**Trees, Gleason Spaces, and Coabsolutes of #N #N**

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# TREES, GLEASON SPACES, AND COABSOLUTES OF $\beta N \sim N$

BY

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ABSTRACT. For a regular Hausdorff space $X$, let $\mathcal{E}(X)$ denote its absolute, and call two spaces $X$ and $Y$ coabsolute ($\mathcal{G}$-absolute) when $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ ($\beta\mathcal{E}(X)$ and $\beta\mathcal{E}(Y)$) are homeomorphic. We prove $X$ is $\mathcal{G}$-absolute with a linearly ordered space iff the POSET of proper regular-open sets of $X$ has a cofinal tree; a Moore space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense metrizable subspace; a dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable; if $X$ is a locally compact noncompact metric space, then $\beta X \sim X$ is coabsolute with a compact linearly ordered space having a dense set of $P$-points; CH implies but is not implied by "if $X$ is a locally compact noncompact space of $\pi$-weight at most $2^{\omega}$ and with a compatible complete uniformity, then $\beta X \sim X$ and $\beta N \sim N$ are coabsolute."

A *tree* $T$ is a POSET (partially ordered set) in which $]\leftarrow, t[$, the set of predecessors of $t$, is well ordered for each $t \in T$. The trees most familiar to topologists are the Cantor tree, the Souslin trees, and the Aronszajn trees **[Ku]**, **[Ru]**. In §1 we study conditions under which a given POSET contains a cofinal tree.

Recall **[Po]**, **[P.S.]** that if $X$ is a space,[1] then the *absolute* $\mathcal{E}(X)$ of $X$ is the unique (up to a homeomorphism) extremally disconnected space that can be mapped irreducibly onto $X$ by a perfect map. Following **[C.N.2]** call $\beta\mathcal{E}(X)$ the *Gleason space* of $X$ and denote it by $\mathcal{G}(X)$. Two spaces $X$ and $Y$ are *coabsolute* (*$\mathcal{G}$-absolute*) whenever $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ (respectively, $\mathcal{G}(X)$ and $\mathcal{G}(Y)$) are homeomorphic. Designate $\mathcal{R}(X)$ for the Boolean algebra of regular-open sets of $X$–then it is known that $\mathcal{G}(X) \cong \mathcal{G}(Y)$ iff $\mathcal{R}(X) \equiv \mathcal{R}(Y)$.

In §2, we begin an application of §1 to topology with several theorems. We prove:

(2.1) $X$ is $\mathcal{G}$-absolute with a linearly ordered space if, and only if, $(\mathcal{R}(X) \sim \{X\}, \supseteq)$ contains a cofinal tree.

(2.3) ((2.8)) A first countable (Moore) space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense linearly ordered (metrizable) subspace.

(2.10) A dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable.

We also give (2.6 and 2.7) sufficient conditions (dependent on certain cardinal functions) for a space $X$ to have a dense linearly ordered subspace.

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[1] All spaces are infinite Hausdorff and regular.


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In §3 we consider coabsolutes of Stone-Čech remainders. For a noncompact completely regular space $X$, let $X^* = \beta X \sim X$. We prove

(3.5) A locally compact noncompact metric space $X$ has $X^*$ coabsolute with a linearly ordered space having a dense set of $P$-points.

(3.9) If $X$ is a locally compact noncompact metric space of density at most $2^\omega$, then $X^*$ is coabsolute with one of $\mathbf{N}^*$, $\mathbf{R}^*$, or $\mathbf{N}^* + \mathbf{R}^*$.

(3.13) The following is implied by **CH**, and consistent with and independent of $\neg$**CH**: If $X$ is locally compact noncompact, has $\pi w(X) \leqslant 2^\omega$, and if $X$ admits a complete uniformity, then $X^*$ is coabsolute with $\mathbf{N}^*$.

A result of importance in §§2 and 3 is the Stone duality theorem [**C.N.2**]. The Stone space of a Boolean algebra $B$ is denoted by $\mathrm{St}(B)$. "$\equiv$" between POSETS or Boolean algebras means "is isomorphic to".

We assume **ZFC**. **CH** is the continuum hypothesis, **SH** is Souslin's hypothesis, and **MA** is Martin's axiom. All cardinals and ordinals are von Neumann ordinals, so $\beta < \alpha$ means $\beta \in \alpha$. $\omega_0$ is denoted by $\omega$ and if $\kappa$ is a cardinal, $2^\kappa$ is the cardinal number of the set of subsets of $\kappa$. $|X|$ means the cardinality of $X$. If $A$ and $B$ are sets ${}^AB$ denotes the set of functions from $A$ to $B$. The standard binary tree [**Ku**] of height $\lambda$, an ordinal, is $\{f \in {}^\alpha 2\colon \alpha \in \lambda\}$ ordered by $f \leqslant g$ when $f = g\restriction \mathrm{dom}(f)$ and is denoted by $\mathrm{TREE}(\lambda)$.

**N** is the space of natural numbers, **Irr** is the space of irrationals, and **R** is the space of reals. "int" and "cl" are the interior and closure operators. "$\cong$" between spaces means "is homeomorphic" and $A \sim B$ means the complement of $B$ in $A$. "$\Sigma$" and "$+$" denote free union.

**1. Trees in POSETS.** Suppose $P$ is a POSET, $p \in P$, and $Q \subseteq P$. $Q$ is *cofinal* if $r \in P \Rightarrow \exists q \in Q$ with $r \leqslant q$. $Q$ is a *filter* if $p > q \in Q \Rightarrow p \in Q$. $p \in P$ is *compatible*[2] with $Q$ if, for each $q \in Q$, $\exists r$ with $p \leqslant r$ and $q \leqslant r$. $P$ is *separative* if, for each pair $p, q \in P$, $p \not\leqslant q \Rightarrow \exists r \geqslant q$ with $r$ and $p$ incompatible. If $F$ is a cofinal filter in a separative POSET $P$, then each maximal incompatible family of $F$ is a maximal incompatible family of $P$. On the other hand, if $I$ is a maximal incompatible family of $P$, then $\{p \in P\colon \exists i \in I, i \leqslant p\}$ is a cofinal filter of $P$.

Suppose $T$ is a tree and $\alpha$ is an ordinal, then

$$\mathrm{lv}(T, \alpha) = \{t \in T\colon \,]\leftarrow, t[ \text{ has order type } \alpha\}$$

is the *$\alpha$th level* of $T$ also denoted by $\mathrm{lv}(\alpha)$ when there is no confusion. $T(\alpha) = \bigcup \{\mathrm{lv}(\beta)\colon \beta \in \alpha\}$ is the *$\alpha$th subtree*. The *height* of $T$ is

$$h(T) = \inf\{\alpha\colon \mathrm{lv}(\alpha) = \varnothing\}.$$

A *branch* $b$ of $T$ is a maximal linearly ordered subset of $T$ and $\mathrm{ord}(b)$ denotes the order type of $b$.

If every linearly ordered subset of a POSET $P$ is bounded above, then Zorn's lemma provides $P$ with a cofinal tree of height 1. However, within any POSET $P$ we may build a tree $T$, recursively, by the subtrees $T(\alpha)$, having special properties:

[2] This is the first of several traditional [**Bu**], [**Je**] definitions (also separative, $\kappa$-distributive, $\kappa$-closed) for which we have reversed the usual order relation to maintain the orientation upwards for trees.

