34d)$$

Similarly, assuming $q(t,z) - q_+(z) \in L^{1,2}(\mathbb{R}_t^+)$ and using (3.32) we have:

$$a(\zeta,z) = \mathrm{Wr}(\psi_{+,1}(t,z), \phi(t,z,\zeta)) \\ + \mathrm{Wr}(\psi_{+,2}(t,z), \phi(t,z,\zeta))(\zeta - iA) + o(\zeta - iA)\,, \quad \zeta \to iA\,, \tag{3.35a}$$

$$\bar{a}(\zeta,z) = \mathrm{Wr}(\bar{\phi}(t,z,\zeta), \bar{\psi}_{+,1}(t,z)) \\ + \mathrm{Wr}(\bar{\phi}(t,z,\zeta), \bar{\psi}_{+,2}(t,z))(\zeta + iA) + o(\zeta + iA), \quad \zeta \to -iA\,, \tag{3.35b}$$

$$b(\zeta,z) = \mathrm{Wr}(\bar{\phi}(t,z,\zeta), \psi_{+,1}(t,z)) \\ + \mathrm{Wr}(\bar{\phi}(t,z,\zeta), \psi_{+,2}(t,z))(\zeta + iA) + o(\zeta + iA), \quad \zeta \to -iA\,, \tag{3.35c}$$

$$\bar{b}(\zeta,z) = \mathrm{Wr}(\bar{\psi}_{+,1}(t,z), \phi(t,z,\zeta)) \\ + \mathrm{Wr}(\bar{\psi}_{+,2}(t,z), \phi(t,z,\zeta))(\zeta - iA) + o(\zeta - iA), \quad \zeta \to iA\,. \tag{3.35d}$$

One could continue this analysis by placing further restrictions on the potential if higher order terms of the scattering coefficients are needed.

Finally, we discuss the limiting behavior of the reflection coefficients near the branch points. Recalling the definition of the reflection coefficient from the left and right, (3.9a) and (3.9b) imply that the branch points $\pm iA$ are in the domain of the latter, while the reflection coefficients from the left are only defined for $\zeta \in \mathbb{R}$. To find the branch behavior of $r_+(\zeta,z)$ (resp. $\bar{r}_+(\zeta,z)$) near $\zeta = iA$ (resp. $\zeta = -iA$), we first compare the Wronskian representations of the scattering coefficients (3.7a) and (3.7b) with the proportionality relations (3.33) and obtain

$$a(iA,z) = \mathrm{e}^{-i\arg q_+(z)}\bar{b}(iA,z), \qquad b(-iA,z) = -\mathrm{e}^{-i\arg q_+(z)}\bar{a}(-iA,z). \tag{3.36}$$

Thus, if $q(t,z) - q_+(z) \in L^{1,1}(\mathbb{R}_t^+)$, (3.9b) yields

$$\lim_{\zeta\to iA} r_+(\zeta,z) = -\mathrm{e}^{-i\arg q_+(z)}, \qquad \lim_{\zeta\to -iA} \bar{r}_+(\zeta,z) = \mathrm{e}^{-i\arg q_+(z)}. \tag{3.37}$$

4. Inverse Problem

We now discuss the inverse problem in the IST, namely the reconstruction of the solution of the MBEs (2.3) from the knowledge of the scattering data. We formulate the inverse problem of the IST in terms of a matrix Riemann-Hilbert problem (RHP) for a suitable set of sectionally meromorphic functions in $D^+ \cup D^-$, with assigned jumps across $\mathbb{R} \cup \mathscr{C}$, i.e., the oriented contour in the complex ζ-plane as in Fig. 1. We show that one can obtain two different RHP formulations, namely a RHP "from the left" and one "from the right", depending on which scattering relation is used. The jump across $\mathscr{C}$ is then removed using the symmetries of the eigenfunctions.

4.1. Riemann-Hilbert problem from the left. We begin by introducing the following meromorphic matrix-valued function $M(t,z,\zeta)$ based on the analyticity properties of the Jost eigenfunctions and scattering coefficients discussed in Sect. 3:

Lemma 4.1. *Let $M(t,z,\zeta)$ be the sectionally meromorphic matrix defined as*

$$M(t,z,\zeta)=\begin{cases}\left(\dfrac{\psi(\zeta)}{a(\zeta)}e^{-i\lambda(\zeta)t},\ \phi(\zeta)e^{ik(\zeta)t}\right), & \zeta\in D^+_{out},\\ \left(\dfrac{\bar\psi(-\zeta^*)}{\bar b(-\zeta^*)}e^{-i\lambda(\zeta^*)t},\ \phi(-\zeta^*)e^{-ik(\zeta^*)t}\right), & \zeta\in D^-_{in},\\ \left(\bar\phi(-\zeta^*)e^{ik(\zeta^*)t},\ \dfrac{\psi(-\zeta^*)}{b(-\zeta^*)}e^{i\lambda(\zeta^*)t}\right), & \zeta\in D^+_{in},\\ \left(\bar\phi(\zeta)e^{-ik(\zeta)t},\ \dfrac{\bar\psi(\zeta)}{\bar a(\zeta)}e^{i\lambda(\zeta)t}\right), & \zeta\in D^-_{out}.\end{cases} \tag{4.1a}$$

where the t and z dependence of the eigenfunctions and the z dependence of the scattering coefficients in the right-hand side was omitted for brevity. The matrix $M(t,z,\zeta)$ satisfies the jump condition

$$M^+(t,z,\zeta)=M^-(t,z,\zeta)\,J_1(t,z,\zeta),\qquad \zeta\in\mathbb{R}, \tag{4.1b}$$

where the superscripts $\pm$ denote the limit being taken from the left/right of the side of the oriented contour in the complex ζ-plane, respectively, and where

$$J_1(t,z,\zeta)=\begin{cases}J_o(t,z,\zeta), & \zeta\in(-\infty,-A)\cup(A,\infty),\\ J_o(t,z,-\zeta), & \zeta\in(-A,A),\end{cases} \tag{4.1c}$$

with

$$J_o(t,z,\zeta)=\begin{cases}\begin{pmatrix}(1-r_-(z,\zeta)\,\bar r_-(z,\zeta))\mathrm{e}^{i(k(\zeta)-\lambda(\zeta))t} & -\bar r_-(z,\zeta)\mathrm{e}^{2ik(\zeta)t}\\ r_-(z,\zeta)\,\mathrm{e}^{-2i\lambda(\zeta)t} & \mathrm{e}^{i(k(\zeta)-\lambda(\zeta))t}\end{pmatrix}, & \zeta\in(-\infty,-A)\cup(A,\infty),\\ \begin{pmatrix}\mathrm{e}^{-i\zeta t} & \dfrac{1}{r_-(z,\zeta)}\mathrm{e}^{-2i\lambda(\zeta)t}\\ -\dfrac{1}{\bar r_-(z,\zeta)}\,\mathrm{e}^{-2ik(\zeta)t} & \left(1-\dfrac{1}{r_-(z,\zeta)\,\bar r_-(z,\zeta)}\right)\mathrm{e}^{-i\zeta t}\end{pmatrix}, & \zeta\in(-A,A).\end{cases} \tag{4.1d}$$

Proof. One can use the scattering relation form the left (3.5a) as well as the second symmetry (3.14b), (3.13) along with the fact that $-A^2/\zeta=-\zeta^*$ when $\zeta\in\mathscr{C}$, to obtain the above jump condition across $\zeta\in\mathbb{R}\cup\mathscr{C}$. Note that, as a result of having augmented the RHP to circumvent the nonlocality, there is no jump across $\mathscr{C}$. □

Recall that the asymptotic behavior of the eigenfunctions and the scattering matrix is given by (3.27) and (3.28). Accordingly, we have the following:

Lemma 4.2. *The sectionally meromorphic matrix $M(t,z,\zeta)$ in (4.1a) has the following asymptotic behavior:*

$$M^\pm(t,z,\zeta)=M_\infty+o(1),\qquad \zeta\to\infty,\quad \zeta\in D^{out}_\pm, \tag{4.2a}$$

$$M^\pm(t,z,\zeta)=M_o+o(1),\qquad \zeta\to 0,\quad \zeta\in D^{in}_\pm, \tag{4.2b}$$

where $M_\infty=M_o=I_{2\times2}$.

Lemma 4.3. *The meromorphic matrices defined in Lemma 4.1 satisfy the following residue conditions at the discrete eigenvalues:*

$$Res_{\zeta=\zeta_n} M(t,z,\zeta) = \Big(C_n(z) e^{-i\zeta_n t} M_2(t,z,\zeta_n)\,,\ \mathbf{0} \Big), \tag{4.3a}$$

$$Res_{\zeta=\zeta_n^*} M(t,z,\zeta) = \Big(\mathbf{0}\,,\ \bar{C}_n(z) e^{i\zeta_n^* t} M_1(t,z,\zeta_n^*) \Big), \tag{4.3b}$$

$$Res_{\zeta=-\hat{\zeta}_n^*} M(t,z,\zeta) = \Big(\mathbf{0}\,,\ \hat{C}_n(z) e^{i\zeta_n^* t} M_1(t,z,-\hat{\zeta}_n^*) \Big), \tag{4.3c}$$

$$Res_{\zeta=-\hat{\zeta}_n} M(t,z,\zeta) = \Big(\bar{\hat{C}}_n(z) e^{-i\zeta_n t} M_2(t,z,-\hat{\zeta}_n)\,,\ \mathbf{0} \Big), \tag{4.3d}$$

where M_1 and M_2 denote the columns of $M(t,z,\zeta)$ and the constants C_n, $\bar{C}_n$, $\hat{C}_n$ and $\bar{\hat{C}}_n$ were defined in (3.19a)*,* (3.19b) *and* (3.22)*, respectively.*

Remark 4.4. Summarizing, the RHP from the left consists in determining a matrix function $M(t,z,\zeta)$, meromorphic in $\mathbb{C}\setminus\mathbb{R}$, satisfying the jump condition (4.1b), the normalization condition (4.2) and the residue conditions (4.3). The minimal set of scattering data needed to define the RHP is comprised of: (i) the reflection coefficient $r_-(\zeta,0)$ for $\zeta \in (-\infty,-A)\cup(A,\infty)$, which (as discussed in detail in Sect. 6) combined with the "boundary conditions" $D_-(\zeta,z)$ and $P_-(\zeta,z)$ determines $r_-(z,\zeta)$ via (6.30a) (and also $D_+(\zeta,z)$ and $P_+(\zeta,z)$ via (6.10b)) for all $\zeta \in \mathbb{R}$ and all $z>0$, (ii) the discrete eigenvalues $\zeta_1,\ldots,\zeta_N$ and (iii) the associated norming constants $C_n(z)$, $C_n^*(z)$, $\bar{C}_n(z)$, $\hat{C}_n(z)$ and $\bar{\hat{C}}_n(z)$ satisfying the symmetries (3.19b), (3.23) and (3.26) (again, see Sect. 6 for a detailed discussion of the propagation of the norming constants).

