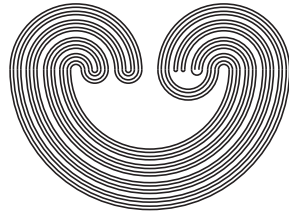


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## ON THE COUNTABLE BOX PRODUCT OF COMPACT ORDINALS

by

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**ON THE COUNTABLE BOX PRODUCT  
OF COMPACT ORDINALS**

**Soulian Yang<sup>1</sup> and Scott W. Williams<sup>2</sup>**

If  $X$  is a topological space, then  $\sigma^{\kappa}X$  (the box product of  $\kappa$  many copies of  $X$ ) denotes the product  $\prod^{\kappa}X$  with the topology induced by the family of all sets of the form  $\prod_{\alpha \in \kappa} U_{\alpha}$ , where each  $U_{\alpha}$  is an open set in  $X$ . For a recent survey on box products, see [Wi2].

Consider the following theorem due to M. E. Rudin:

*0.1 Theorem. Assume the Continuum Hypothesis holds. Then, for each ordinal  $\lambda$ ,  $\sigma^{\omega} \lambda + 1$  is paracompact.*

The conclusion to this theorem has been expanded to the larger class of compact spaces ([Kul]) and  $\omega_1$  many factors ([Wi3]). Under the set-theoretic statement-- there is  $\kappa$ -scale in  ${}^{\omega}\omega$ --the best result was " $\sigma^{\omega} \omega_1 + 1$  is paracompact" ([Wil]). We offer our main result:

*0.2 Theorem. Suppose that for some cardinal  $\kappa$  there is a  $\kappa$ -scale in  ${}^{\omega}\omega$ . Then, for each ordinal  $\lambda$ ,  $\sigma^{\omega} \lambda + 1$  is paracompact.*

**1. Preliminaries**

Given a set  $X$ ,  ${}^{\omega}X$  is the set of functions from  $\omega$  to the set  $X$ . For  $f$  and  $g$  in  ${}^{\omega}X$ , define  $f =^* g$  if they differ on

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only finitely many coordinates. We denote the resulting quotient set by  $\nabla^\omega X$  and write  $[f] = \{g: g =^* f\}$ .

Suppose  $X$  is an ordinal set. There are two very different but similarly defined orders on  $\nabla^\omega X$ . First of all, define  $f \leq^* g$  ( $f, g \in \square^\omega X$ ) provided that  $f(n) > g(n)$  for only finitely many  $n \in \omega$ ; define  $f <^* g$  provided that  $f(n) \geq g(n)$  for only finitely many  $n \in \omega$ . Then we define

$$[f] \leq^* [g] \text{ if } f \leq^* g;$$

$$[f] <^* [g] \text{ if } f <^* g.$$

It is trivial that  $[f] = [g]$  iff  $f =^* g$  and both orders,  $\leq^*$  and  $<^*$ , are partial orders on  $\nabla^\omega X$ .

*In this paper, for each  $x \in \nabla^\omega X$ , we fix some  $f_x \in x$  and identify  $x = [f_x]$  with  $f_x$ .*

Suppose,  $f, g \in \nabla^\omega X$ . We define

$$[f, g] = \{h \in \nabla^\omega X: f \leq^* h \leq^* g\} = \bigvee_{n \in \omega} [f(n), g(n)],$$

and call  $[f, g]$  *basic set* iff both sets  $\{n: g(n) \text{ is limit ordinal, } f(n) = g(n)\}$  and  $\{n: f(n) \text{ is a limit ordinal}\}$  are finite.

Suppose  $\kappa$  is a cardinal. The statement there is a  $\kappa$ -scale in  ${}^\omega \omega$  means there is an order preserving injection from  $\kappa$  into  ${}^\omega \omega$  whose range is confinal in  $({}^\omega \omega, <^*)$ .