1.1. LEMMA. *If $P$ is a POSET, then there is a tree $T \subseteq P$ satisfying:*

(1) $\mathrm{lv}(0)$ *is a maximal incompatible family of $P$.*

(2) *If, for $\alpha \in h(T)$, $b$ is a branch of $T(\alpha)$ bounded above in $P$, then $\mathrm{lv}(\alpha)$ contains a maximal incompatible family of $\bigcap \{]t, \to [: t \in b\}$, where each $]t, \to [ = \{p \in P: t < p\}$.*

(3) $\bigcap \{]t, \to [: t \in b\} \subseteq T$ *for each branch $b$ of $T$.*

Whenever a tree $T$ satisfies 1.1(1)–(3) in a POSET $P$, $T$ will be called an *unbounded tree* of $P$. Since a cofinal tree in a POSET will be unbounded it is useful to define the ordinal invariants (under isomorphism)

$$\#(P) = \inf\{h(T): T \text{ is an unbounded tree of } P\}$$

and $w\#(P) = \inf\{\#([p, \to [): p \in P\}$.

1.2. THEOREM. *For a tree $T$ in a POSET $P$ the following hold:*

(1) *$T$ is unbounded if $p \in P$, $(p \not\leqslant t \forall t \in T) \Rightarrow p$ is compatible with at least two incompatible elements of $T$.*

(2) *If $P$ is separative and $T$ is unbounded, then $(p \not\leqslant t \forall t \in T) \Rightarrow p$ is compatible with two elements of some level of $T$.*

(3) *If $P$ is separative, then $\#([p, \to [) \leqslant \#(P) \forall p \in P$.*

(4) [Ny] *If every compatible pair of elements of $P$ is linearly ordered and if $T$ is unbounded, then $T$ is cofinal in $P$.*

PROOF. (1) Certainly $T$ satisfies 1.1(1), for otherwise some element of $P$ would be compatible with no elements of $T$. If $\lambda \leqslant h(T)$ and if $p$ is a successor of each member of a branch $b$ of $T(\lambda)$, then $b = \{t \in T(\lambda): p \text{ and } t \text{ are compatible}\}$. So 1.1(3) is immediate while 1.1(2) uses, as well, the observations for 1.1(1).

(2) We note that if $I$ is a maximal incompatible family of a separative POSET $P$ and if $p \in P$ is compatible with precisely one $i \in I$, then $i \leqslant p$. Otherwise, we may find $q \in P$ so that $p < q$ with $q$ and $i$ (and hence $I \cup \{q\}$) incompatible.

So if $p \in P$ is compatible with at most one element of each level, then 1.1(1) and (2) force $p$ to be an upper bound of a branch of $T$. So $p \in T$, by 1.1(3).

(3) Suppose $U$ is an unbounded tree of $P$, $\#(P) = h(U)$, and $p \in P$. If $p \leqslant u$ for some $u \in U$, then $U \cap [p, \to [$ is an unbounded tree of $[p, \to [$. So assume $p \not\leqslant u \forall u \in U$. From (2) $\exists$ a first $\alpha_0 \in h(U)$ such that $p$ is compatible with two elements of $\mathrm{lv}(U, \alpha_0)$. We may find a maximal incompatible family $I$ of $\{q \in P: \exists u \in \mathrm{lv}(U, \alpha_0), u < q, p < q\}$. Starting with

$$T(\alpha_0 + 1) = U(\alpha_0) \cup I \cup \{u \in \mathrm{lv}(U, \alpha_0): p \text{ and } u \text{ are incompatible}\}$$

we can build, using (2) to obtain each level, an unbounded tree $T$ of $P$ such that if $\alpha \in h(T)$ and $t \in \mathrm{lv}(T, \alpha)$, then there is a $u \in \mathrm{lv}(U, \alpha) \ni t \leqslant u$. So $h(U) = h(T)$. Since $I \subseteq [p, \to [$, 1.1 shows that $T \cap [p, \to [$ is an unbounded tree of $[p, \to [$. Therefore,

$$h([p, \to [) \leqslant h(T) = \#(P).$$

(4) This generalization of "every linearly ordered set has a well-ordered cofinal subset" is proved similarly to (2). □

For a POSET $P$ and a cardinal $\kappa$, $P$ is called *$\kappa$-distributive* if each intersection of at most $\kappa$ cofinal filters in $P$ is cofinal; $P$ is *$\kappa$-closed* whenever each increasing sequence of length at most $\kappa$ is bounded above. In forcing arguments the above properties have seen considerable activity [**Bu**].

1.3. LEMMA [**Bu**, 3.11]. *Let $P$ be a POSET and $\kappa$ a cardinal.*

(1) *If $P$ is $\kappa$-closed, it is $\kappa$-distributive.*

(2) *If $\lambda$ is the first cardinal such that $P$ is not $\lambda$-distributive (or $\lambda$-closed), then either $\lambda = \omega$ or $\lambda$ is a successor cardinal.*

1.4. LEMMA. *Suppose $P$ is a POSET with no maximal elements, $T$ is an unbounded tree of $P$, and $\kappa$ is a cardinal.*

(1) *If $P$ is $\kappa$-distributive and has no maximal elements, then $\kappa^+ \leqslant h(T)$ and $\mathrm{lv}(T, \alpha)$ is a maximal incompatible family of $P\ \forall \alpha \in \kappa^+$.*

(2) *If $P$ is $\kappa$-closed, $\kappa < \mathrm{cf}(\mathrm{ord}(b))\ \forall$ branches $b$ of $T$.*

PROOF. As (2) is obvious, we suppose $P$ is $\kappa$-distributive. Let $\alpha$ be the first ordinal such that $\mathrm{lv}(\alpha)$ is not a maximal incompatible family of $P$. If $\alpha \in \kappa^+$

$$C = \bigcap \left\{ \bigcup \{ ]t, \to [ : t \in \mathrm{lv}(\beta) \} : \beta \in \alpha \right\}$$

is a cofinal filter of $P$. Given $c \in C$ we may choose $t_\beta \in \mathrm{lv}(\beta)\ \forall \beta \in \alpha$ such that $t_\beta \leqslant c$. As $T$ is a tree, $c$ is an upper bound of the branch $b = \{t_\beta : \beta \in \alpha\}$ of $T(\alpha)$. From 1.1(2), $c$ is compatible with some element of $\mathrm{lv}(\alpha)$ bounding $b$ from above. Thus, $\mathrm{lv}(\alpha)$ is a maximal incompatible family of $C$, and hence, of $P$. As this is absurd, we must have $\alpha \geqslant \kappa^+$. □

1.5. THEOREM. *If $P$ is a separative POSET without maximal elements, then $w\#(P) = \sup\{\kappa^+ : P$ is $\kappa$-distributive$\}$.*

PROOF. We denote the right-hand side of the above equation by $\lambda$. Then 1.4(1) shows $\lambda \leqslant \#(P)$. We consider two cases.

*Case* 1. $w\#(P) = \#(P)$. Suppose $\{F_\alpha : \alpha \in \lambda\}$ is a family of cofinal filters in $P$. Let $p \in P$ be arbitrary and $J = \{q \in P : p$ and $q$ are incompatible$\}$. We build, recursively, a tree $T$ in $]p, \to [$.