Next we show how the above RHP can be converted to a set of linear algebraic-integral equations. We introduce the standard Cauchy projectors:

$$\big(P^{\pm} f\big)(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-\zeta}\mathrm{d}s, \qquad \zeta \in \mathbb{C}\setminus\mathbb{R}, \tag{4.4}$$

which are well-defined for any function $f \in L^1(\mathbb{R})$. If $f^{\pm}$ are analytic in $\mathbb{C}^{\pm}$ and $f^{\pm} = O(1/\zeta)$ as $\zeta \to \infty$ in the appropriate half-plane, then $P^{\pm}(f^{\pm}) = \pm f^{\pm}$ and $P^{\mp}(f^{\pm}) = 0$. Applying the Cauchy projectors (4.4) to the RHP defined by Eqs. (4.1a), (4.1b), and (4.2) yields the solution of the RHP in terms of the following system of linear algebraic-integral equations:

Theorem 4.5. *The solution of the RHP defined by Lemmas 4.1, 4.2, and 4.3 is given by*

$$M(t,z,\zeta) = M_\infty + \frac{1}{2\pi i}\int_{\mathbb{R}} M^-(s)L(s)\frac{ds}{s-\zeta} + \sum_{n=1}^{N}\left(\frac{Res_{\zeta=\zeta_n} M(\zeta)}{\zeta-\zeta_n} + \frac{Res_{\zeta=\zeta_n^*} M(\zeta)}{\zeta-\zeta_n^*} + \frac{Res_{\zeta=-\hat{\zeta}_n} M(\zeta)}{\zeta+\hat{\zeta}_n} + \frac{Res_{\zeta=-\hat{\zeta}_n^*} M(\zeta)}{\zeta+\hat{\zeta}_n^*}\right), \tag{4.5a}$$

$$M_1(t,z,\omega) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi i}\int_{\mathbb{R}} \big(M^-(s)L(s)\big)_1 \frac{ds}{s-\omega} + \sum_{n=1}^{N}\left(\frac{C_n e^{-i\zeta_n t} M_2(\zeta_n)}{(\omega-\zeta_n)} + \frac{\bar{\hat{C}}_n e^{-i\zeta_n t} M_2(-\hat{\zeta}_n)}{(\omega+\hat{\zeta}_n)}\right), \qquad \omega = \zeta_n^*, -\hat{\zeta}_n^*, \tag{4.5b}$$

$$M_2(t,z,\omega) = \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\pi i}\int_{\mathbb{R}} \big(M^-(s)L(s)\big)_2 \frac{ds}{s-\omega} + \sum_{n=1}^{N}\left(\frac{\bar{C}_n e^{i\zeta_n^* t} M_1(\zeta_n^*)}{(\omega-\zeta_n^*)} + \frac{\hat{C}_n e^{i\zeta_n^* t} M_1(-\hat{\zeta}_n^*)}{(\omega+\hat{\zeta}_n^*)}\right), \qquad \omega = \zeta_n, -\hat{\zeta}_n\,, \tag{4.5c}$$

where $L(\zeta) = J_1(\zeta) - I_{2\times 2}$, and the (t,z)-dependence in the right-hand side was again omitted for brevity.

Once the solution of the above RHP has been obtained, one can reconstruct the potential in terms of the scattering data by comparing the resulting asymptotics of the eigenfunctions in (4.5b) to (3.27), yielding:

Theorem 4.6. (Reconstruction formula). *Let $M(t,z,\zeta)$ be the solution of the RHP in Theorem 4.5. The corresponding solution $q(t,z)$ of the MBEs with one-sided NZBG is reconstructed as*

$$q(t,z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \big(M^-(t,z,s)L(z,s)\big)_{12}\, ds - \sum_{n=1}^{N}\left(1+\frac{iA^2}{(\zeta_n^*)^2}\right)\bar{C}_n(z) e^{i\zeta_n^* t} M_{11}(t,z,\zeta_n^*). \tag{4.6}$$

4.2. Riemann-Hilbert problem from the right. The inverse problem of the IST can also be formulated as a RHP from the right, as we discuss next. The reason why this could also be useful is that the MBEs are not symmetric under $t \mapsto -t$, and in the case of one-sided NZBG, the two RHPs have different properties, depending on whether the NZBG are given in the past or in the future. This is reflected in the fact that, as we will see below, in the RHP from the right, the jump matrix will be given in terms of $r_+(\zeta,z)$ [cf. (4.8) below], whereas, in the RHP from the left, the jump matrix (4.1c) is given in terms of $r_-(\zeta,z)$.

Like the RHP from the left, the RHP from the right can be formulated as a local RHP for a 2×2 matrix involving only the eigenfunctions evaluated at ζ and the reflection coefficients from the right on $\mathscr{C}$. Specifically, we introduce the following 2×2 matrix of modified eigenfunctions:

$$M(t,z,\zeta) = \begin{cases} \left(\psi(\zeta)\mathrm{e}^{-i\lambda(\zeta)t},\ \dfrac{\phi(\zeta)}{c(\zeta)}\mathrm{e}^{ik(\zeta)t}\right), & \zeta \in D_{\text{out}}^+, \\ \left(\dfrac{iq_+^*\zeta^*}{A^2}\bar{\psi}(-\zeta^*)\mathrm{e}^{-i\lambda(\zeta^*)t},\ \dfrac{iq_+\zeta^*}{A^2}\dfrac{\phi(-\zeta^*)}{d(-\zeta^*)}\mathrm{e}^{-ik(\zeta^*)t}\right), & \zeta \in D_{\text{in}}^-, \\ \left(\dfrac{iq_+^*\zeta^*}{A^2}\dfrac{\bar{\phi}(-\zeta^*)}{\bar{d}(-\zeta^*)}\mathrm{e}^{ik(\zeta^*)t},\ \dfrac{iq_+\zeta^*}{A^2}\psi(-\zeta^*)\mathrm{e}^{i\lambda(\zeta^*)t}\right), & \zeta \in D_{\text{in}}^+, \\ \left(\dfrac{\bar{\phi}(\zeta)}{\bar{c}(\zeta)}\mathrm{e}^{-ik(\zeta)t},\ \bar{\psi}(\zeta)\mathrm{e}^{i\lambda(\zeta)t}\right), & \zeta \in D_{\text{out}}^-, \end{cases} \tag{4.7}$$

which should be compared with (4.1a), and where the t and z dependence in the right-hand side was again omitted for brevity. Now using the scattering relation from the right (3.5b) and the second symmetry (3.13), along with the fact that $-A^2/\zeta = -\zeta^*$

when $\zeta \in \mathscr{C}$, we obtain that the jump condition across $\zeta \in \mathbb{R}$ is still expressed by (4.1b), with $J_1(z,\zeta)$ still given by (4.1c), except that now

$$J_o(t,z,\zeta) = \begin{cases} \begin{pmatrix} e^{i(k(\zeta)-\lambda(\zeta))t} & r_+(\zeta,z)e^{2ik(\zeta)t} \\ -\bar{r}_+(\zeta,z)\,e^{-2i\lambda(\zeta)t} & (1-r_+(\zeta,z)\,\bar{r}_+(\zeta,z))e^{i(k(\zeta)-\lambda(\zeta))t} \end{pmatrix}, & \zeta \in (-\infty,-A)\cup(A,\infty), \\ \begin{pmatrix} e^{i\zeta t} & \dfrac{q_+(z)}{q_+^*(z)\,r_+(\zeta,z)}e^{2ik(\zeta)t} \\ -\dfrac{q_+^*(z)}{q_+(z)\,\bar{r}_+(\zeta,z)}\,e^{2i\lambda(\zeta)t} & \left(1-\dfrac{1}{r_+(\zeta,z)\,\bar{r}_+(\zeta,z)}\right)e^{i\zeta t} \end{pmatrix}, & \zeta \in (-A,A). \end{cases} \tag{4.8}$$

Again, note that no jump is present across $\mathscr{C}$. As before, the above jump condition must be supplemented by appropriate normalization condition and residue conditions. Since these are obtained using similar methods as above, we omit the details for brevity.

4.3. Trace formulae. As usual, one can also obtain "trace formulae" to recover the analytic scattering coefficients in terms of the scattering data. We begin by defining the following functions:

$$\beta^+(\zeta,z) = a(\zeta,z)\prod_{n=1}^N \frac{\zeta-\zeta_n^*}{\zeta-\zeta_n}, \tag{4.9a}$$

$$\beta^-(\zeta,z) = \bar{a}(\zeta,z)\prod_{n=1}^N \frac{\zeta-\zeta_n}{\zeta-\zeta_n^*},$$

$$\alpha^+(\zeta,z) = -\frac{i\zeta}{q_+^*(z)}\,b(\zeta,z)\prod_{n=1}^N \frac{\zeta_n(\zeta-\hat{\zeta}_n^*)}{\zeta_n^*(\zeta-\hat{\zeta}_n)}, \tag{4.9b}$$

$$\alpha^-(\zeta,z) = -\frac{i\zeta}{q_+(z)}\,\bar{b}(\zeta,z)\prod_{n=1}^N \frac{\zeta_n^*(\zeta-\hat{\zeta}_n)}{\zeta_n(\zeta-\hat{\zeta}_n^*)}.$$

Recalling (3.7), one can see that $\beta^\pm$ are analytic in $D^\pm_{\text{out}}$, while $\alpha^\pm$ analytic in $D^\pm_{\text{in}}$.

Moreover, using the asymptotic behavior of the scattering coefficients (3.28) one can show that

$$\lim_{\zeta\to\infty}\beta^\pm = 1, \quad \zeta \in D^\pm_{\text{out}}, \qquad \lim_{\zeta\to 0}\alpha^\pm = 1, \quad \zeta \in D^\pm_{\text{in}}. \tag{4.10}$$

Also, by construction $\beta^\pm$ and $\alpha^\pm$ have no zeros. Now we define the following sectionally analytic vector function:

Lemma 4.7. *Consider the meromorphic vector function $N(\zeta,z)$ defined as*

$$N(\zeta,z) = \begin{cases} \left(\log\left(\beta^+(\zeta,z)\right),\ \log\left(\beta^+(-\zeta^*,z)\right)\right), & \zeta \in D^+_{\text{out}}, \\ \left(\log\left(\alpha^-(-\zeta^*,z)\right),\ \log\left(\alpha^-(\zeta,z)\right)\right), & \zeta \in D^-_{\text{in}}, \\ \left(-\log\left(\alpha^+(-\zeta^*,z)\right),\ -\log\left(\alpha^+(\zeta,z)\right)\right), & \zeta \in D^+_{\text{in}}, \\ \left(-\log\left(\beta^-(\zeta,z)\right),\ -\log\left(\beta^-(-\zeta^*,z)\right)\right), & \zeta \in D^-_{\text{out}}. \end{cases} \tag{4.11a}$$