Suppose  $Z$  is a topological space. Then  $\nabla^\omega Z$  denoted the quotient space induced by  $=^*$  on  $\square^\omega Z$ . This is known as the nabla product. We make strong use of an important lemma due to K. Kunen (see [Wi2]):

1.1 *Lemma.* *If  $Z$  is locally compact and paracompact then*

- (1)  $\nabla^\omega Z$  is paracompact iff  $\square^\omega Z$  is paracompact;
- (2)  $\nabla^\omega Z$  is a P-space (every  $G_\delta$ -set is open).

So we need only to prove  $\nabla^{\omega\lambda} + 1$  is paracompact in order to prove 0.2.

1.2 *Definition.* A space  $X$  is called specially paracompact provided that each open cover of  $X$  has a refinement consisting of pairwise disjoint basic sets.

The symbol  $\#(\alpha)$  denotes the statement:  $\nabla^{\omega\alpha} + 1$  is specially paracompact. According to 1.1  $\#(\alpha)$  implies  $\nabla^{\omega\alpha} + 1$  is paracompact.

1.3 *Theorem.* If there is a  $\kappa$ -scale in  ${}^{\omega}\omega$ , then  $\#(\omega_1)$  is true.

This theorem has been proved by Williams in [Wil]. But he stated in [Wil] a weaker proposition:  $\nabla^{\omega\omega_1} + 1$  is paracompact if  $\exists$  is a  $\kappa$ -scale in  ${}^{\omega}\omega$ . In fact, his proof really is a stronger one.

1.4 *Lemma.* Suppose  $\lambda$  is an ordinal. Then every clopen set in  $\nabla^{\omega\lambda} + 1$  is a union of pairwise disjoint basic sets.

This result is implicitly proved in [Ru]. But it is not so easy to extract from Rudin's paper. Fortunately, in this paper, we need only some particular cases of the lemma: first case, the clopen set is a difference set between two basic sets; second case, the clopen set is an intersection of countably many basic sets. Both are not so hard to prove. We leave it to the readers.

## 2. Tops Refinement

2.1 *Definition.* A set  $M \subset \omega \times (\lambda + 1)$  is called a matrix provided that there is  $b \in \lambda + 1$  for each  $n < \omega$  such that  $\langle n, b \rangle \in M$ .

2.2 *Definition.* Suppose  $f \in \mathcal{V}^\omega \lambda + 1$ . If there is a member  $\langle n, b \rangle$  of  $M$  for all but finitely many  $n$ , such that  $f(n) = b$ , then we say that  $f$  is on the matrix  $M$ . The set  $\{f \in \mathcal{V}^\omega \lambda + 1 : f \text{ is on the matrix } M\}$  is denoted by  $D(M)$ .

2.3 *Lemma.* Assume  $\#(\omega_1)$  is true. If a matrix  $M$  is countable and  $D(M)$  is closed, then  $D(M)$  is specially paracompact.

*Proof.* Since  $M$  is countable and  $D(M)$  is closed, it is easy to find an embedding map  $E$  from a basic set  $[\bar{0}, g] \subset \mathcal{V}^\omega \omega_1 + 1$  into  $\mathcal{V}^\omega \lambda + 1$ , such that  $E([\bar{0}, g]) = D(M)$ , where  $\bar{0} = \langle 0, 0, \dots \rangle$  and  $g \prec^* \bar{\omega}_1 = \langle \omega_1, \omega_1, \dots \rangle$ . In fact,  $M_n = \{b \in \lambda + 1 : \langle n, b \rangle \in M \cap (\{n\} \times (\lambda + 1))\}$  is countable and is a closed set for all but finitely many  $n$ . Let  $\eta_n$  be the order type of  $M_n$ . If  $M_n$  is closed, then  $\eta_n$  is a successor ordinal,  $\eta_n = \mu_n + 1$ . Let  $g = \langle \mu_0, \mu_1, \dots, \mu_n, \dots \rangle$ . Obviously, we can define an embedding map  $E: [\bar{0}, g] \rightarrow \mathcal{V}^\omega \lambda + 1$  satisfying  $E([\bar{0}, g]) = D(M)$  in natural way. Moreover,  $g \prec^* \bar{\omega}_1$  since  $\mu_n < \omega_1$ .  $[\bar{0}, g]$  is specially paracompact since  $\#(\omega_1)$  and  $[0, g]$  is closed in  $\mathcal{V}^\omega \omega_1 + 1$ . It implies  $D(M)$  is specially paracompact.