Let $T(0) = \varnothing$. Suppose we are given an ordinal $\alpha$, $\alpha \leqslant \lambda$, such that for each $\beta < \lambda$, we have found $T(\beta)$ subject to the restriction

(*) If $\gamma < \beta$, then $\mathrm{lv}(T, \gamma)$ is a maximal incompatible family of $F_\gamma \cap ]p, \to [$.

If $\alpha$ is a limit ordinal, we must set $T(\alpha) = \bigcup \{T(\beta) : \beta < \alpha\}$. If $\alpha$ is a successor ordinal, then $|\alpha| < \lambda$ since $\lambda$ is a cardinal. Set

$$F = F_\alpha \cap \left( \bigcap \left\{ \bigcup \{ ]t, \to [ : t \in \mathrm{lv}(T, \beta) \} : \beta + 1 < \alpha \right\} \right).$$

So $J \cup F$ is a cofinal filter of $P$. Now suppose $r \in J \cup F$ and $r > p$.

Clearly $r \in F$ and $\exists r_\beta$ for each $\beta < \alpha$ such that $r_\beta < r$ and $r \in \mathrm{lv}(T, \beta)$. Since $T(\alpha - 1)$ is a tree, $\{r_\beta : \beta + 1 < \alpha\}$ is a branch of $T(\alpha - 1)$. Since $F$ is now a cofinal filter of $]p, \to [$, there is a maximal incompatible family $I$ of $]p, \to [$ contained in $F$. Set $T(\alpha) = F \cup T(\alpha - 1)$. As our hypothesis (*) is now met, the construction of $T$ is complete when we let $T = T(\lambda)$.

Now let us suppose that $\lambda < \#(P)$. Then $T$ is not an unbounded tree of $]p, \to[$. From $(*)$, each $\mathrm{lv}(T, \alpha)$ is a maximal incompatible family of $]p, \to[$, so 1.1 implies that $T$ has a branch $b$ bounded above. If $\mathrm{ord}(b) < \lambda$, then there is a $t \in \mathrm{lv}(T, \mathrm{ord}(b))$ compatible with every element of $b$. Since $T$ is a tree, $t$ is an upper bound for $b$. As the latter is impossible, $\mathrm{ord}(b) = \lambda$. Now if $s$ is an upper bound for $b$, then $(*)$ implies $p < s$ and $s \in \bigcap \{F_\alpha : \alpha < \lambda\}$. As $p$ was chosen arbitrarily, $\bigcap \{F_\alpha : \alpha < \lambda\}$ is a cofinal filter of $P$. So $\lambda \nleq \lambda^+$. This is a contradiction. Therefore, $\lambda = \#(P)$.

*Case* 2. $w\#(P) < \#(P)$. Choose a maximal incompatible family $I \subseteq P$ with

$$w\#(]p, \to[) = \#(]p, \to[) \; \forall p \in I.$$

Since $\bigcup \{]p, \to[ : p \in I\}$ is a cofinal filter of $P$, $\#(P) = \sup\{\#(]p, \to[) : p \in I\}$. On the other hand, we may add $\bigcup \{]p, \to[ : p \in I\}$ to any family of cofinal filters. So, from case 1, $\#(]p, \to[) \leqslant \lambda \; \forall p \in I$. Therefore, $\lambda = \#(P)$. □

We observe that case 2 of 1.5 also shows

1.6. COROLLARY. *$\#(P)$ is a cardinal whenever $P$ is a separative POSET.*

1.7. THEOREM. *Suppose $P$ is a separative POSET for which $\#(P) = w\#(P)$. If $P$ has a cofinal family the union of $\#(P)$ incompatible families, then $P$ has a cofinal tree of height $\#(P)$.*

PROOF. WLOG we assume $\bigcup \{I_\alpha : \alpha \in \#(P)\}$ is cofinal in $P$ where each is a maximal incompatible family of $P$ containing no maximal elements of $P$. We construct an unbounded tree $T$, modifying the successor ordinal steps in 1.1, as follows:

If $\alpha < \#(P)$, $|\alpha| < \#(P)$ from 1.5. By 1.4

$$C = \left(\bigcup \{[i, \to[ : i \in I_\alpha\}\right) \cap \left(\bigcap \left\{\bigcup \{[t, \to[ : t \in \mathrm{lv}(T(\alpha - 1), \beta)\} : \beta < \alpha - 1\right\}\right)$$

is a cofinal filter of $P$. So there is a maximal incompatible family $I$ of $P \ni I \subseteq C$. Set $T(\alpha) = T(\alpha - 1) \cup I$.

It is clear that $T$ is cofinal and $h(T) = \#(P)$. □

1.8. COROLLARY. (1) (*Weiss*) *If $P$ is a separative $\kappa$-closed POSET $\forall \kappa < \lambda$ and $P$ has a cofinal family which is the union of $\lambda$ incompatible families, then $P$ has a cofinal tree.* (2) (*Davies*) *If $P$ is a separative $\omega$-distributive POSET and $|P| \leqslant \omega_1$, then $P$ has a cofinal tree.*

PROOF. Observe that both results follow from 1.7 even if "separative" is removed from their hypothesis since we only used "$P$ is $\kappa$-distributive $\forall \kappa < \#(P)$" in the proof of 1.7. □

1.9. THEOREM. *If a separative POSET has a cofinal tree, then it has a cofinal tree of height $\#(P)$.*

PROOF. In proving 1.5, we observed that there is a maximal incompatible family $I$ of $P$ such that, $\forall i \in I$, $w\#([i, \to[) = \#([i, \to[)$ and

$$\#(P) = \sup\{\#([i, \to[) : i \in I\}.$$

So WLOG we may assume $P$ has no maximal elements and $\#(P) = w\#(P)$. Suppose $T$ is a cofinal tree in $P$, $U$ is an unbounded tree in $P$, and $h(U) = \#(P)$. From 1.4 each $\mathrm{lv}(U, \alpha)$ is a maximal incompatible family of $P$. So we may choose a maximal incompatible family of $P$,

$$I_\alpha \subseteq \left(\bigcup \{[t, \to [ : t \in \mathrm{lv}(U, \alpha)\}\right) \cap T.$$

If $\bigcup \{I_\alpha : \alpha \in \#(P)\}$ is cofinal, the theorem follows from 1.7. So suppose $t \in T$ such that $t \not\leqslant i \forall i \in I_\alpha \forall \alpha \in \#(P)$. Since $P$ is separative, we have, from 1.2(2), $\exists \alpha \in h(U)$ and $u_0$, $u_1 \in \mathrm{lv}(U, \alpha) \ni t$ is compatible with each $u_n$. Again using separativity $\exists$, for each $n \in \{0, 1\}$, a $t_n \in T$ such that $t < t_n$ and $u_n < t_n$. $\exists$ for each $n$, $i_n \in I_\alpha$ such that $u_n < i_n$ and $t_n$ is compatible with $i_n$. Since $T$ is a tree, either $i_n \geqslant t_n \geqslant t$ (a contradiction) or $i_n < t_n$. In the latter case, $t$ and $i_n$ are compatible so $i_0 < t$ and $i_1 < t$ (a contradiction). □

From 1.7, $2^\kappa = \kappa^+$ implies each $\kappa$-distributive POSET of cardinality at most $2^\kappa$ has a cofinal tree. Our next theorem represents an attempt at removing the set-theoretic hypothesis from this result.