$N(\zeta,z)$ satisfies the jump condition

$$N^+(\zeta,z) - N^-(\zeta,z) = K(\zeta,z), \qquad \zeta \in \mathbb{R}, \tag{4.11b}$$

where the superscripts ± denote the limit being taken from the left/right of the negative/positive side of the oriented contour in complex ζ-plane, respectively, and

$$K(\zeta, z) = \begin{cases} \Big(K_o(\zeta, z)\ ,\ K_o(-\zeta, z)\Big), & \zeta \in (-\infty, -A) \cup (A, \infty), \\ \Big(K_o(-\zeta, z)\ ,\ K_o(\zeta, z)\Big), & \zeta \in (-A, A), \end{cases} \tag{4.11c}$$

with

$$K_o(\zeta, z) = \begin{cases} -\log\left[\frac{\zeta^2}{\zeta^2+A^2}(1 + |r_-(\zeta, z)|^2)\right], & \zeta \in (-\infty, -A) \cup (A, \infty), \\ \log\left[\frac{A^2}{\zeta^2+A^2}\left(1 + \frac{1}{|r_-(\zeta,z)|^2}\right)\right], & \zeta \in (-A, A). \end{cases} \tag{4.11d}$$

Note that using the determinant of the scattering matrix form the left (3.5a) and (3.6) as well as the second symmetry of the scattering coefficients (3.14b), along with the fact that $-A^2/\zeta = -\zeta^*$ when $\zeta \in \mathscr{C}$, one can derive the above jump condition across $\zeta \in \mathbb{R}$. Applying the Cauchy projectors (4.4) to the RHP defined by Eqs. (4.11a) and (4.11b) yields

$$N(\zeta, z) = \frac{1}{2\pi i} \int_\Sigma \frac{K(s, z)}{s - \zeta} \mathrm{d}s, \qquad \zeta \in \mathbb{C} \setminus \mathbb{R}, \tag{4.12}$$

where

$$\Sigma = (-\infty, -A) \cup (A, \infty) \cup (A, -A).$$

Now using the solution of the RHP, one can recover the analytic scattering coefficient from the knowledge of the reflection coefficients and discrete eigenvalues, as follows:

Theorem 4.8. *The analytic scattering coefficients defined in (3.5) are given by*

$$a(\zeta, z) = \prod_{n=1}^{N} \frac{\zeta - \zeta_n}{\zeta - \zeta_n^*} \exp\left\{\frac{1}{2\pi i} \int_\Sigma \frac{K_1(s, z)}{s - \zeta} \mathrm{d}s\right\}, \quad \zeta \in D_{out}^+, \tag{4.13a}$$

$$\bar{a}(\zeta, z) = \prod_{n=1}^{N} \frac{\zeta - \zeta_n^*}{\zeta - \zeta_n} \exp\left\{-\frac{1}{2\pi i} \int_\Sigma \frac{K_1(s, z)}{s - \zeta} ds\right\}, \quad \zeta \in D_{out}^-, \tag{4.13b}$$

$$b(\zeta, z) = \frac{i q_+^*(z)}{\zeta} \prod_{n=1}^{N} \frac{\zeta_n^*(\zeta - \hat{\zeta}_n)}{\zeta_n(\zeta - \hat{\zeta}_n^*)} \exp\left\{-\frac{1}{2\pi i} \int_\Sigma \frac{K_2(s, z)}{s - \zeta} \mathrm{d}s\right\}, \quad \zeta \in D_{in}^+, \tag{4.13c}$$

$$\bar{b}(\zeta, z) = \frac{i q_+(z)}{\zeta} \prod_{n=1}^{N} \frac{\zeta_n(\zeta - \hat{\zeta}_n^*)}{\zeta_n^*(\zeta - \hat{\zeta}_n)} \exp\left\{\frac{1}{2\pi i} \int_\Sigma \frac{K_2(s, z)}{s - \zeta} ds\right\}, \quad \zeta \in D_{in}^-, \tag{4.13d}$$

where $K(\zeta, z)$ is given by (4.11c) *and the subscript $j = 1, 2$ denotes its j-th column.*

We reiterate that, unlike what happens in the IST for the focusing NLS equation and for the MBEs with zero background, the above trace formulae are needed in order to obtain the correct propagation equation for the norming constants (cf. Sect. 6.4).

5. Asymptotics of the Density Matrix as $t \to \pm\infty$

The behavior of the density matrix is one of the novel aspects of the IST for the MBEs compared to that for the NLS equation.

Proposition 5.1. *If $v(t, z, \zeta)$ is any fundamental matrix solution of the scattering problem (2.4a), the quantity $v^{-1}(t, z, \zeta)\rho(t, z, \zeta)v(t, z, \zeta)$ is time-independent.*

As a consequence, it is convenient to define asymptotic values for the density matrix as follows:

$$\begin{aligned} \rho_-(\zeta, z) &= \Phi^{-1}(t, z, \zeta)\rho(t, z, \zeta)\Phi(t, z, \zeta), \\ \rho_+(\zeta, z) &= \Psi^{-1}(t, z, \zeta)\rho(t, z, \zeta)\Psi(t, z, \zeta), \end{aligned} \tag{5.1}$$

and conversely

$$\rho(t, z, \zeta) = \Phi(t, z, \zeta)\rho_-(\zeta, z)\Phi^{-1}(t, z, \zeta) \equiv \Psi(t, z, \zeta)\rho_+(\zeta, z)\Psi^{-1}(t, z, \zeta). \tag{5.2}$$

Taking into account the asymptotics as $t \to \pm\infty$ of the Jost eigenfunctions we obtain:

Proposition 5.2. *For all $\zeta \in \mathbb{R}$ and all $z \geq 0$, one has:*

$$\begin{aligned} \rho_-(\zeta, z) &= \lim_{t\to-\infty} e^{-ikt\sigma_3}\rho(t, z, \zeta)e^{ikt\sigma_3}, \\ \rho_+(\zeta, z) &= \lim_{t\to+\infty} e^{-i\lambda t\sigma_3}Y_+^{-1}(\zeta, z)\rho(t, z, \zeta)Y_+(\zeta, z)e^{i\lambda t\sigma_3}. \end{aligned} \tag{5.3}$$

Note, however, that the density matrix does not have a limit per se, i.e., $\rho_\pm(\zeta, z)$ are not simply the limits of $\rho(t, z, \zeta)$ as $t \to \pm\infty$. One can also check by direct calculation that $S\rho_+ = \rho_- S = \Phi^{-1}\rho\Psi$. As a result, one has:

Proposition 5.3. *For all $\zeta \in \mathbb{R}$ and all $z \geq 0$, the asymptotic values of the density matrix are related as follows:*

$$\rho_+(z, \zeta) = S^{-1}(\zeta, z)\rho_-(\zeta, z)S(\zeta, z). \tag{5.4}$$

Equation (5.4) relates the asymptotic values of the density matrix as $t \to \pm\infty$, and allows one to obtain ρ_+ from knowledge of ρ_- and S (in turn, the latter can be completely determined by $q(t, z)$]. Thus, one can only choose one between $\rho_\pm$, and, due to causality, it makes sense to choose ρ_-. Note also that the density matrix is a single-valued function of k, which means that $\rho(t, z, -A^2/\zeta) = \rho(t, z, \zeta)$, and the same holds for $\rho_-(\zeta, z)$ (which depends on the Jost eigenfunction Φ), but not for $\rho_+(\zeta, z)$, which is defined via Ψ.

The properties of the density matrix $\mathrm{tr}\rho = 0$ and $\det\rho = -1$ imply that $\mathrm{tr}\rho_\pm = 0$ and $\det\rho_\pm = -1$. Also, since $\rho^\dagger = \rho$ for $\zeta \in \mathbb{R}$, the same holds for $\rho_\pm = \rho_\pm^\dagger$. Thus we can denote the entries of $\rho_\pm$ as

$$\rho_\pm(\zeta, z) = \begin{pmatrix} D_\pm & P_\pm \\ P_\pm^* & -D_\pm \end{pmatrix} \qquad \zeta \in \mathbb{R}, \tag{5.5}$$

with $P_\pm(\zeta, z) = P_\pm(\zeta^*, z)$. As for the density matrix itself, $D_\pm$ and $P_\pm$ are not limits of D and P as $t \to \pm\infty$, as such limits in general do not exist. Combining the second symmetry of the eigenfunctions with the definition (5.3) we obtain:

Proposition 5.4. *For all $\zeta \in \mathbb{R}$, the matrices $\rho_\pm(\zeta, z)$ satisfy the following symmetries:*

$$\rho_-(-A^2/\zeta, z) = \rho_-(\zeta, z), \qquad \rho_+(-A^2/\zeta, z) = \sigma_3 Q_+(z)\rho_+(\zeta, z)Q_+^{-1}(z)\sigma_3. \quad (5.6)$$

Proposition 5.4 implies that one does not have the freedom to pick the asymptotic states $\rho_\pm$ for all values of k (or ζ). One can only pick $\rho_\pm$ for $k \in \mathbb{R}$ on first sheet, or, equivalently, for $\zeta \in (-\infty, -A] \cap [A, \infty)$. Specifically,

$$D_\pm \in \mathbb{R}, \quad P_\pm \in \mathbb{C}, \quad D_\pm^2 + |P_\pm|^2 = 1, \quad (5.7a)$$

$$D_-(-A^2/\zeta, z) = D_-(\zeta, z), \qquad D_+(-A^2/\zeta, z) = -D_+(\zeta, z), \quad (5.7b)$$

$$P_-(-A^2/\zeta, z) = P_-(\zeta, z), \qquad P_+(-A^2/\zeta, z) = (q_+(z)/q_+^*(z))P_+^*(\zeta, z). \quad (5.7c)$$

Equation (5.4) can then be used to obtain D_+ and P_+ from D_- and P_-, respectively. Note that in principle $D_\pm$ and $P_\pm$ depend on k on each sheet.