Suppose  $B \subset \mathcal{V}^\omega \lambda + 1$ . Remember that we have fixed  $f_x \in x$  for each  $x \in \mathcal{V}^\omega \lambda + 1$ . Let

$$B_n = \{f_x(n) : x \in B\}.$$

If  $B$  is countable, then the matrix

$$M(B) = \cup_{n < \omega} \{(n, p) : p \in \overline{B}_n\}$$

is also countable, where  $\overline{B}_n$  denotes the closure of  $B_n$  in  $\mathcal{V}^{\omega_\lambda + 1}$ . Moreover,  $D(M(B))$  is a closed set since  $D(M(B)) = \bigcap_{n \in \omega} \overline{B}_n$ .

*2.4 Definition.* Suppose that  $[a, b]$  is a basic set in  $\mathcal{V}^{\omega_\lambda + 1}$ ,  $B \subset \mathcal{V}^{\omega_\lambda + 1}$  is countable and  $\mathcal{R}$  is an open cover of  $[a, b]$ . The notion of tops refinement of  $\mathcal{R}$  relative to  $B$  and  $[a, b]$  is defined by the following cases:

*Case 1.*  $D(M(B)) \cap [a, b] = \emptyset$ . There is an open set  $U \in \mathcal{R}$  such that  $b \in U$ . We choose a basic set  $V$  satisfying  $b \in V$  and  $V \subset U \cap [a, b]$ . In this case, the tops refinement is a singleton basic set  $\{V\}$ .

*Case 2.*  $D(M(B)) \cap [a, b] \neq \emptyset$ . Assume  $\#(\omega_1)$ . By 2.3,  $D(M(B))$  is specially paracompact since  $D(M(B))$  is closed. Then  $D(M(B)) \cap [a, b]$  also is specially paracompact. Hence there is a refinement  $\mathcal{U}$  of  $\mathcal{R}$  consisting of pairwise disjoint basic sets which cover  $D(M(B)) \cap [a, b]$ . We call  $\mathcal{U}$  tops refinement of  $\mathcal{R}$  relative to  $B$  and  $[a, b]$ .

Since a tops refinement might not cover  $[a, b]$ , we notice that a tops refinement need not be a refinement in the usual sense. However, we wish to make use of those sets not belonging to the tops refinement.

*2.5 Lemma.* Suppose  $B \subset \mathcal{V}^{\omega_\lambda + 1}$  is countable,  $[a, b]$  is a basic set in  $\mathcal{V}^{\omega_\lambda + 1}$  and  $\mathcal{R}$  is an open cover of  $[a, b]$ . Then there is a tops refinement  $\mathcal{U}$  of  $\mathcal{R}$  relative to  $B$  and  $[a, b]$ , and there is a partition  $\mathcal{P}$  of  $[a, b]$  into basic sets such that  $\mathcal{U} \subset \mathcal{P}$ .

*Proof.* By 2.3  $D(M(B))$  is specially paracompact. Then  $K = D(M(B)) \cap [a, b]$  is specially paracompact. For  $K$ , as a subspace of  $V^\omega \lambda + 1$ , there is a refinement  $\mathcal{V}$  of  $\{U \cap K : U \in \mathcal{R}\}$  consisting of pairwise disjoint basic sets in  $K$ . The tops refinement  $\mathcal{U}$  of  $\mathcal{R}$  relative to  $B$  and  $[a, b]$  can be induced by  $\mathcal{V}$  in the following way.