1.10. LEMMA. *Suppose $\kappa$ is an infinite cardinal, $P$ is a separative $\kappa$-closed POSET, and $p \in P$ has no maximal successor; then $p$ has $2^\kappa$ incompatible successors.*

PROOF. We can construct a tree in $[p, \to [$ isomorphic to TREE$(\kappa + 1)$ by applying "separative" at successor ordinals to get two incompatible elements, and "$\kappa$-closed" at limit ordinals to get a single successor of a branch. The final level of TREE$(\kappa + 1)$ contains $2^\kappa$ incompatible elements. □

1.11. LEMMA. *Suppose $\kappa$ is an infinite cardinal and $P = \{p(\zeta) : \zeta \in 2^\kappa\}$ is a $\kappa$-closed separative POSET. If $I = \{i(\alpha) : \alpha \in 2^\kappa\}$ is an incompatible family in $P$, then there is a family $J \subseteq P$ subject to*

(1) *$J = \{j(\alpha, \xi) : (\alpha, \xi) \in 2^\kappa \times 2^\kappa\}$, where $i(\alpha) < j(\alpha, \xi)$ for each $\xi$.*

(2) *If $p(\zeta)$ is compatible with an element of $J$ but $p(\zeta) \not\leqslant j \forall j \in J$, then*

$$|\{i \in I : p(\zeta) \text{ and } i \text{ are compatible}\}| \leqslant |\zeta|.$$

PROOF. $J$ is constructed recursively via a diagonalization argument–we examine the $\beta$th step:

Suppose $\delta$ is the first element of $2^\kappa$ such that, $\forall \alpha \in 2^\kappa$, $p(\delta) \not\leqslant i(\alpha)$ and $\forall \gamma \in \beta \forall \xi \in 2^\kappa$ $p(\delta) \not\leqslant j(\gamma, \xi)$, but $p(\delta)$ and $i(\beta)$ are compatible. For some $q \in P$ with $p(\delta) < q$ and $i(\beta) < q$ we choose a family $\{j(\beta, \xi) : \xi \in 2^\kappa\}$ of maximal incompatible successors of $i(\beta)$ to which $q$ belongs. □

1.12. THEOREM. *Let $\kappa$ be an infinite cardinal. If $P$ is a separative $\kappa$-closed POSET with $|P| \leqslant 2^\kappa$, then $P$ has a cofinal tree.*

PROOF. From 1.10, $|P| < 2^\kappa \Rightarrow P$ has a cofinal tree of height 1. So WLOG assume $P$ has no maximal elements and we have a listing of $P$, $\{p(\zeta) : \zeta \in 2^\kappa\}$.

We can construct, as in 1.1, using 1.10 and 1.11, an unbounded tree $T$ of $P$ subject to the additional conditions:

(4) lv(0) contains a successor of $p(0)$.

(5) If $\alpha \in h(T)$ and $b$ is a branch of $T(\alpha)$ bounded above in $P$, then $|\operatorname{lv}(\alpha) \cap (\bigcap \{[t,\rightarrow[: t \in b\}) | = 2^{\kappa}$.

(6) If $p(\zeta) \not\leqslant t \forall t \in T(\alpha)$ for $\alpha \in h(T)$, then $\alpha \leqslant \zeta$ and $|\{t \in T(\alpha): p(\zeta)$ and $t$ are incompatible$\}| \leqslant |\zeta|$.

Suppose $p = p(\zeta) \in P$ and $p \not\leqslant t \forall t \in T$; then there is, by 1.2(2), a first $\alpha_0$ such that $p$ is compatible with two elements of $\operatorname{lv}(\alpha_0)$. Using 1.4(2) and following the proof of 1.10, we may build a tree $S \subseteq T$ consisting of elements compatible with $p$, each of whose levels is contained in a level of $T$, and which is isomorphic to TREE$(\kappa + 1)$. Since $\exists \lambda \in h(T)$ such that the last level of $S$ is contained in $\operatorname{lv}(\lambda)$, $p$ causes (6) to fail for $\alpha = \lambda$. □

1.13. LEMMA [**Je**, 29B]. *If $P$ is a separative POSET, then there exists a unique (up to an isomorphism) complete Boolean algebra $\mathfrak{B}(P)$ for which $P$ is cofinally embedded in $(\mathfrak{B}(P) - \{0\}, \geqslant)$.*

If $B$ is a Boolean algebra such that $(B - \{1\}, \leqslant)$ or, equivalently, $(B - \{0\}, \geqslant)$, possesses (resp. a cofinal set $P$ satisfying) the properties defined in this section for a POSET, we say for simplicity that $B$ (resp. *cofinally*) possesses said property; therefore:

(i) Every Boolean algebra is separative.

(ii) No atomless $\sigma$-complete Boolean algebra is $\omega$-closed.[3]

(iii) $\mathfrak{B}(\text{TREE}(\omega_1))$ is cofinally $\omega$-closed.

1.14. COROLLARY. *Suppose $\kappa$ is an infinite cardinal. If $\kappa^+ = 2^{\kappa}$, then $\mathfrak{B}(\text{TREE}(2^{\kappa}))$ is the only complete atomless Boolean algebra which is cofinally $\kappa$-closed and has a cofinal set of cardinal $2^{\kappa}$. If $2^{\kappa^+} = 2^{\kappa}$, then there are at least two complete atomless Boolean algebras cofinally $\kappa$-closed and having cofinal subsets of cardinal $2^{\kappa}$.*

PROOF. The Pressing Down Lemma [**Ku**] shows that, for each $\kappa$, $\mathfrak{B}(\text{TREE}(\kappa^+)) \ncong \mathfrak{B}(\text{TREE}(\kappa^{+\,+}))$. On the other hand (v) in the construction of $T$ in 1.12 shows that if $\kappa^+ = 2^{\kappa}$ and $P$ is a $\kappa$-closed separative POSET without maximal elements of cardinality $2^{\kappa}$, then $P$ has a cofinal tree isomorphic to $\cup \{\operatorname{lv}(\text{TREE}(\kappa^+), \lambda): \lambda$ is a limit ordinal in $\kappa^+\}$. □

1.15. *On products.* Suppose $\kappa$ is a cardinal and $P(\alpha)$ is a POSET for each $\alpha \in \kappa$; there are two traditional definitions for partial orders on the Cartesian product $\prod = \prod\{P(\alpha): \alpha \in \kappa\}$:

(1) The *lexicographic product*, $\operatorname{lex} \prod$, is ordered by "$f < g$ whenever $\exists \alpha \in \kappa$ with $f(\alpha) < g(\alpha)$ and $f(\beta) = g(\beta) \forall \beta \in \alpha$". It is easy to see that $\operatorname{lex} \prod$ *has a cofinal tree whenever $P(\alpha)$ has a cofinal tree $\forall \alpha \in \kappa$.*

(2) The *usual product* on $\prod$, denoted by $\times\{P(\alpha): \alpha \in \kappa\}$, is ordered by "$f \leqslant g$ whenever $f(\alpha) \leqslant g(\alpha) \forall \alpha \in \kappa$". An easy application of the Pressing Down Lemma shows TREE$(\omega) \times$ TREE$(\omega_1)$ has no cofinal tree. However, 1.7 shows that $P \times Q$ has a cofinal tree whenever each of $P$ and $Q$ have a cofinal tree and $w\#(P) = w\#(Q)$.