Equation (5.3) yield an explicit relation between $\rho_\pm$ and ρ, which, in component form, is

$$D_+(z, \zeta) = \lim_{t\to+\infty} \left[\frac{k}{\lambda} D - \frac{1}{\lambda}\operatorname{Im}(q_+^* P)\right], \quad (5.8a)$$

$$P_+(z, \zeta) = \frac{1}{2\lambda} \lim_{t\to+\infty} \mathrm{e}^{-2i\lambda t} \left[2iq_+ D + \zeta P + \frac{q_+^2}{\zeta} P^*\right]. \quad (5.8b)$$

Conversely, (5.2) also implies that

$$\rho(t, z, \zeta) = \mathrm{e}^{ikt\sigma_3}\rho_-(\zeta, z)\mathrm{e}^{-ikt\sigma_3} + o(1), \qquad t \to -\infty, \quad (5.9a)$$

$$\rho(t, z, \zeta) = Y_+(\zeta, z)\mathrm{e}^{i\lambda t\sigma_3}\rho_+(\zeta, z)\mathrm{e}^{-i\lambda t\sigma_3}Y_+^{-1}(\zeta, z) + o(1), \qquad t \to -\infty, \quad (5.9b)$$

i.e.

$$D(t, z, \zeta) = D_- + o(1), \quad P(t, z, \zeta) = \mathrm{e}^{-2ikt}P_- + o(1), \qquad t \to -\infty, \quad (5.10a)$$

$$D(t, z, \zeta) = \frac{k}{\lambda}D_+ - \frac{1}{\lambda}\operatorname{Im}\left(\mathrm{e}^{-2i\lambda t}P_+^* q_+\right) + o(1), \qquad t \to \infty, \quad (5.10b)$$

$$P(t, z, \zeta) = -\frac{i}{\lambda}q_+ D_+ + \frac{\zeta}{2\lambda}\mathrm{e}^{2i\lambda t}P_+ + \frac{q_+}{2\lambda\zeta}\mathrm{e}^{-2i\lambda t}P_+^* + o(1), \qquad t \to \infty. \quad (5.10c)$$

Remark 5.5. Note that, as in the case of zero background (ZBG) and of symmetric NZBG, P does not have a limit as $t \to \pm\infty$, but instead it oscillates in time. The quantity D, on the other hand, has a constant limit D_- as $t \to -\infty$, but due to the nonzero background radiation as $t \to +\infty$, it also does not have a limit and instead oscillates. Moreover, $P \neq 0$ as $t \to +\infty$ even in the particular case in which D and P are time-independent and do tend to a limit as $t \to +\infty$, which is when $P_+ = 0$. The nonzero contribution arises from the polarization induced by the limiting value q_+ of the optical field via D_+. [Recall that the normalization $\det \rho_- = -1$ implies the constraint $D_\pm^2 + |P_\pm|^2 = 1$, which in turn implies that one does not have the freedom to assign $D_\pm$ and $P_\pm$ independently. In particular, in the special case when ρ_- is diagonal, then $D_- = \pm 1$ and $P_- = 0$].

Remark 5.6. Recall that the density matrix $\rho(t, z, k)$ describes the physical properties of the medium. Therefore its value is independent of the choice of sign for λ and is therefore the same for $\zeta \in (-\infty, -A] \cup [A, \infty)$ (i.e., the continuous spectrum on sheet I) or $\zeta \in [-A, A]$ (i.e., the continuous spectrum on sheet II). The same is true for the first set of Jost solutions, $\Phi(t, z, \zeta)$. Conversely, the Jost solutions $\Psi(t, z, \zeta)$ are defined explicitly in terms of λ, and therefore take on different values for $\zeta \in (-\infty, -A] \cup [A, \infty)$ or $\zeta \in [-A, A]$ (as discussed in Sect. 3.2). Similarly, $\rho_-(\zeta, z)$, which is defined in terms of $\Phi(t, z, \zeta)$, is independent of the choice of sign for λ, whereas $\rho_+(\zeta, z)$, which is defined in terms of $\Psi(t, z, \zeta)$, depends on the sign of λ, and is therefore different on different sheets. Equation (5.7) reflect this difference. In turn, this sheet dependence is also reflected in (5.10), and it should be taken into account when reconstructing the asymptotic behavior of the medium. For example, if $P_\pm = 0$, the sign of D_+ coincides with that of $D(z, \zeta)$ on the first sheet, not the second one.

6. Propagation

Recall that in the MBEs the role of the evolution variable is played by the physical propagation distance z, and therefore we will refer to the z-dependence as the propagation. The evolution of the scattering data is another aspect of the IST formalism where the treatment for the MBEs differs significantly from (and is considerably more complicated than) that for the NLS equation.

6.1. Propagation of the background. Let us first discuss the z-dependence of the asymptotic values of the optical field.

Lemma 6.1. *The limiting values $q_\pm(z) = \lim_{t\to\pm\infty} q(z, t)$ are given by*

$$q_-(z) = 0, \qquad q_+(z) = \mathrm{e}^{2iW_+(z)} q_+(0), \tag{6.1a}$$

where

$$W_+(z) = \int_0^z w_+(s)\, ds, \qquad w_+(z) = \frac{1}{2} \int_{\mathbb{R}} \frac{D_+(\xi, z) g(\xi)}{\lambda(\xi)} \mathrm{d}\xi. \tag{6.1b}$$

Proof. The propagation of $q_\pm(z)$ is governed by the limits as $t \to \pm\infty$ of (2.3), i.e.

$$\frac{\partial Q_\pm}{\partial z} = -\frac{1}{2} \lim_{t\to\pm\infty} \int_{\mathbb{R}} [\sigma_3, \rho(t, z, k)] g(k) dk.$$

Using (5.2) we can express

$$\rho(t, z, \zeta) = \mathrm{e}^{ikt\sigma_3} \rho_-(z, \zeta) \mathrm{e}^{-ikt\sigma_3} + o(1) \qquad \text{as } t \to -\infty, \tag{6.2}$$

$$\rho(t, z, \zeta) = Y_+(\zeta, z) \mathrm{e}^{i\lambda t\sigma_3} \rho_+(z, \zeta) \mathrm{e}^{-i\lambda t\sigma_3} Y_+^{-1}(\zeta, z) + o(1) \qquad \text{as } t \to +\infty. \tag{6.3}$$

Consequently, one has

$$\frac{\partial Q_-}{\partial z} = 0, \qquad \frac{\partial Q_+}{\partial z} = i w_+ [\sigma_3, Q_+], \tag{6.4}$$

with $w_+(z)$ as in (6.1b). Next, recall that $\mathrm{tr}\rho_\pm = \mathrm{tr}\rho = 0$ and D_+ has opposite sign on sheets I and II. Since λ also changes sign on opposite sheets, w_+ is single-valued, as it should be. Integrating (6.4) we find

$$Q_-(z) = Q_-(0) \equiv 0, \qquad Q_+(z) = \mathrm{e}^{iW_+(z)\sigma_3} Q_+(0) \mathrm{e}^{-iW_+(z)\sigma_3}, \tag{6.5}$$

whose components yield (6.1a). □

Equation (6.1a) provide the explicit z-dependence in the symmetries in Sect. 3.2. In particular, the propagation of the asymptotic eigenvector matrix $Y_+(\zeta, z)$ is given by:

$$Y_+(\zeta, z) = I_2 + (i/\zeta)\sigma_3 Q_+(z) = \mathrm{e}^{iW_+(z)\sigma_3} Y_+(\zeta, 0) \mathrm{e}^{-iW_+(z)\sigma_3}, \tag{6.6}$$

and consequently the asymptotic behavior of the Jost solutions is

$$\Phi(t, z, \zeta) = I_2 \mathrm{e}^{ik(\zeta)t\sigma_3}(1 + o(1)), \qquad \text{as } t \to -\infty \tag{6.7a}$$
$$\Psi(t, z, \zeta) = \mathrm{e}^{iW_+(z)\sigma_3} Y_+(\zeta, 0) \mathrm{e}^{i(\lambda(\zeta)t - W_+(z))\sigma_3}(1 + o(1)), \quad \text{as } t \to +\infty. \tag{6.7b}$$

We can use (5.4) to express $2w_+(z)\sigma_3 = \int (\rho_{+,d}(k, z) g(k)/\lambda) dk$ in terms of $\rho_{-,d}$ as:

$$\rho_{+,d} = (S^{-1}\rho_- S)_d = \rho_{-,d} + \frac{\zeta}{2\lambda}\left(2b\bar{b}\, D_- + P_- \bar{a} b - P_-^* a \bar{b}\right)\sigma_3. \tag{6.8}$$

Note that the above expression has an extra term proportional to $b\bar{b}$ compared to [14], and this extra term is also present in the case of symmetric NZBG studied there.

Lemma 6.2. *The asymptotic values D_+ and P_+ can be obtained from the initial state of the medium as follows:*

$$D_+ = \frac{1}{1 + |r_-|^2}\Big((1 - |r_-|^2) D_- + 2\,Re(r_-\, P_-)\Big), \tag{6.9a}$$
$$P_+ = \frac{\mathrm{e}^{-2i\,arg(a)}}{1 + |r_-|^2}\Big(P_- - (r_-^2\, P_-)^* - 2r_-^*\, D_-\Big), \tag{6.9b}$$

where we omitted the dependence on ζ and z for brevity.

Proof. Considering (6.8) and (5.4), and using the symmetries (3.10), we can express the diagonal and off-diagonal entries of ρ_+ in terms of those of ρ_-:

$$D_+ = \frac{\zeta}{2\lambda}\Big((|a|^2 - |b|^2) D_- + a^* b P_- + a b^* P_-^*\Big). \tag{6.10a}$$
$$P_+ = \frac{\zeta}{2\lambda}\Big((a^*)^2 P_- - (b^2 P_-)^* - 2a^* b^* D_-\Big). \tag{6.10b}$$

Using (3.9a) and (3.12) one can then rewrite above D_+ and P_+ explicitly as in (6.9b). □

Equations (6.9a) and (6.9b), which are the same as in the case of ZBG except for the additional presence of the factor $\zeta/(2\lambda)$ (which is a direct consequence of the fact that $\det S \neq 1$, and which reduces to 1 in the case of ZBG), show that even if the medium is initially prepared so that $P_- = 0$ and $D_- = \pm 1$, in general one has $P_+ \neq 0$, since r_- cannot be chosen to be identically zero. (We note that in the case of symmetric NZBG, $P_- = 0$ does not imply $P_+ = 0$ except in the reflectionless case.) The limiting values of these quantities as $z \to \infty$ will be discussed in Sect. 7.

6.2. Simultaneous solutions of the Lax pair. To obtain propagation equations for the scattering data, which will be done in Sects. 6.3 and 6.4, one needs to introduce simultaneous solutions of both parts of the Lax pair. Since the asymptotic behavior of the Jost solutions $\Phi(t, z, \zeta)$ and $\Psi(t, z, \zeta)$ as $t \to \pm\infty$ is independent of z, in general they will not be solutions of (2.4c). However, the Jost solutions can be used to define simultaneous solutions of the Lax pair as follows.

Lemma 6.3. *If $\Xi(t, z, \zeta)$ is any fundamental matrix solution of the scattering problem (2.4a), it can be written as*

$$\Xi(t, z, \zeta) = \Psi(t, z, \zeta)C_+(\zeta, z) = \Phi(t, z, \zeta)C_-(\zeta, z), \qquad \zeta \in \mathbb{R}, \tag{6.11}$$

where $C_\pm(\zeta, z)$ are 2×2 matrices independent of t which evolve according to

$$\frac{\partial C_\pm}{\partial z} = \frac{i}{2} R_\pm C_\pm, \tag{6.12}$$

with

$$R_-(\zeta, z) = -2i\Phi^{-1}[T\Phi - \partial_z\Phi], \qquad R_+(\zeta, z) = -2i\Psi^{-1}[T\Psi - \partial_z\Psi]. \tag{6.13}$$

Proof. Since both Φ and Ψ are fundamental matrix solutions of the scattering problem, any other solution $\Xi(t, z, \zeta)$ of (2.4a) can then be written as (6.11) with suitable matrices $C_\pm(\zeta, z)$.