First of all, we define a map from  $\mathcal{V}$  into the family of all basic sets in  $V^\omega \lambda + 1$ . If  $V = [r, s] \cap K \in \mathcal{V}$ ,  $r, s \in K$ , we define  $\phi(V) = [\bar{r}, s] \subset [a, b]$  by the following clauses:

(1)  $r(n) < s(n)$ . Define  $\bar{r}(n)$  from  $r(n)$ . If  $r(n)$  is a successor ordinal; define

$$\bar{r}(n) = \sup\{p \in \lambda + 1 : p < r(n) \ \& \ p \in K_n\} + 1$$

if  $r(n)$  is a limit ordinal and  $S = \{p \in \lambda + 1 : p < r(n) \ \& \ p \in K_n\} \neq \emptyset$ ; define  $\bar{r}(n) = a(n)$  if  $r(n)$  is a limit ordinal and  $S = \emptyset$ .

(2)  $r(n) = s(n)$ . In this case  $s(n)$  is an isolated point of  $K_n$ . Define  $\bar{r}(n) = s(n)$  if  $s(n)$  is still an isolated point in  $\lambda + 1$ . If  $s(n)$  is a limit ordinal, then we define  $\bar{r}(n)$  in the same way as we did in (1).

We claim that:

- (a)  $V = \phi(V) \cap K$  for every  $V \in \mathcal{V}$ ;
- (b)  $\phi(\mathcal{V}) = \{\phi(V) : V \in \mathcal{V}\}$  is pairwise disjoint;
- (c)  $\cup \phi(\mathcal{V})$  is a basic set.

The clauses (a) and (b) are trivial. We prove (c).

In fact, let  $h_n = \sup K_n$ , then

$$[a, h] = \cup \phi(\mathcal{V}),$$

where  $h = X_{n < \omega} h_n$ . Let us prove this equality. Suppose  $x \in [a, h]$ . Then  $a(n) \leq x(n) \leq h(n)$  for almost all  $n$ . Let

$$m(n) = \min P, \text{ if } P = \{\alpha \in K_n : \alpha > x(n)\} \neq \emptyset;$$

$$m(n) = h(n), \text{ if } P = \emptyset; m = \langle m(0), m(1), \dots, m(n), \dots \rangle.$$

Since  $m \in K$ , there is a  $V = [r, s] \cap K$  such that  $m \in V$ . Then

$$\bar{r}(n) \leq r(n) \leq x(n) \leq m(n) \leq s(n)$$

according to (1), (2) and  $r, m, s \in K$ . It means  $x \in \phi(V)$ .

Then  $x \in \cup \phi(V)$ . So far we have proved the inclusion " $\subset$ ".

The other inclusion " $\supset$ " is trivial.

Notice that every  $\phi(V)$  contains only one member  $V$  of  $\mathcal{V}$ , and  $V \subset U$  for some  $U \in \mathcal{R}$ . Without loss of generality, we suppose  $U$  is a basic set. Then we can find naturally a basic set  $\psi(V)$  in  $\mathcal{V}^{\omega} \lambda + 1$  such that

$$V \subset \psi(V) \subset \phi(V) \text{ and } \psi(V) \subset U.$$

Let

$$U = \{\psi(V) : V \in \mathcal{V}\}, H = \{\phi(V) \setminus \psi(V) : V \in \mathcal{V}\} \cup \{[a, b] \setminus [a, h]\}$$

Then

$$[a, b] = [a, h] \cup ([a, b] \setminus [a, h]) = (U \cup H) \cup ([a, h]).$$

By 1.4, every member of  $H$  is a union of pairwise disjoint basic sets. Hence there is a collection  $\mathcal{G}$  of pairwise disjoint basic sets such that  $U \cup \mathcal{G} = U \cup H$ . Then  $\mathcal{P} = U \cup \mathcal{G}$  is a partition of  $[a, b]$  and  $U \subset \mathcal{P}$ .

*2.6 Definition.* Every element of  $\mathcal{P} \setminus U$  is called an uncovered tape.

### 3. The Proof of the Main Theorem

We assume that there is a  $\kappa$ -scale in  ${}^\omega \omega$ . We are going to prove that  $\#(\lambda)$  is true for every ordinal  $\lambda$ .