1.16. REMARKS. (1) Is it consistent that "every $\omega$-distributive POSET of cardinality $\omega_1$ is $\omega$-closed?" Not in a model of **ZFC** + **SH**; however, Franklin Tall has

[3] In [**Wo2**], [**Wo3**] cofinally $\omega$-closed Boolean algebras are called *Cantor-separable*.

communicated to the author Peter Davies' affirmative answer under the assumption of the consistency of certain large cardinal axioms.

(2) For the POSET $P$ of nonempty clopen subsets of $\beta\mathbf{N} \sim \mathbf{N}$ (under "$\supseteq$") 1.5, 1.9, and 1.12 were proved, independently, in [**B.P.S.**].

(3) 1.8(2) is due to Peter Davies. 1.8(1) is an observation William Weiss made from one of our early results.

(4) Is it consistent with **ZFC** + **CH** that "there is precisely one complete atomless cofinally $\omega$-closed Boolean algebra with a cofinal set of cardinality $2^{\omega}$?" See 3.13.

(5) For many POSETS $P$, $\#(P)$ is well defined by considering cofinal subsets of $P$. With proof similar to 1.2(3) and (4), this is true when $P$ is either separative or when every compatible pair of elements of $P$ are linearly ordered or when $P$ is directed.

**2. $\mathcal{G}$-absolutes of linearly ordered spaces.** Recall [**Ju2**] if $(X, \tau)$ is a space, then a cofinal subset of $(\tau - \{\varnothing\}, \supseteq)$ is known as a *$\pi$-base* (*pseudobase* in [**C.N.2**]) and the *$\pi$-weight*, $\pi w(X)$, is the least cardinal possessed by a $\pi$-base for $X$. The *weight*, $w(X)$, is the least cardinal possessed by a base for $\tau$.

2.1. THEOREM. *For a space $X$, the following are equivalent*:

(1) *$X$ has a $\pi$-base with a cofinal tree.*

(2) *Every $\pi$-base of $X$ has a cofinal tree.*

(3) *$X$ is $\mathcal{G}$-absolute with a linearly ordered space.*

PROOF. If $Y$ is the set of isolated points of $X$, then $Y$ is a subset of every $\pi$-base for $X$, $Y$ is the set of atoms of $\mathcal{R}(X)$, and

$$\mathcal{R}(X) \equiv \mathcal{R}\big(Y \dot{\cup} \operatorname{int}(X \sim Y)\big).$$

Since the free union of linearly ordered spaces is linearly ordered, we need only prove the theorem for $X \sim Y$. WLOG we assume $X$ has no isolated points.

(1) $\Rightarrow$ (2). Let $P$ be a $\pi$-base for $X$. Since a cofinal subset of a $\pi$-base is a $\pi$-base, we suppose that $(T, \supseteq)$ is a tree of nonempty open subsets of $X$ such that $T$ has the minimum possible height for a tree $\pi$-base for $X$. Since $X$ has no isolated points, $h(T)$ and $\operatorname{ord}(b)$ are limit ordinals whenever $b$ is a branch of $T$. We now construct, recursively, two trees $S_1 \subseteq T$ and $S_2 \subseteq P$.

For $i \in \{1, 2\}$ let $S_i(0) = \varnothing$. Suppose we are given an ordinal $\alpha \leqslant h(T)$ such that $S_i(\beta)$ has been found, for each $\beta < \alpha$ and each $i \in \{1, 2\}$, subject to the restriction $\gamma < \beta \Rightarrow$

(a) $\operatorname{lv}(S_i, \gamma)$ is a pairwise-disjoint family of nonempty open sets.

(b) If $b$ is a branch of $S_1(\gamma)$ and if $\operatorname{int}(\bigcap b) \neq \varnothing$, then

$$\operatorname{int}\Big(\bigcap b\Big) \subseteq \operatorname{cl}\Big(\big\{p \in \operatorname{lv}(S_2, \gamma) : p \subseteq \bigcap b\big\}\Big).$$

(c) $\operatorname{lv}(S_1, \gamma) \subseteq T \sim T(\gamma)$.

(d) If $p \in \operatorname{lv}(S_2, \gamma)$, then $p \subseteq \operatorname{cl}(\{t \in \operatorname{lv}(S_1, \gamma) : t \subseteq p\})$.

If $\alpha$ is a limit ordinal, we set $S_i(\alpha) = \bigcup \{S_i(\beta) : \beta < \alpha\}\ \forall i$. If $\alpha$ is a successor ordinal, the choice of $\operatorname{lv}(S_i, \alpha - 1)$ is straightforward (given a $\pi$-base $Q$ of a space $Y$

and a nonempty open set $G$ of $Y$, $G$ has a dense set which is the union of a family of pairwise-disjoint members of $Q$). Let $S_i(\alpha) = S_i(\alpha - 1) \cup \mathrm{lv}(S_i, \alpha - 1)$. Our recursion hypothesis is clearly met. So our construction of $S_i \forall i \in \{1, 2\}$ is complete when we set each $S_i = S_i(h(T))$.

Now (b) and (d) imply $S_2$ is a $\pi$-base for $X$ iff $S_1$ is cofinal in $T$. (b) and (d) also imply that if $\alpha < h(T)$ and if $b$ is a branch of $S_1(\alpha)$, then

$$\mathrm{int}(\cap b) \subseteq \mathrm{cl}\big(\{t \in \mathrm{lv}(S_1, \alpha): t \subseteq \cap b\}\big).$$

From (c) and 1.1, $S_1$ is an unbounded tree of $T$. 1.2(4) shows $S_1$ is cofinal in $T$.

(2) $\Rightarrow$ (3). Since an infinite Hausdorff space contains an infinite family of nonempty pairwise-disjoint open sets, we suppose $T$ is a cofinal tree in $\mathfrak{R}(X)$ satisfying

(iv) if $\alpha \in h(T)$ and $b$ is a branch of $T(\alpha)$ bounded above in $\mathfrak{R}(X)$, then $\mathrm{lv}(T, \alpha)$ contains infinitely many elements each of whose closure is a subset of $\cap b$.

Following the standard **[Ku]** collapsing of Souslin and Aronszajn trees, order each $\mathrm{lv}(T, \alpha)$ so that $\mathrm{lv}(\alpha)$ and in the case of (iv), the successors of $b$ in $\mathrm{lv}(\alpha)$, form a discrete linearly ordered set without endpoints.

Set $L = L(T) = \{b: b \text{ is a branch of } T\}$ and for $b_0, b_1 \in L$ define $b_0 < b_1$ if for some $\alpha \in h(T)$

$$\big(b_0 \cap \mathrm{lv}(T, \alpha)\big) < \big(b_1 \cap \mathrm{lv}(T, \alpha)\big) \quad \text{while } b_0 \cap T(\alpha) = b_1 \cap T(\alpha).$$

Thus, $L$ is linearly ordered. Set $\psi(t) = \{b \in L: t \in b\}$ for each $t \in T$.

Since $\psi(t)$ has no endpoints, $\psi(t)$ is clopen; further, if in $L$, $b_0 < b_1 < b_2$, then by (iv), $\exists t \in b_1 \sim (b_0 \cup b_1) \ni \psi(t) \subseteq ]b_0, b_1[$. So $\psi$ embeds $T$ cofinally within $\mathfrak{R}(L)$. Now apply 1.13 and the Stone duality.