Then, if $\Xi(t, z, \zeta)$ is a simultaneous solution of both parts of the Lax pair (2.4), (i.e., it also satisfies $\Xi_z = T\Xi$), (6.12) follows, where we have taken into account that for our one-sided NZBG Φ can be chosen independent of z. Although it is not obvious a priori that the RHS of (6.13) is independent of t, (6.12) shows that it must be. Moreover, even though $g(k)$ and $\rho(t, z, k)$ are only defined for $k \in \mathbb{R}$, $R_\pm$ in (6.13) can be evaluated for all $\zeta \in \mathbb{R}$. □

Lemma 6.4. *Denoting by $R_{\pm,ij}$ the (i, j)-th entry of the matrices $R_\pm(\zeta, z)$, these entries are given by:*

$$R_{+,11} = -R_{+,22} = \fint_{\mathbb{R}} \frac{(\lambda(k) + \xi - k)D_+(\xi, z)g(\xi)}{\lambda(\xi)(\xi - k)}\mathrm{d}\xi \quad \zeta \in \mathbb{R}\,, \tag{6.14a}$$

$$R_{-,11} = -R_{-,22} = \fint_{\mathbb{R}} \frac{D_-(\xi, z)g(\xi)}{\xi - k}\mathrm{d}\xi \quad \zeta \in \mathbb{R}\,, \tag{6.14b}$$

$$R_{+,12} = R^*_{+,21} = \begin{cases} i\sigma\pi g(k(\zeta))P_+(\zeta, z) & \zeta \in \mathbb{R}\,, \\ 0 & \zeta \in \mathscr{C} \setminus \{\pm A\}\,, \end{cases} \tag{6.14c}$$

$$R_{-,12} = R^*_{-,21} = -i\pi g(k(\zeta))P_-(\zeta, z) \qquad \zeta \in \mathbb{R}\,. \tag{6.14d}$$

Proof. Note that, in general, both $C_\pm$ and $R_\pm$ have different values on either sheet of the Riemann surface. Using (5.2), we can write the second operator in the Lax pair as

$$\begin{aligned} T(t, z, \zeta) &= \frac{i\pi}{2}\mathscr{H}_k[\Phi(t, z, \zeta(\xi))\rho_-(\xi, z)\Phi^{-1}(t, z, \zeta(\xi)\, g(\xi)] \\ &= \frac{i\pi}{2}\mathscr{H}_k[\Psi(t, z, \zeta(\xi))\rho_+(\xi, z)\Psi^{-1}(t, z, \zeta(\xi)\, g(\xi)], \end{aligned} \tag{6.15}$$

where the subscript k in the Hilbert transform is to be intended as $k(\zeta)$ henceforth. As in [14], even though the individual terms on the RHS of the second equality are only single-valued for $k \in \mathbb{R}$, the whole RHS is a single-valued function for $k \in \mathbb{C}$. In fact, one can use the symmetries of the Jost solution Ψ and of ρ_+ to show that $T(t, z, \zeta) = T(t, z, -A^2/\zeta)$.

Next, we can evaluate the RHS of the first of (6.13) in the limit $t \to -\infty$. Assuming that ∂_z and the limit $t \to -\infty$ of the Jost eigenfunctions can be interchanged, i.e., that the Jost eigenfunctions approach their asymptotic values as $t \to \pm\infty$ uniformly for all $z \geq 0$, we obtain

$$\begin{aligned} R_-(\zeta, z) &= -2i \lim_{t\to-\infty} \mathrm{e}^{-ikt\sigma_3} \left[T(t, z, \zeta)\mathrm{e}^{ikt\sigma_3} - \partial_z \mathrm{e}^{ikt\sigma_3} \right] \\ &= \pi \mathscr{H}_k \left[\mathrm{e}^{-i(k-\xi)t\sigma_3} \rho_-(\xi, z)\mathrm{e}^{i(k-\xi)t\sigma_3}\, g(\xi) \right]. \end{aligned} \tag{6.16}$$

Now the second of (6.13) in the limit $t \to +\infty$ yields:

$$\begin{aligned} R_+(\zeta, z) = \lim_{t\to+\infty} \Big\{ & 2i\Psi^{-1}(t, z, \zeta)\partial_z\Psi(t, z, \zeta) + \pi \mathrm{e}^{-i\lambda(k)t\sigma_3} Y_+(k, z) \\ & \times \mathscr{H}_k \Big[Y_+(\xi, z)\mathrm{e}^{i\lambda(\xi)t\sigma_3} \rho_+(\xi, z)\mathrm{e}^{-i\lambda(\xi)t\sigma_3} Y_+^{-1}(\xi, z) g(\xi) \Big] Y_+(k, z)\mathrm{e}^{i\lambda(k)t\sigma_3} \Big\}. \end{aligned} \tag{6.17}$$

The limits can be computed explicitly by noting that

$$\lim_{t\to\pm\infty} \fint_{\mathbb{R}} \mathrm{e}^{2i(k-\xi)t} \frac{f(\xi, k)}{\xi - k} \mathrm{d}\xi = \mp i\pi f(k, k) \qquad k \in \mathbb{R} \tag{6.18}$$

$$\lim_{t\to\pm\infty} \fint_{\mathbb{R}} \mathrm{e}^{\pm i(\lambda(\xi)-\lambda(k))t} \frac{f(\xi, k)}{\xi - k} \mathrm{d}\xi = \begin{cases} \pm i\sigma\pi f(k, k) & k \in \mathbb{R} \\ 0 & k \in i[-A, A] \end{cases} \tag{6.19}$$

where $\sigma = \pm 1$ when k is on sheet I or II, respectively. Then, if $R_\pm = R_{\pm,d} + R_{\pm,o}$ where as before the subscripts "d" and "o" denote the diagonal and off-diagonal parts of the corresponding matrices, we find:

$$\begin{aligned} R_{-,d}(\zeta, z) &= \pi \mathscr{H}_k[\rho_{-,d}(\xi, z)g(\xi)], \qquad \zeta \in \mathbb{C} \\ R_{+,d}(\zeta, z) &= \pi\lambda(\zeta)\mathscr{H}_k[\rho_{+,d}(\xi, z)g(\xi)/\lambda(\xi)] + 2w_+(z)\sigma_3, \qquad \zeta \in \mathbb{C} \end{aligned} \tag{6.20a}$$

$$\begin{aligned} R_{-,o}(\zeta, z) &= i\pi g(k(\zeta))\rho_{-,o}(\zeta, z)\sigma_3, \\ R_{+,o}(\zeta, z) &= \begin{cases} -i\sigma\pi g(k(\zeta))\rho_{+,o}(\zeta, z)\sigma_3 & \zeta \in \mathbb{R} \\ 0 & \zeta \in \mathscr{C} \setminus \{\pm A\} \end{cases} \end{aligned} \tag{6.20b}$$

where $\sigma = 1$ for $\zeta \in (-\infty, -A] \cup [A, +\infty)$ and $\sigma = -1$ for $\zeta \in (-A, A)$. In component form, the above equations are given by (6.14). □

Note that, in (6.20), both $\rho_{+,d}$ and λ inside the Hilbert transform take opposite signs on sheets I and II, so the Hilbert transform yields the same result on the two sheets, as it should be. We conclude that the matrix $R_{+,d}$ is determined independently of the choice of the integration variable. Importantly notice that, since $R_\pm(\zeta, z)$ are defined as principal value integrals, even when they admit extension to the complex k-plane, their values are going to be discontinuous across the real k-axis.

In general, it is not possible to extend all the entries of $R_\pm$ off the continuous spectrum. Similarly to what happens in the case of symmetric nonzero background in [14], however, we have:

Lemma 6.5. *The following off-diagonal entries of $R_\pm$ admit analytic extension off the real ζ axis:*

$$R_{+,12}(\zeta,z)=0,\quad \zeta\in D^+,\qquad R_{+,21}(\zeta,z)=0,\quad \zeta\in D^-, \tag{6.21a}$$
$$R_{-,21}(\zeta,z)=0,\quad \zeta\in \mathbb{C}^+,\qquad R_{-,12}(\zeta,z)=0,\quad \zeta\in \mathbb{C}^-. \tag{6.21b}$$

Proof. The off-diagonal parts of $R_\pm$ can be written as

$$R_{-,o}(\zeta,z)=\pi\lim_{t\to-\infty}\mathscr{H}_k\left[g(\xi)\mathrm{e}^{-i(k-\xi)t\sigma_3}\rho_{-,o}(t,z,\xi)\mathrm{e}^{i(k-\xi)t\sigma_3}\right],$$
$$R_{+,o}(\zeta,z)=\pi\lim_{t\to+\infty}\mathscr{H}_k\left[g(\xi)\mathrm{e}^{-i(\lambda(k)-\lambda(\xi))t\sigma_3}\rho_{+,o}(t,z,\xi)\mathrm{e}^{i(\lambda(k)-\lambda(\xi))t\sigma_3}\right].$$

For each matrix element, the Hilbert transform is analytic and bounded in the complex ζ-plane wherever the exponential inside tends to zero as $t\to\pm\infty$. Hence, looking at the regions where $\operatorname{Im}k\lessgtr 0$ and $\operatorname{Im}\lambda\lessgtr 0$, (6.21) follows. □

6.3. Propagation of the reflection coefficients.

Lemma 6.6. *For all $\zeta\in\mathbb{R}$ and $z\geq 0$, the scattering matrix $S(\zeta,z)$ obeys the following propagation equation:*

$$\frac{\partial S}{\partial z}=\frac{i}{2}(R_-S-SR_+). \tag{6.22}$$

Proof. In light of (6.11), (3.5a) implies

$$S(\zeta,z)=C_-(\zeta,z)C_+^{-1}(\zeta,z),\qquad \forall\zeta\in\mathbb{R}.$$

Using (6.12), one then obtains (6.22). □

Recall that we have introduced two sets of reflection coefficients: the reflection coefficients from the left (i.e., $r_-=b/a$ and $\bar r_-=\bar b/\bar a$) and the reflection coefficients from the right (i.e., $r_+=-\bar b/a$ and $\bar r_+=-b/\bar a$). In order to obtain the propagation equations for both sets of reflection coefficients, we introduce

$$B(\zeta,z)=S_o\,(S_d)^{-1}=\begin{pmatrix}0&\bar r_-\\ r_-&0\end{pmatrix},\qquad \tilde B(\zeta,z)=(S_d)^{-1}S_o=-\begin{pmatrix}0&r_+\\ \bar r_+&0\end{pmatrix},$$

and observe that

$$\frac{\partial B}{\partial z}=(S_o)_z(S_o)^{-1}B-B(S_d)_z(S_d)^{-1},\qquad \frac{\partial\tilde B}{\partial z}=\tilde B(S_o)^{-1}(S_o)_z-(S_d)^{-1}(S_d)_z\tilde B. \tag{6.23}$$

Separating (6.22) into its diagonal and off-diagonal parts and substituting into the propagation equations for B and $\tilde B$ yields

$$-2i\frac{\partial B}{\partial z}=R_{-,o}+[R_{-,d},B]-BR_{-,o}B-S_dR_{+,o}(S_d)^{-1}+BS_oR_{+,o}(S_d)^{-1}, \tag{6.24a}$$

$$-2i\frac{\partial\tilde{B}}{\partial z} = -R_{+,o} + [R_{+,d}, \tilde{B}] + \tilde{B}R_{+,o}\tilde{B} + (S_d)^{-1}R_{-,o}S_d - (S_d)^{-1}R_{-,o}S_o\tilde{B}. \tag{6.24b}$$