For simplicity, let  $X = \nabla^{\omega_\lambda} + 1$ . Suppose  $\mathcal{R}$  is an open cover of  $X$ . We intend to build a tree  $T$  consisting of basic sets (exactly, of uncovered tapes). The tree  $T$  is ordered by  $>$  and the height of  $T$  is  $\omega_1$ , where the order  $>$  is defined by

$$[a,b] > [r,s] \text{ iff } a \leq^* r \leq^* s \not\leq^* b$$

( $s \not\leq^* b$  means  $s \leq^* b$  and  $s \neq^* b$ ). Simultaneously, we will construct a collection  $G_\alpha$  of basic sets for each ordinal  $\alpha < \omega_1$  so that  $\mathcal{U} = \cup\{G_\alpha : \alpha < \omega_1\}$  is a refinement of  $\mathcal{R}$  covering  $X$ . All of them subject to the following restrictions:

(3.1) The level 0, which is denoted by  $T_0$ , of  $T$  is  $\{X\}$  and  $G_0 = \emptyset$ .

(3.2)  $T_\alpha$  denotes the  $\alpha$ 'th level of  $T$ . Then  $(\cup T_\alpha) \cup (G_\alpha) = X$  and the elements of  $T_\alpha \cup G_\alpha$  are pairwise disjoint for every  $\alpha < \omega_1$ .

(3.3) For each  $\alpha < \omega_1$  and  $V \in G_\alpha$  there is an  $U \in \mathcal{R}$  such that  $V \subset U$ .

(3.4) The elements of  $\cup_{\alpha < \eta} G_\alpha$  are pairwise disjoint for all  $\eta < \omega_1$ .

(3.5)  $\alpha < \beta < \omega_1$  implies  $G_\alpha \subset G_\beta$  (Then we have  $\cup(G_\beta \setminus G_\alpha) \subset \cup T_\alpha$ ,  $\alpha < \beta$ , which follows from (3.2) and the inclusion  $G_\alpha \subset G_\beta$ ).

(3.6) Suppose  $A$  is a branch of  $\eta$ 'th subtree  $\cup\{T_\alpha : \alpha < \eta\}$ ,  $\eta < \omega_1$ , then the length of  $A$  is  $\eta$ , and the intersection  $\cap A$  is non-empty.

Suppose  $\mu < \omega_1$ . We assume inductively that, for each  $\xi < \mu$ ,  $T_\alpha$ ,  $G_\alpha$ ,  $\alpha < \xi$  have been built and satisfy (3.1)-(3.6)

by taking  $\xi$  instead of  $\omega_1$  in the statements. It is easy to check that  $T_\alpha, G_\alpha, \alpha < \mu$ , also satisfy (3.1)-(3.6) by taking  $\mu$  instead of  $\omega_1$  in the statements.

Now we built  $T_\mu$  and  $G_\mu$  by the following way.

First of all, we claim that

$$\bigcap_{\alpha < \mu} (UT_\alpha) = U\{\bigcap A : A \in \beta\}, \tag{1}$$

where  $\beta$  is the collection of all branches of the  $\mu$ 'th subtree  $U\{T_\alpha : \alpha < \mu\}$ . It follows trivially from (3.6) and (3.2).

Suppose  $A = \{V_\alpha = [a_\alpha, b_\alpha] \in T_\alpha : \alpha < \mu\}$  is a branch. We conclude that  $A = \bigcap A \neq \emptyset$ , because there is no  $(\omega, \omega^*)$  gap in  ${}^\omega$  and  $\mu < \omega_1$ . Moreover, the set  $A$  is clopen since  $\mu < \omega_1$  and  $X$  is a P-space (by 1.1). Then  $A = \bigcup S_A$  where  $S_A$  is a collection of pairwise disjoint basic sets (by 1.4).