(3) $\Rightarrow$ (1). Suppose $L$ is a linearly ordered space without isolated points; then $P = \{G \in \mathfrak{R}(L): G \text{ is an open interval of } L\}$ is cofinal in $\mathfrak{R}(L)$. Suppose $T$ is any unbounded tree of $P$ and $]x, y[ \in P$ such that $t \not\subseteq ]x, y[ \forall t \in T$. According to 1.2(2) $\exists \beta \in h(T)$, $]x_i, y_i[ \in \mathrm{lv}(T, \beta)$ for $i \in \{0, 1\}$, such that $]x, y[ \cap ]x_i, y_i[ \neq \varnothing$,

$$(*) \qquad x_0 < x < y_0 < y, \qquad x < x_1 < y < y_1,$$

and there are no points between $y_0$ and $x_1$. Since $\mathfrak{R}(L)$ is closed under intersection, $]x, y_0[ \in P$. Again applying 1.2(2) $\exists \alpha \in h(T)$, $\alpha > \beta$, $\exists ]x_j, y_j[ \in \mathrm{lv}(T, \alpha)$ for $j \in \{2, 3\}$, such that $]x_0, y_0[ \cap ]x_j, y_j[ \neq \varnothing$, $x_2 < x < y_2 < y_0$, $x < x_3 < y_0 < y_3$, and there are no points between $y_2$ and $x_3$. In particular, $y_0 \in ]x_3, y_3[ \sim ]x_0, y_0[$. Since $T$ is a tree, $\beta < \alpha$, and $L$ has no isolated points, $]x_3, y_3[ \cap ]x_0, y_0[ = \varnothing$ which contradicts $(*)$. So $T$ is cofinal in $P$. Now apply the Stone duality. □

2.2. COROLLARY. *For a space $X$, the following are equivalent:*

(1) *$X$ has a $\sigma$-disjoint $\pi$-base.*

(2) *$\mathfrak{R}(X)$ has a cofinal tree and $\#(\mathfrak{R}(X)) \leqslant \omega$.*

(3) *$\exists$ a metric space $M$ $\mathcal{G}$-absolute with $X$.*

PROOF. (1) $\Rightarrow$ (2). This is a corollary of 1.7 since $\mathfrak{R}(X)$ is $n$-closed $\forall$ finite cardinals $n$.

(2) $\Rightarrow$ (3). In 2.1 (2) $\Rightarrow$ (3) the tree $T$ may be assumed to have height $\omega$ (from 1.9). Thus, the space $L$ is metrizable via $\rho(b_0, b_1) = 2^{-n}$ when $b_0 \cap \mathrm{lv}(n) \neq b_1 \cap \mathrm{lv}(n)$ and $b_0 \cap T(n) = b_1 \cap T(n)$.

(3) $\Rightarrow$ (1). Every metric space has a $\sigma$-disjoint base. $\square$

2.3. THEOREM. *Suppose $X$ is a space whose every point has a well-ordered local base; then $X$ is $\mathcal{G}$-absolute with a linearly ordered space iff $X$ has a dense linearly ordered subspace.*

PROOF. We need only show "$\Rightarrow$". WLOG assume $X$ has no isolated points. For each $x \in X$ let $\mathcal{U}(x) \subseteq \mathcal{R}(X)$ be a well-ordered, by "$\supseteq$", local base at $x$. From 2.1, $\mathcal{R}(X)$ has a cofinal tree $U$. We construct a tree $T$ and a function $\psi\colon T \to X$ as follows:

Let $T(1) = U(1)$ and $\psi(t) \in t$ be arbitrarily chosen for each $t \in T(1)$. Suppose we are given an ordinal $\lambda$ such that $T(\alpha)$ and $\psi(t) \in t$ have been constructed $\forall \alpha \in \lambda\, \forall t \in T(\alpha)$ subject to the restrictions:

(i) $T(\alpha)$ satisfies 1.1(1) and (2).

(ii) $T(\alpha)$ satisfies (iv) of 2.1(2) $\Rightarrow$ (3).

(iii) If $\beta < \alpha$ and $b$ is a branch of $T(\beta)$, then

$$|\{t \in \mathrm{lv}(T(\alpha), \beta)\colon t \subseteq \cap b \text{ and } t \notin U\}| \leqslant 1,$$

where equality holds only if $\psi$ is constant on a tail of $b$.

(iv) If $\beta < \alpha$ and $t \in T(\beta)$, then $T(\alpha)$ has a branch $b$ with $t \in b$ and $\psi(b) = \{\psi(t)\}$.

(v) If $s, t \in T(\alpha)$, $t \subsetneqq s$, and if $\psi(t) = \psi(s)$, then $t \in \mathcal{U}(\psi(s))$.

If $\lambda$ is a limit ordinal, set $T(\lambda) = \cup\{T(\alpha)\colon \alpha \in \lambda\}$. If $\lambda$ is a successor ordinal, then we will assign to each branch $b$ of $T(\lambda - 1)$ a family $I(b)$ and we will set

$$T(\lambda) = T(\lambda - 1) \cup \big(\cup\{I(b)\colon b \text{ is a branch of } T(\lambda - 1)\}\big).$$

For $t \in T(\lambda) \sim T(\lambda - 1)$ and $t \subseteq \cap b$, we assign $\psi(t) = \psi(s)$ if $\psi$ is constantly $\psi(s)$ on a tail of $b$ and if $t \in \mathcal{U}(\psi(s))$; otherwise, $\psi(t) \in t$ may be arbitrarily chosen.

Let $I(b) = \varnothing$ whenever $\mathrm{int}(\cap b) = \varnothing$. If $\mathrm{int}(\cap b) \neq \varnothing$ and $\psi$ is not constant on a tail of $b$, we may choose, since $X$ has no isolated points, a pairwise disjoint subfamily $I(b)$ of $U$ with $\cup\{\mathrm{cl}(u)\colon u \in I(b)\}$ a dense subset of $\mathrm{int}(\cap b)$. If $\mathrm{int}(\cap b) \neq \varnothing$ and $\psi$ is constantly $\psi(s)$ on a tail of $b$ with $s \in b$, then (v) implies $\mathrm{int}(\cap b)$ is a nbhd of $\psi(s)$. Choose $t \in \mathcal{U}(\psi(s))$ with $\mathrm{cl}(t) \supsetneqq \mathrm{int}(\cap b)$. Then $I(b)$ will be the union of $\{t\}$ and an infinite pairwise disjoint subfamily of $U$ such that $\cup\{\mathrm{cl}(u)\colon u \in I(b) \sim \{t\}\}$ is a dense subset of $\mathrm{int}(\cap b) \sim \mathrm{cl}(t)$.

As the induction hypothesis is clearly satisfied, we can continue it until we have an unbounded tree $T$. Since $\psi(t) \in t\, \forall t \in T$, $\psi(T)$ is dense if $T$ is cofinal in $\mathcal{R}(X)$. So we suppose $u \in U$ such that $t \nsubseteq u\, \forall t \in T$. By 1.2(2) there is a first ordinal $\beta \in h(T)$ such that $u \cap t_0$, $u \cap t_1 \in \mathcal{R}(X) \sim \{\varnothing\}$ for two distinct elements $t_0$, $t_1 \in \mathrm{lv}(T, \beta)$. So $u \subseteq \cap b$ for a branch $b$ of $T(\beta)$, and $t_0 \cup t_1 \subseteq \cap b$. Since $U$ is a tree, neither $t_0$ nor $t_1$ is in $U$. This contradicts (iii) for $\alpha = \beta + 1$. So $T$ is cofinal in $\mathcal{R}(X)$.