First, we express the RHS of (6.24a) in terms of the limiting values as $t \to -\infty$. In order to do so, we look at the last three terms in the RHS. Recall that $R_{\pm,0}$ is given by (6.20b), and ρ_+ is expressed in terms of ρ_- via (5.4). Also, the first symmetry implies that $S^{-1} = S^\dagger/\det S$ with $\det S$ given by (3.6). Decomposing (5.4) into its diagonal and off-diagonal parts yields:

$$\rho_{+,o} = \frac{1}{\det S}\left(S_d^\dagger\,\rho_{-,o}\,S_d + S_o^\dagger\,\rho_{-,o}\,S_o + S_d^\dagger\,\rho_{-,d}\,S_o + S_o^\dagger\,\rho_{-,d}\,S_d\right). \tag{6.25a}$$

Moreover, since $S^\dagger S = SS^\dagger = (\det S)I$, we have

$$S_d^\dagger\,S_d + S_o^\dagger\,S_o = S_d\,S_d^\dagger + S_o\,S_o^\dagger = (\det S)I, \quad S_d^\dagger\,S_o + S_o^\dagger\,S_d = S_d\,S_o^\dagger + S_o\,S_d^\dagger = O_{2\times 2}. \tag{6.25b}$$

Substituting the above expressions into the last three terms in the RHS of (6.24a), after simplifications we obtain:

$$-2i\frac{\partial B}{\partial z} = (1+\sigma)R_{-,o} + [R_{-,d}, B] - (1-\sigma)\,BR_{-,o}B + i\nu\pi g[\rho_{-,d}, B]\sigma_3, \tag{6.26}$$

where $\sigma = 1$ for $\zeta \in (-\infty, -A] \cup [A, +\infty)$, and $\sigma = -1$ for $\zeta \in (-A, A)$. Then the (2, 1)-entry of the above matrix equation yields the propagation equation for $r_-(\zeta, z)$:

$$\frac{\partial r_-}{\partial z} = \begin{cases} -iK^{\text{out}}\,r_- - \pi g P_-^* & \zeta \in (-\infty, -A] \cup [A, +\infty), \\ -r_-\big[\pi g\,P_-\,r_- + iK^{\text{in}}\big] & \zeta \in (-A, A), \end{cases} \tag{6.27}$$

where

$$K^{\text{out/in}}(\zeta, z) = \pi\mathscr{H}_k[D_-(\zeta, z)\,g(\zeta)] \pm i\pi D_-(\zeta, z)\,g(\zeta), \tag{6.28}$$

with K^{out} and K^{in} corresponding, respectively, to the positive and negative signs in the right-hand side of (6.28). Explicitly, we have:

$$\frac{\partial r_-}{\partial z} = \begin{cases} -i\pi\mathscr{H}_k[D_-\,g]\,r_- + \pi g D_- r_- - \pi g P_-^* & \zeta \in (-\infty, -A] \cup [A, +\infty), \\ -i\pi\mathscr{H}_k[D_-\,g]\,r_- - \pi P_- r_-^2 - \pi g D_- r_- & \zeta \in (-A, A). \end{cases} \tag{6.29}$$

One can verify that the propagation equation (6.29) for r_- is consistent with the symmetry (3.16). Both of the equations in (6.29) can be solved explicitly. In particular, solving the linear, non-homogenous equation for $\zeta \in (-\infty, -A] \cup [A, +\infty)$ then finally yields:

Lemma 6.7. *The reflection coefficient $r_-(\zeta, z)$ obeys the following propagation equation:*

$$r_-(\zeta, z) = e^{-i\chi(\zeta,z)}\left[r_-(\zeta, 0) - \pi g(\zeta)\int_0^z P_-^*(\zeta, y)e^{i\chi(\zeta,y)}dy\right],$$
$$\zeta \in \mathbb{R} \setminus (-A, A), \tag{6.30a}$$

where

$$\chi(\zeta, z) = \int_0^z K^{\text{out}}(\zeta, y)\mathrm{d}y, \tag{6.30b}$$

and K^{out} *is as in* (6.28). *The value of* $r_-(\zeta, z)$ *in the segment* $(-A, A)$ *can be obtained using the symmetry relation* (3.16) *The value of the second reflection coefficient* $\bar{r}_-(\zeta, z)$ *can be computed through the first symmetry relation* (3.11b).

The propagation equations for the scattering coefficients from the right can be computed using a similar approach. Namely, we express the RHS of (6.24b) in terms of the limiting values as $t \to \infty$. Like before, it is useful to write $\rho_{-,o}$ in terms of ρ_+ as follows:

$$\rho_{-,o} = \frac{1}{\det S}\left(S_d\,\rho_{+,o}\,S_d^\dagger + S_o\,\rho_{+,o}\,S_o^\dagger + S_d\,\rho_{+,d}\,S_o^\dagger + S_o\,\rho_{+,d}\,S_d^\dagger\right). \tag{6.31}$$

Using the above equation along with (6.25b), one can rewrite the Eq. (6.24b) as

$$\begin{aligned} -2i\frac{\partial \tilde{B}}{\partial z} = & -R_{+,o} + [R_{+,d}, \tilde{B}] + \tilde{B}R_{+,o}\tilde{B} + i\pi g\rho_{+,o}\sigma_3 - i\pi g\tilde{B}\rho_{+,o}\tilde{B}\sigma_3 \\ & + i\pi g[\tilde{B}, \rho_{+,d}]\sigma_3. \end{aligned} \tag{6.32}$$

Considering the $(1, 2)$-component of the above equation, we can find the propagation equation for $r_+(\zeta, z)$. Specifically, we have

$$\frac{\partial r_+}{\partial z} = \begin{cases} -i\Lambda\, r_+ - \pi g P_+ & \zeta \in (-\infty, -A] \cup [A, +\infty), \\ r_+\left[\pi g\, P_+^* \,r_+ - i\Lambda\right] & \zeta \in (-A, A), \\ \frac{1}{2}\left(\pi g P_+^* r_+^2 - 2i\Lambda r_+ - \pi g P_+\right) & \zeta \in \mathscr{C}^+, \end{cases} \tag{6.33}$$

where

$$\Lambda(\zeta, z) = -\pi\lambda(\zeta)\mathscr{H}_k[D_+(\zeta, z)\, g(\zeta)/\lambda(\zeta)] - 2\, w_+(z) - i\pi g(\zeta)D_+(\zeta, z), \tag{6.34}$$

with $w_+(z)$ as in (6.1b). The above equation can be written as:

$$\begin{aligned} \frac{\partial r_+}{\partial z} = & \quad i\pi\lambda\mathscr{H}_k[D_+\, g/\lambda] + 2i\, w_+ r_- \pi g D_+ r_+ \\ & + \begin{cases} -\pi g P_+ & \zeta \in (-\infty, -A] \cup [A, +\infty), \\ +\pi g P_+^* r_+^2 & \zeta \in (-A, A), \\ +\frac{1}{2}\pi g P_+^* r_+^2 - \pi g P_+ & \zeta \in \mathscr{C}^+. \end{cases} \end{aligned} \tag{6.35}$$

As before, the propagation equation for $\bar{r}_+$ can be obtained using the symmetry relation (3.11c).

Next, we verify that the propagation equation for r_+ is consistent with the symmetries. First, we differentiate the symmetry relation (3.17) and obtain

$$r_+(-A^2/\zeta, z)\frac{\partial r_+(\zeta, z)}{\partial z} + r_+(\zeta, z)\frac{\partial r_+(-A^2/\zeta, z)}{\partial z} = 4i\, w_+(z)\frac{q_+(0)}{q_+^*(0)}\mathrm{e}^{4i W_+(z)}. \tag{6.36}$$

Then we show that (6.33) is consistent with the above symmetry. Without loss of generality, suppose $\zeta \in (-A, A)$. Then the LHS of the above equation becomes

$$r_+(-A^2/\zeta)r_+(\zeta)\left[\pi g(\zeta)\, P_+^*(\zeta)\, r_+(\zeta) - i\Lambda(\zeta)\right]$$

$$+r_+(\zeta)\big[-i\Lambda(-A^2/\zeta)r_+(-A^2/\zeta)-\pi g(-A^2/\zeta)P_+(-A^2/\zeta)\big]. \tag{6.37}$$

Using the definition of $\Lambda(\zeta)$ in (6.33) with Eq. (5.7) we have the following relations:

$$\Lambda(-A^2/\zeta,z)=-\Lambda(\zeta,z)-4w_+(z),\quad P_+(-A^2/\zeta,z)=\frac{q_+(0)}{q_+^*(0)}\mathrm{e}^{4iW_+(z)}P_+^*(\zeta,z), \tag{6.38}$$

where $w_+(z)$ and $W_+(z)$ defined in (6.1b) and (6.5), respectively. Combining Eqs. (6.38) and (6.37) one can verify the symmetry relation (6.36).

6.4. Propagation of the norming constants. We now derive the propagation equation for the norming constants.

Lemma 6.8. *For all $n=1,\dots,N$, the norming constant C_n obeys the following propagation equation:*

$$\frac{\partial C_n}{\partial z}=-\frac{i}{2}\big(R_{+,11}(\zeta_n)-R_{-,22}(\zeta_n)+\eta(\zeta_n)\big)\,C_n, \tag{6.39a}$$

with

$$\eta(\zeta_n)=\int_{(-\infty,-A)\cup(A,\infty)}\frac{\zeta_n(s^2+A^2)-2s\,A^2}{s\,(s-\zeta_n)(s\,\zeta_n-A^2)}g(s)\big(D_-(s,z)-D_+(s,z)\big)\,\mathrm{d}s. \tag{6.39b}$$

Proof. Using the definition of the norming constant C_n with $n=1,\dots,N$ we have

$$C_n(z)=b_n(z)\lim_{\zeta\to\zeta_n}\frac{\zeta-\zeta_n}{a(\zeta,z)}. \tag{6.40}$$

Differentiating the above equation with respect to z and assuming the limit and the derivative commute, namely that $a(\zeta,z)$ is uniformly continuous near each $\zeta=\zeta_n$ for all $z\geq 0$, we obtain:

$$\frac{\partial C_n}{\partial z}=\frac{\partial b_n}{\partial z}\frac{1}{a'(\zeta_n,z)}-C_n(z)\lim_{\zeta\to\zeta_n}\left(\frac{1}{a(\zeta,z)}\frac{\partial a(\zeta,z)}{\partial z}\right). \tag{6.41}$$