Let  $B = \{b_\alpha : [a_\alpha, b_\alpha] \in A\}$ . By 2.5 there is a tops refinement  $\mathcal{U}_C$  of  $\mathcal{R}$  relative to  $B$  and  $C, C \in S_A$ , and there is a partition  $\mathcal{P}_C$  of  $C$  such that  $\mathcal{U}_C \subset \mathcal{P}_C$ . Let

$$W = \{A = \bigcap A : A \in \beta\}.$$

We define

$$\begin{aligned} G_\mu &= (U\{\mathcal{U}_C : C \in S_A \ \& \ A \in W\}) \cup (U\{G_\alpha : \alpha < \mu\}), \\ T_\mu &= U\{\mathcal{P}_C \setminus \mathcal{U}_C : C \in S_A \ \& \ A \in W\}. \end{aligned} \tag{2}$$

If  $[r, s] \in T_\mu$ , then  $[r, s] \subset V_\alpha$  for all  $V_\alpha \in \hat{A}$ . In fact,  $[r, s] \in \mathcal{P}_C$  for some  $C \in S_A$  and some  $A \in W$ , then  $[r, s] \subset A \subset V_\alpha$  for every  $V_\alpha \in \hat{A}$ . On the other hand,  $[r, s] \notin \mathcal{U}_C$  implies  $s \notin \bigcup \mathcal{U}_C$ . But the top of  $V_\alpha = [a_\alpha, b_\alpha], b_\alpha$ , is an element of  $\bigcup \mathcal{U}_C$ . Hence  $s \neq^* b$ .

The rest of the job is to check if  $T_\alpha, G_\alpha, \alpha < \mu + 1$  still satisfy (3.1)-(3.6) by taking  $\mu + 1$  instead  $\omega_1$ . We check the clause (3.2) and leave the rest to the readers.

It is easy to check that

$$(\bigcap_{\alpha < \mu} (UT_\alpha)) \cup (\bigcup_{\alpha < \mu} (UG_\alpha)) = X. \quad (3)$$

Now we prove

$$(UT_\mu) \cup (UG_\mu) = X,$$

i.e. (3.2) holds. In fact, if  $x \in X$ , then either  $x \in \bigcap_{\alpha < \mu} (UT_\alpha)$  or  $x \notin \bigcap_{\alpha < \mu} (UT_\alpha)$ . If  $x \in \bigcap_{\alpha < \mu} (UT_\alpha)$ , then  $x \in \bigcup (\bigcap A : A \in \beta)$  by the equality (1). Thus

$$x \in \bigcap A = A,$$

$$x \in C,$$

where  $A \in \beta$ ,  $C \in S_A$  and  $A = US_A$ . Because  $C = U\mathcal{P}_C$  and  $\mathcal{U}_C \subset \mathcal{P}_C$ , we have

$$x \in (\bigcup (\mathcal{P}_C \setminus \mathcal{U}_C)) \cup (\bigcup \mathcal{U}_C).$$

So

$$x \in (UT_\mu) \cup (UG_\mu). \quad (4)$$

If  $x \notin \bigcap_{\alpha < \mu} (UT_\alpha)$ , then the fact (4) follows from (3) and (2).

The clauses (3.2), (3.3) imply that  $\mathcal{U} = \{G_\alpha : \alpha < \omega_1\}$  is a refinement of  $\mathcal{R}$  and the elements of  $\mathcal{U}$  are pairwise disjoint. Is  $\mathcal{U}$  a cover of  $X$ ? We have to prove the following theorem in order to answer the question.

**3.7 Theorem.** Suppose  $A = \{V_\alpha = [a_\alpha, b_\alpha] \in T_\alpha : \alpha < \omega_1\}$  is a branch of the tree  $T$ . Then  $\bigcap A = \emptyset$ .

*Proof.* If the conclusion is false, then there is a point  $x \in \bigcap A$ . Let  $B_\mu$  denote the set  $\{b_\alpha : \alpha < \mu\}$ .

$M(B_\mu) \upharpoonright x$  denote such a submatrix of  $M(B_\mu)$  that

$$M(B_\mu) \upharpoonright x = \{\langle n, j \rangle \in M(B_\mu) : n < \omega, j > x(n)\}.$$

We say that a point  $b \in X$  extends a matrix  $M$  if there is an infinite set  $E \subset \omega$  such that  $b(n) < j$  for all  $n \in E$  and  $\langle n, j \rangle \in M$ .