From (iv) and (v) there is for each $t \in T$ precisely one branch $b(t)$ of $T$ such that $t \in b(t)$ and $b(t)$ is a local base at $\psi(t)$. So when $\{b(t)\colon t \in T\}$ inherits the order given in 2.1(2) $\Rightarrow$ (3), the map $b(t) \to \psi(t)$ gives $\psi(T)$ a linear order generating the subspace topology inherited from $X$. $\square$

2.4. COROLLARY [**Wh**]. *A first countable space has a $\sigma$-disjoint $\pi$-base iff it has a dense metrizable subspace.*

PROOF. In 2.3 each $\mathcal{R}(x)$ can be assumed to have order type $\omega$, and, according to 2.2, $h(U) = \omega$. So $h(T) = \omega$. Following the proof of 2.2(2) $\Rightarrow$ (3) we see that $\psi(T)$ is metrizable. □

As a Souslin line has no dense metrizable subspace, it is consistent with **ZFC** that "$\sigma$-disjoint $\pi$-base" cannot be replaced by "$\mathcal{G}$-absolute with a linearly ordered space" in 2.4. However, it is replaceable for the class of first countable spaces $X$ which have $\#(\mathcal{R}(X)) \leqslant \omega$. Therefore, we consider some cardinal functions which affect $\#(\mathcal{R}(X))$.

Henceforth, *we shall use* $\#[w\#]$ *to denote* $\#(\mathcal{R}(X))$ $[w\#(\mathcal{R}(X))]$ when there is no confusion. A topological translation of 1.5 and 1.9 yields

2.5. LEMMA. *Let $X$ be a space and $\{I_\alpha\colon \alpha \in \kappa\}$ be a collection of families of pairwise-disjoint nonempty open sets such that* $\mathrm{cl}(\bigcup I_\alpha) = X \forall \alpha \in \kappa$. *If $\kappa \leqslant w\#$, then there is an unbounded tree $T$ of $\mathcal{R}(X)$ satisfying*:

(1) $h(T) = \#$.

(2) $s, t \in T, s \subsetneq t \Rightarrow \mathrm{cl}(s) \subseteq t$.

(3) $\mathrm{cl}(\bigcup \mathrm{lv}(\alpha)) = X \forall \alpha \in w\#$.

(4) *$T$ satisfies* (iv) *of* 2.1(2) $\Rightarrow$ (3).

(5) $i \in I_\alpha \Rightarrow \exists\, t \in \mathrm{lv}(\alpha) \ni t \subseteq i$.

*Further, if $X$ is $\mathcal{G}$-absolute with a linearly ordered space, then $T$ may be assumed to be cofinal in $\mathcal{R}(X)$.*

Recall [**C.N.2**] that for a cardinal $\kappa$, a space $X$ is *$\kappa$-Baire* if the intersection of at most $\kappa$ many open dense subsets of $X$ is dense [so Baire = $\omega$-Baire]; and $x \in X$ is a *$P_\kappa$-point* if it has a local base $\lambda$-closed $\forall \lambda < \kappa$ [so $P$-point = $P_{\omega_1}$-point]. $X$ is an *almost $P_\kappa$-space* [**Le**] if the intersection of less than $\kappa$ many nonempty open subsets of $X$ has nonempty interior (equivalently, if $\mathcal{R}(X)$ is cofinally $\lambda$-closed $\forall \lambda < \kappa$). A base (or $\pi$-base) for a space will be called *$\kappa$-disjoint* when it is the union of a collection of $\kappa$ many families of pairwise disjoint sets (WLOG each family may be assumed to have union dense in $X$).

The following should be compared to [**C.N.1,** 3.1] and [**C.N.2**, 6.15].

2.6. THEOREM. *For a space $X$ with a $\kappa$-disjoint $\pi$-base $B$,*

(1) $\kappa \geqslant \#$ *(so* $\pi w(X) \geqslant \#$*),*

(2) *if $\kappa = w\#$, $X$ is $\kappa$-Baire, and if $B$ is a base, then $X$ has a dense subset of $P_\kappa$-points which is linearly orderable.*

PROOF. (1) From 1.6 there is a family $J$ of pairwise-disjoint nonempty regular-open subsets of $X$ such that $\bigcup J$ is dense in $X$, $w\#(\mathcal{R}(G)) = \#(\mathcal{R}(G)) \forall G \in J$, and $\# = \sup\{\#(\mathcal{R}(G))\colon G \in J\}$. From 2.5(5) $\kappa \geqslant \#(\mathcal{R}(G))$ for each $G$.

(2) Let $B = \bigcup\{I_\alpha\colon \alpha \in \kappa\}$, where each $I_\alpha$ is pairwise-disjoint and has dense union. Let $T$ be the tree guaranteed in 2.5 and set $D = \{\bigcap b\colon b$ is a branch of $T$, $\mathrm{ord}(b) = \kappa$, and $\bigcap b \neq \varnothing\}$. Since $X$ is $\kappa$-Baire, 2.5(3) implies $\bigcup D$ is dense. Since $B$ is a base and $T$ is a tree, 2.5(5) implies $b$ is a well-ordered local base at $\bigcap b$ and

$|\bigcap b| = 1$ whenever $\bigcap b \in D$. To see that $\bigcup D$ is linearly orderable, follow the last paragraph of 2.3. □

2.7. THEOREM. *Suppose $X$ is a space whose diagonal is the intersection of $\kappa$ many open subsets of $X \times X$; then $\kappa \geq \#$. Further, if $\kappa = w\#$, $X$ is $\kappa$-Baire, and if $X$ is the intersection of at most $\kappa$ many open subsets of $\beta X$, then $X$ has a dense subset of $P_\kappa$-points which is linearly orderable.*

PROOF. Let $\{\mathcal{O}(\alpha): \alpha \in \kappa\}$ be a collection of open sets of $X \times X$ whose intersection is $\Delta = \{(x, x): x \in X\}$. Given $x \in X$ and $\alpha \in \kappa$, there is a nbhd $G$ of $x$ such that $G \times G \subseteq \mathcal{O}(\alpha)$. Hence, for each $\alpha \in \kappa$ there is a pairwise-disjoint collection $I_\alpha \subseteq \mathcal{R}(X)$ such that $\mathrm{cl}(\bigcup I_\alpha) = X$ and $G \times G \subseteq \mathcal{O}(\alpha) \forall G \in I_\alpha$. Let $T$ be the tree of 2.5 constructed from $\{I_\alpha: \alpha \in \kappa\}$. If $\lambda < w\#$, then 2.5(5) implies $\Delta \neq \bigcap \{\mathcal{O}(\alpha): \alpha \in \lambda\}$. Following 2.5(1) shows $\# \leq \kappa$.