As we show next, this is another instance in which the formalism deviates significantly from the case of zero background. This is because, in order to derive the correct propagation equation for C_n one needs to evaluate (6.41) from the expression for $a(\zeta,z)$ obtained using the trace formulae, which are derived in Sect. 4.3. Specifically, we begin by differentiating (4.13a) with respect to z, which yields

$$\frac{\partial a}{\partial z}=\frac{a}{2\pi i}\int_\Sigma\frac{\partial K_1}{\partial z}(s,z)\frac{\mathrm{d}s}{s-\zeta},\qquad \zeta\in D^+_{\mathrm{out}}. \tag{6.42}$$

Using Eqs. (4.11c) and (6.27), one can show the following:

$$-\frac{1}{2\pi}\int_\Sigma\frac{\partial K_1}{\partial z}(s,z)\frac{\mathrm{d}s}{s-\zeta}\mathrm{d}s=$$
$$=\int_{(-\infty,-A)\cup(A,\infty)}g(s)\left(\frac{D_-(s,z)|r_-(s,z)|^2-\mathrm{Re}[r_-(s,z)P_-(s,z)]}{1+|r_-(s,z)|^2}\right)\frac{\mathrm{d}s}{s-\zeta}$$

$$- \int_{(-A,A)} g(s) \left(\frac{D_-(s,z) + \mathrm{Re}[r_-(s,z) P_-(s,z)]}{1 + |r_-(s,z)|^2} \right) \frac{\mathrm{d}s}{s+\zeta}. \tag{6.43}$$

One can eliminate $\mathrm{Re}[r_-(s,z)P_-(s,z)]$ using (6.9b) and obtain

$$\frac{\partial a}{\partial z} = \frac{i}{2}\,\eta(\zeta,z)\,a(\zeta,z)\,, \qquad \zeta \in D^+_{\mathrm{out}}, \tag{6.44}$$

where

$$\eta(\zeta,z) = \int_{(-\infty,-A)\cup(A,\infty)} \frac{D_-(s,z) - D_+(s,z)}{s-\zeta} g(s)\,\mathrm{d}s \tag{6.45}$$
$$- \int_{(-A,A)} \frac{D_-(s,z) + D_+(s,z)}{s+\zeta} g(s)\,\mathrm{d}s.$$

Performing a change of variable $s \to -A^2/s$ and using the symmetries of $D_\pm$ (Eqs. 5.7), one can combine the above two integrals and obtain:

$$\eta(\zeta,z) =$$
$$= \int_{(-\infty,-A)\cup(A,\infty)} \left[\frac{D_-(s,z) - D_+(s,z)}{s-\zeta} g(s) - \frac{A^2\, g(-A^2/s)\big(D_-(s,z) - D_+(s,z)\big)}{s(s\,\zeta - A^2)} \right] \mathrm{d}s$$
$$= \int_{(-\infty,-A)\cup(A,\infty)} \left(\frac{1}{s-\zeta} - \frac{A^2}{s(s\,\zeta - A^2)} \right) g(s)\big(D_-(s,z) - D_+(s,z)\big)\,\mathrm{d}s$$
$$= \int_{(-\infty,-A)\cup(A,\infty)} \frac{\zeta(s^2 + A^2) - 2s\,A^2}{s\,(s-\zeta)(s\,\zeta - A^2)} g(s)\big(D_-(s,z) - D_+(s,z)\big)\,\mathrm{d}s\,. \tag{6.46a}$$

Now recall from Eq. (6.13), for $z \in \mathbb{R}$,

$$\frac{\partial \Phi}{\partial z} = -\frac{i}{2}\Phi R_- + T\Phi, \qquad \frac{\partial \Psi}{\partial z} = -\frac{i}{2}\Psi R_+ + T\Psi. \tag{6.47}$$

Now observe that some columns of the above equations can be extended into the UHP. Namely,

$$\frac{\partial \phi}{\partial z} = -\frac{i}{2}\phi R_{-,22} + T\phi, \qquad \frac{\partial \psi}{\partial z} = -\frac{i}{2}\psi R_{+,11} + T\psi, \quad \mathrm{Im}\, z > 0. \tag{6.48}$$

Since at $\zeta = \zeta_n$ one has $\psi(t,z,\zeta_n) = b_n(z)\,\phi(t,z,\zeta_n)$, differentiating with respect to z we obtain

$$\frac{\partial \psi(\zeta_n)}{\partial z} = \frac{\partial b_n(z)}{\partial z}\phi(\zeta_n) + b_n(z)\frac{\partial \phi(\zeta_n)}{\partial z}. \tag{6.49}$$

In turn, using (6.48), this yields

$$\frac{\partial b_n(z)}{\partial z} = -\frac{i}{2}(R_{+,11}(\zeta,z) - R_{-,22}(\zeta,z))\big|_{\zeta=\zeta_n}\, b_n(z). \tag{6.50}$$

Finally, using Eqs. (6.50), (6.41) and (6.44) one can derive the propagation Eq. (6.39a) for the norming constants. Importantly, (6.44) also shows that the zeros of $a(\zeta)$, i.e., the discrete eigenvalues of the scattering problem, are independent of z. □

Remark 6.9. We emphasize that the above derivation of the propagation equations for $a(\zeta, z)$ and for the norming constants does not assume any analytic continuation beyond what has been established in the direct problem. We show below that the propagation equation that one obtains for $a(\zeta, z)$ assuming that Eq. (6.21) can be extended off the real ζ-axis does not appear to coincide with (6.44).

Recall that, for $\zeta \in \mathbb{R}$, the propagation of the scattering matrix is given by (6.22). The (1, 1) and (2, 1) entries of (6.21) yield, respectively,

$$\frac{\partial a}{\partial z} = \tfrac{i}{2}\Big((R_{-,11} - R_{+,11})\, a + b\, R_{-,12} - \bar{b}\, R_{+,21}\Big), \quad \zeta \in \mathbb{R}, \tag{6.51}$$

$$\frac{\partial b}{\partial z} = \tfrac{i}{2}\Big((R_{-,22} - R_{+,11})\, b + a\, R_{-,21} - \bar{a}\, R_{+,21}\Big), \quad \zeta \in \mathbb{R}.$$

If one assumes that (6.21) can be extended into an arbitrarily small strip around the real ζ axis, one also has:

$$\frac{\partial a}{\partial z} = \frac{i}{2}(R_{-,11} - R_{+,11})\, a\,, \quad \zeta \in D^{+}_{\text{out}}. \tag{6.52}$$

In order to compare it with (6.44), note that

$$R_{-,11} - R_{+,11} = \fint_{\mathbb{R}} \frac{D_{-}(\xi, z)}{\xi - k} g(\xi)\mathrm{d}\xi - \fint_{\mathbb{R}} \frac{\lambda(k) + \xi - k}{\lambda(\xi)\,(\xi - k)} D_{+}(\xi, z)\, g(\xi)\mathrm{d}\xi, \tag{6.53}$$

and the latter has to coincide with $\eta(z)$. First we perform a variable change $\xi = \frac{1}{2}(s - A^2/s)$, which implies

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \tfrac{1}{2}\Big(1 + \tfrac{A^2}{s^2}\Big), \qquad \lambda(\xi) = s - \xi = \tfrac{1}{2}\Big(s + \tfrac{A^2}{s}\Big), \tag{6.54}$$

$$\xi - k = \tfrac{1}{2}(s - \zeta)\frac{(s\,\zeta + A^2)}{s\,\zeta}.$$

Substituting the above expressions into (6.53) yields

$$
\begin{aligned}
R_{-,11} - R_{+,11} &= \int_{(-\infty,-A)\cup(A,\infty)} \left(\frac{\zeta(s^2+A^2)}{s\,(s-\zeta)(s\,\zeta+A^2)} D_-(s,z)\,g(s) \right. \\
&\quad \left. - \frac{(s^2+A^2)\zeta + 2A^2(s-\zeta)}{s\,(s-\zeta)(s\,\zeta+A^2)} D_+(s,z)\,g(s) \right) \mathrm{d}s \\
&= \int_{(-\infty,-A)\cup(A,\infty)} \frac{\zeta(s^2+A^2)}{s\,(s-\zeta)(s\,\zeta+A^2)} g(s)\big(D_-(s,z) - D_+(s,z)\big)\mathrm{d}s \\
&\quad - 2A^2 \int_{(-\infty,-A)\cup(A,\infty)} \frac{g(s)D_+(s,z)}{s\,(s\,\zeta+A^2)}\mathrm{d}s\,.
\end{aligned} \tag{6.55}
$$

Comparing the above equation with (6.46) shows that the two expressions coincide when $A = 0$, but not, in general, in the case of nonzero background. Therefore, when dealing with a nontrivial background, one has to take (6.44) as the correct equation for the propagation of $a(\zeta, z)$.

7. Asymptotic States of Propagation

We now show how the IST formalism developed in the previous sections allows one to immediately obtain certain features about the asymptotic state of the medium, as well as information on the asymptotic behavior of the optical pulse in the medium.

Asymptotic value of the scattering coefficients. We begin by looking at the asymptotic value of the reflection coefficient for large z. Recall that the evolution (i.e., propagation inside the medium) of the reflection coefficient $r_-(\zeta, z)$ as a function of z is given by (6.30a), with $\chi(\zeta, z)$ given by (6.30b) and $K^{\text{out}}(\zeta, z)$ in turn by (6.28). Therefore, its behavior as a function of z is determined by the sign of the imaginary part of K^{out}, which is given by (6.28). Since $D_-(\zeta, z)$ and $g(\zeta)$ are real-valued, so is the Hilbert transform in (6.28). Therefore, since $g(\zeta)$ is non-negative, the growth or decay of $r_-(\zeta, z)$ is completely determined by the sign of $D_-(\zeta, z)$.

Let us consider first the case $P_-(\zeta, z) \equiv 0$, since it is the simplest one. Inspection of (6.30a) shows that if the medium is initially in the stable pure state (i.e., it is prepared so that $P_- = 0$ and $D_- = -1$), $r_-(\zeta, z)$ is exponentially decaying as $z \to +\infty$ for $\zeta \in (-\infty, -A] \cup [A, +\infty)$, and exponentially growing for $\zeta \in (-A, A)$. Conversely, if the medium is initially in the unstable pure state (i.e., it is prepared with $P_- = 0$ and $D_- = 1$), then $r_-(\zeta, z)$ is exponentially growing as $z \to +\infty$ for $\zeta \in (-\infty, -A] \cup [A, +\infty)$, and exponentially decaying for $\zeta \in (-A, A)$.

Finally, it is straightforward to see from (6.30a) that similar considerations apply when $P_-(\zeta, z) \not\equiv 0$. More precisely, whenever $D_-(\zeta, z) > 0$, the reflection coefficient $r_-(\zeta, z)$ has a similar kind of exponential growth in z as when $D_-(\zeta, z) = 1$, and whenever $D_-(\zeta, z) < 0$, $r_-(\zeta, z)$ exhibits exponentially decay to a non-zero value if $P_- \neq 0$.

Asymptotic state of the medium. Next we look at the asymptotic state of the medium as $t \to \infty$, as given by D_+ and P_+, which are determined by the reflection coefficient $r_-(\zeta, z)$ via (6.9b).