We conclude that, for each  $\mu < \omega_1$ ,  $b_\mu$  extends  $M(B_\mu) \upharpoonright x$ . In fact, since  $[a_\mu, b_\mu] \in T_\mu$ , there is a basic set  $C \subset \cap \{V_\alpha \in T_\alpha : \alpha < \mu\}$  such that

$$[a_\mu, b_\mu] \in \mathcal{P}_C \setminus \mathcal{U}_C.$$

Because  $\mathcal{U}_C$  covers  $D(M(B_\mu)) \cap C$ , and  $\mathcal{P}_C$  is a partition of  $C$ , we have

$$[a_\mu, b_\mu] \cap D(M(B_\mu)) = \emptyset.$$

Hence

$$[x, b] \cap D(M(B_\mu)) = \emptyset. \tag{5}$$

If the assertion fails then for every infinite  $E \subset \omega$  there is some  $n \in E$  and  $\xi \in \overline{(B_\mu)_n}$  with  $x(n) < \xi \leq b(n)$ . So we can find an  $m \in \omega$ , for each  $n > m$ , there is  $\xi_n \in \overline{(B_\mu)_n}$  with  $x(n) < \xi_n \leq b(n)$ . Let  $f(n) = \xi_n$  for every  $n > m$ . Then

$$f \in [x, b] \cap D(M(B_\mu)).$$

It is contradictory to (5).

There are only  $\omega_1$  many  $b_\mu$ 's. So the extending will go  $\omega_1$  many times. It is impossible. Why? Suppose  $\mu_{nj} < \omega_1$  is an ordinal. We define inductively

$$\mu_{nj+1} = \begin{cases} \min\{\eta : x(n) < b_\eta(n) < b_{\mu_{nj}}(n)\}, \\ \text{if } \exists \eta > \mu_{nj} (x(n) < b_\eta(n) < b_{\mu_{nj}}(n)) \\ \mu_{nj}, \text{ if } \nexists \eta > \mu_{nj} (x(n) < b_\eta(n) < b_{\mu_{nj}}(n)) \end{cases}$$

and  $b_{\mu_{n0}} = b_0$ . Then

$$x(n) < \dots \leq b_{\mu_{nj}}(n) \leq \dots \leq b_{\mu_{n1}}(n) \leq b_{\mu_{n0}}(n), \tag{*}$$

for every  $n < \omega$ . There just are finitely many " $<$ " appearing in the line (\*) since every  $b_{\mu_{nj}}$  is an ordinal. So there is a minimum among  $b_{\mu_{nj}}$ 's ( $j = 0, 1, \dots$ ). Let

$$b(n) = \min\{b_{\mu_{nj}}(n) : j < \omega\},$$

$$\mu_n = \min\{\mu_{nj} : b_{\mu_{nj}}(n) = b(n)\}.$$

It is clear that for each  $n < \omega$  there is not any ordinal  $\eta > \mu_n$  such that

$$x(n) < b_\eta(n) < b_{\mu_n}(n).$$

Let

$$\gamma = \sup\{\mu_n : n < \omega\}.$$

$\gamma < \omega_1$  since every  $\mu_n < \omega_1$ . If the extending goes  $\omega_1$  many times, then  $b_\gamma$  extends  $M(B_\gamma)$ . It implies that there is an infinite set  $E \subset \omega$  such that

$$x(n) < b_\gamma(n) < b_{\mu_n}(n), n \in E,$$

since every  $b_{\mu_n}$  is on the matrix  $M(B_\gamma)$ . It is a contradiction.

It is similar to the equality (3) that the following equality holds

$$\bigcap_{\alpha < \omega_1} (UT_\alpha) \cup \bigcup_{\alpha < \omega_1} (UG_\alpha) = X.$$

The theorem 3.7 implies

$$\bigcap_{\alpha < \omega_1} (UT_\alpha) = \emptyset.$$

So

$$\bigcup_{\alpha < \omega_1} (UG_\alpha) = X.$$

i.e.  $\mathcal{U}$  covers  $X$ .

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