As discussed at the end of Sect. 5, while the values of D_- and P_- are independent of the sheet chosen, i.e., whether $\zeta \in (-\infty, -A] \cup [A, \infty)$ or $\zeta \in [-A, A]$, the values of D_+ and P_+ are different for $\zeta \in (-\infty, -A] \cup [A, \infty)$ or $\zeta \in [-A, A]$. At the same time, whenever $P_+ \equiv 0$ or $P_+ \to 0$ as $z \to \infty$ [which happens, for example, when the system is initially in a pure state, i.e., $P_- \equiv 0$], one can see from (5.10) that the sign of D_+ coincides, at least for sufficiently large z, with the sign of D on $\zeta \in (-\infty, -A] \cup [A, \infty)$. Therefore, for large z one should consider the value of D_+ for $\zeta \in (-\infty, -A] \cup [A, \infty)$ (i.e., the continuous spectrum on the first sheet) as the physical value. Hence, to discuss the asymptotic state of a medium initially in a pure state, it is sufficient to limit ourselves to considering $\zeta \in (-\infty, -A] \cup [A, \infty)$.

Consider first the case in which the medium is initially in the stable pure state (i.e., $P_- = 0$ and $D_- = -1$). In this case, since $r_-(\zeta, z)$ decays exponentially as $z \to \infty$ for all $\zeta \in (-\infty, -A] \cup [A, \infty)$, (6.9b) imply that $D_+ \to -1$ and $P_+ \to 0$ for large z. Therefore, the medium returns to the stable state for sufficiently large propagation distances, justifying the use of the term "stable state".

Conversely, if the medium is initially prepared in the unstable pure state (i.e., $P_- = 0$ and $D_- = 1$), $r_-(\zeta, z)$ is exponentially growing as $z \to \infty$ for all $\zeta \in (-\infty, -A] \cup [A, \infty)$, and (6.10) still give $D_+ \to -1$ and $P_+ \to 0$ for large z. Therefore, the medium reverts to the stable state for sufficiently large propagation distances. This behavior, which is similar to what happens in the MBE system with ZBG [51], may be regarded as a decay process induced by the incident optical pulse.

Finally, the behavior of the reflection coefficient discussed above also allows us to draw some conclusions when $P_-(\zeta, z) \not\equiv 0$. Namely, if $D_-(\zeta, z) < 0$ one has $D_+(\zeta, z) \to D_-(\zeta, z)$ for $\zeta \in (-\infty, -A] \cup [A, \infty)$. Conversely, if $D_-(\zeta, z) > 0$ one has $D_+(\zeta, z) \to -D_-(\zeta, z)$ for $\zeta \in (-\infty, -A] \cup [A, \infty)$. In both cases, one also has $|P_+(\zeta, z)| \to |P_-(\zeta, z)|$. There is an important difference with the previous discussion, however: if the medium is not initially in a pure state [i.e., $P_-(\zeta, z) \not\equiv 0$], (5.10) imply that the behavior of $D(t, z, \zeta)$ as $t \to \infty$ is determined not only by the value of $D_+(\zeta, z)$, but also by $P_+(\zeta, z)$.

Asymptotic values of the optical pulse. We now use the results of the preceding paragraphs to discuss the behavior of the optical pulse inside the medium.

In [51] it was shown that, in the sharp limit, a boundary layer around $z = 0$ arises upon propagation. Specifically, [51] showed that, for causal solutions, a transition arises over an infinitesimally small propagation distance (see also the earlier results of [31–33,54,74]). The analysis of the asymptotic behavior of the optical field and medium density matrix in [51] also revealed a slow decay of the optical field as $t \to \infty$. Both results, however, have been established in the sharp-line limit, and they do not necessarily hold when inhomogeneous broadening effects are taken into account.

On the other hand, as we discuss below, inspection of the RHP derived in Sect. 4 shows that two different asymptotic behaviors arise depending on whether one is considering z near zero or, conversely, the asymptotics at large times with z finite. To appreciate this dichotomy, recall that the jump matrix (4.1c) that defines the jump condition (4.1b) in the RHP is expressed in terms of the reflection coefficient $r_-(\zeta, z)$ via (4.1d). Therefore, the asymptotic behavior of $r_-(\zeta, z)$ discussed in the above paragraphs determines the asymptotic behavior of the solutions of the RHP and in turn of those of the MBE. Specifically, when $D_-(\zeta, z) < 0$, the behavior of the reflection coefficient guarantees that the contribution of the radiation to the solution is exponentially decaying as $z \to \infty$. Therefore, one can expect that, for any finite value of t, $q(t, z) \to 0$ as $z \to \infty$.

A markedly different scenario arises when $z = 0$. In this case the contribution of the reflection coefficient cannot be ignored, and the analysis of the RHP must allow one to recover the IC $q(t, 0)$ of the problem, and in particular the boundary conditions $q(t, 0) \to 0$ as $t \to -\infty$ and $q(t, 0) \to A$ as $t \to \infty$.

Even though a detailed calculation of the long-distance asymptotics of the solutions of the RHP (4.1b) is outside the scope of this work, it should be obvious that a transition region must arise to connect the different limits for $q(t, z)$.

Finally, it is worth mentioning that while the IST has been formulated in terms of the uniformization variable ζ, the physical variable that measures the deviation of the transition frequency of the atoms from its mean value is $k \in \mathbb{R}$. All results involving real values of ζ, including the asymptotic states of propagation discussed in this section, can be rewritten in terms of the physical variable k by replacing $\zeta = k + \sqrt{k^2 + A^2}$, the sign of the square root corresponding to choosing the first branch of λ, i.e., $\zeta \in (\infty, -A) \cup (A, +\infty)$.

8. Concluding Remarks

In summary, we presented the formulation of the IST for two-level systems with inhomogeneous broadening and one-sided nonzero background. The formalism combines some features of the IST with zero background to others of the IST with symmetric nonzero background. We have shown that the reflection coefficient is always nonzero, and therefore no reflectionless solutions exist. This is similar to what happens in the focusing and defocusing NLS equation with asymmetric NZBG [13,24,62], as well as in the Manakov system with non-parallel NZBG [1]. As far as the inverse problem is concerned, the specific choice of 2×2 matrix for the RHP allows one to bypass the nonlocality of the jump condition, as well as to eliminate the jump across the circle $\mathscr{C}$, both of which are novel features compared to [62].

We also briefly discussed the asymptotic behavior of the reflection coefficient for large z, and the asymptotic states of the medium and the limiting values of the optical pulse. In particular, we showed that if $D_-(\zeta, z) < 0$ for all z and for all $\zeta \in \mathbb{R}$, the reflection coefficient decays exponentially as $z \to \infty$. Therefore, for sufficiently large z, the solution becomes effectively reflectionless. We also showed that, for the kinds of boundary conditions considered in this work, if the initial preparation of the medium is a pure state, the medium asymptotically tends to the stable pure state. Finally, we showed that two different asymptotic regimes arise for the optical pulse depending on whether one is considering the limit $z \to \infty$ with t finite or $t \to \infty$ with z finite.

Note that, in the limit $A \to 0$, the formalism of the present work reduces to the one in the case of zero background in a straightforward way. Specifically, when $A = 0$ one simply has $\lambda(k) = k$ and $\zeta = 2k$; the branch cut $[-iA, iA]$ shrinks to a single point (the origin $k = 0$) and the complex ζ plane reduces to the complex k plane up to a factor 2. The second symmetry (which relates values inside and outside the circle $\mathscr{C}$) becomes immaterial, and all integrals in ζ can be trivially converted to integrals in k (in the principal value sense) and viceversa.

We reiterate the importance of studying the system in the presence of inhomogeneous broadening, since this allows us to consider media in arbitrary initial preparations (i.e., not just pure states), unlike what happens in the sharp line limit, where the only initial states of the medium that are compatible with the system when $q(t, z) \to 0$ as $t \to -\infty$ are the pure ones [51].

The results of this work open up a number of interesting problems for future study. One such problem is the question of existence and uniqueness of solutions of the RHP. This question is nontrivial even for the NLS equation. Indeed, it is known that the RHP for the focusing NLS equation with NZBG does not admit a unique solution even in the symmetric case, and even when the uniformization variable is used. In the case of the NLS equation, it was shown in [11] that one can obtain uniqueness results by formulating the IST without a uniformization variable and by augmenting the RHP with suitable growth conditions at the branch points. One can conjecture that the same conditions will also guarantee the uniqueness of solutions for the RHP for the MBE in the formulation of the IST with the uniformization variable as presented here. A rigorous analysis of this question remains as a problem for future work.

A related issue is that of the well-posedness of the Cauchy problem for the MBE. If causality requirement is not imposed, the Cauchy problem for the MBEs without inhomogeneous broadening in the initially unstable case was shown to admit multiple non-causal solutions for the same data (and these solutions decay to both stable and unstable pure states as $t \to +\infty$, see Corollary 4 in [51]). Conversely, given a causal incident pulse, there exists at most one causal solution to the MBE problem without inhomogeneous broadening (see Theorem 1 in [51]). Causality is also imposed in [74] to guarantee uniqueness of solutions of the Gelfand-Levitan-Marchenko equations of the inverse problem (equivalently, this is related to non-uniqueness of solution of the Riemann-Hilbert problem for the eigenfunctions), but the MBEs considered in both [51,74] are restricted to the sharp-line case. On the other hand, at present there is no statement about non-well-posedness of the Cauchy problem for the MBEs with inhomogeneous broadening (or non-uniqueness of solutions of the GLM equations [33]). It should be noted that the proof of uniqueness of a causal solution provided in [51] does not rely on integrability, and will likely carry through if inhomogeneous broadening is considered. But this does not necessarily mean that if causality is not imposed, the Cauchy problem for the MBEs with inhomoegeneous broadening is ill-posed. In [74], Zakharov seems to suggest that the non-uniqueness induced by the spontaneous solutions is due to an arbitrary function analytic in a small neighborhood of the origin, and that the causality requirement forces this arbitrary function to coincide with the analytic extension of the reflection coefficient. Again, this seems to be related to the essential singularity at the origin introduced by the sharp-line limit, and might not happen with inhomogeneous broadening. Moreover, all the above results were established in the case of zero background, and the presence of NZBG introduces yet another layer of complication. Based on the above considerations, a study of the well-posedness of MBEs with inhomogeneous broadening in the case of rapidly decaying optical pulses, as well as for pulses on a NZBG is an interesting open problem.

Finally, another obvious and interesting question concerns a detailed and rigorous study of the “long time” — or more appropriately, in this case, long distance — behavior of solutions. This question, and the others above, are left for future work, and we hope that the results of work will motivate further study on these topics.

Acknowledgement We thank Sitai Li, Peter Miller and Ildar Gabitov for interesting discussions on these topics. We also thank the Isaac Newton Institute for Mathematical Sciences for its support and hospitality during the program Dispersive Hydrodynamics when parts of the present work were undertaken. GB was partially supported by the National Science Foundation under grant DMS-2009487. BP was partially supported by the National Science Foundation under grant DMS-2106488.

Data availability Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

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Communicated by K. Nakanishi