For the further, let $\{H(\alpha): \alpha \in \lambda\}$ be a family of open subsets of $\beta X$ whose intersection is $X$, and suppose $\lambda \leq \kappa$. For $\alpha \in \kappa \sim \lambda$ set $H(\alpha) = \beta X$. For each $\alpha \in \kappa$, let

$$J_\alpha = \left\{H(\alpha) \cap \mathrm{int}_{\beta X}\left(\mathrm{cl}_{\beta X}(G)\right): G \in I_\alpha\right\}.$$

Let $S \subseteq \mathcal{R}(\beta X)$ be the tree of 2.5 constructed for $\beta X$ from $\{J_\alpha: \alpha \in \kappa\}$. Using 2.5(2) we see $\bigcap b$ is a nonempty compact subset of $\beta X$ whenever $b$ is a branch of $S(\eta)$ for $0 < \eta \leq \kappa$. So $b$ is always a local base for $\bigcap b$ when $\eta$ is a limit ordinal. Since $\bigcap \{H(\alpha): \alpha \in \lambda\} = X$, 2.5(5) implies $\bigcap b \subseteq X$ whenever $b$ is a branch of $S(\eta)$ for $\lambda \leq \eta \leq \kappa$. Since $\Delta = \bigcap \{\mathcal{O}(\alpha): \alpha \in \kappa\}$, 2.5(5) implies $|\bigcap b| = 1$ whenever $b$ is a branch of $S$ and $\mathrm{ord}(b) = \kappa$. As $X$ is $\kappa$-Baire, 2.5(3) implies $\{x \in \bigcap b: b \text{ is a branch of } S, \mathrm{ord}(b) = \kappa\}$ is dense in $X$. Now follow the last paragraph of 2.3. □

2.8. COROLLARY. *A Moore space is $\mathcal{G}$-absolute with a linearly ordered space iff it has a dense metrizable subspace.*

PROOF. A Moore space is 1st countable so 2.3 applies. As it also has a $G_\delta$-diagonal 2.7, and hence 2.2, applies. □

2.9. COROLLARY. *For a Čech-complete space $X$, the following hold:*

(1) *If $w\#(\mathcal{R}(X)) > \omega$, then $X$ has a dense open locally compact subspace.*

(2) *If $\#(\mathcal{R}(X)) > \omega$ and $X$ is perfectly normal, then $X$ is not $\omega_1$-Baire.*

(3) *If $X$ has a $G_\delta$-diagonal, then $X$ has a dense metrizable linearly orderable $G_\delta$-set.*

PROOF. (1) Consider $\mathrm{lv}(T, \omega)$ in the "further" of 2.7. (2) The Pressing Down Lemma implies $\mathrm{lv}(T, \omega_1) = \varnothing$ in the "further". (3) This follows immediately from the "further" and 2.2. □

Recall [**Ju1**] the cardinal $\kappa$ is a *caliber* for a space $X$ if each collection of $\kappa$ many nonempty open subsets of $X$ contains a centered subfamily of cardinality $\kappa$. [**Ju1**, A2.2] shows that if $X = {}^{\kappa}2$ is given the Tychonov product topology, then $\omega_1$ is a caliber for $X$ whenever $\kappa \geq \omega_1$.

Suppose $X$ is a space for which $\kappa$ is a caliber; if $Y$ is the image of $X$ under a continuous surjection and $T$ is an unbounded tree of $\mathcal{R}(Y)$ satisfying 2.5(2) for $Y$, then $|T| < \kappa$ since the inverse image of $T$ satisfies 2.5(2).

The next theorem is essentially the theme of [**Gv**]. The preceding paragraph yields for us a shorter proof.

2.10. THEOREM. *A dyadic space is $\mathcal{G}$-absolute with a linearly ordered space iff it is separable and metrizable.*

PROOF. ($\Rightarrow$) $D$ is a dyadic space, if, by definition, $D$ is a dense subset of a continuous image $Y$ of ${}^{\kappa}2$, where, by [**Ju1**, 4.9] $\kappa = \pi w(Y) = w(Y)$. From 2.1 and the above $\kappa = \omega$ and so $Y$ is a compact metric. □

2.11. COROLLARY [**Po**]. *If a dyadic space is coabsolute with a metric space, it is metrizable.*

2.12. REMARKS. (1) A space is non-Archimedean (see [**Ny**] for a survey) provided it has a base in which every pair of members are either related by $\subseteq$ or disjoint. By virtue of 1.2(4), these are spaces having inverted trees as bases. Observe that the space $L$ of 2.1(2) $\Rightarrow$ (3) and $\psi(T)$ of 2.3 are non-Archimedean, while the dense subspaces in 2.6 and 2.7 are actually $\kappa$-metrizable. A technical modification of 2.1(2) $\Rightarrow$ (3) shows that "linearly ordered" in 2.1(3) can be replaced by "subspace of linearly ordered".

(2) Is there (in **ZFC**) a compact first countable space not the compactification of a linearly ordered space? We observe that the Pressing Down Lemma shows that if $X$ is a Souslin line, then $X \times [0, 1]$ has no dense linearly ordered subspace.

(3) In [**Ta**] methods for recognizing Souslin trees in topologies are investigated, and equivalences and implications of **SH** are given. We observe (from 1.6 and 2.9) that SH is equivalent to the statement "if $X$ is an $\omega_1$-Baire Čech-complete perfectly normal space, then $\#(\mathcal{R}(X)) \leqslant \omega$".

(4) Does every compact almost $P$-space of weight $2^{\omega}$ contain a dense linearly ordered subspace? From [**C.N.1**] the answer is yes if **CH** is assumed. From 1.12 and 2.3, a compact almost $P$-space of $\pi$-weight $2^{\omega}$ contains a dense non-Archimedean linearly ordered subspace whenever it contains a dense set of points with well-ordered local bases. The last condition is necessary as every linearly ordered subspace of $\mathcal{G}(\beta\mathbf{N} \sim \mathbf{N})$ contains no isolated points.

(5) 2.7 (for $\kappa = \omega$) and 2.8 were originally proved independently of White's result 2.4; however, 2.4 motivated 2.3. We observe that "Čech-complete" in 2.9 can be replaced by "$p$-space in the sense of Ar'hangelskii", and a similar generality works in 2.7.

**3. Coabsolutes of Stone-Čech remainders.** If $Z$ is a zero set of a completely regular space $X$, then we let $Z^* = \{x \in \beta X \sim X\colon\ Z \in x\}$. So $X^* = \beta X \sim X$. It is well known (see [**En**]) that $X^*$ can be nearly anything for a suitable pseudocompact space, and if $X^*$ is dyadic, then $X$ is pseudocompact. Thus, 2.10 especially encourages us to restrict our attention to a class of spaces in which every pseudocompact closed subspace is compact–in this case we consider the class of spaces with a compatible complete uniformity (which we will call *complete* spaces).[4] Further, if $X$ is nowhere

[4]Shirota's theorem says that when we assume no measurable cardinals exist, a space is complete iff it is realcompact. So the reader may wish to replace "complete" with "realcompact" throughout this section.

locally compact, then $X^*$ and $X$ are dense in $\beta X$. Therefore, we restrict our attention to complete locally compact noncompact spaces. With minor changes in proof the following is [**Wo2**, 3.2]:

3.1. LEMMA. *If $X$ is complete, then $X^*$ and $(\mathcal{C}(X))^*$ are coabsolute.*

3.2. LEMMA. *If $X$ is a locally compact extremally disconnected noncompact space, then $\exists \kappa \geqslant 2^\omega$ and a family $\{D(\alpha): \alpha \in \kappa\}$ satisfying:*

(1) *Each $D(\alpha)$ is the union of countably many pairwise-disjoint compact open subspaces of $X$.*

(2) *Either $D(\alpha) \cong N$ or $D(\alpha)$ has no isolated points.*

(3) $\beta < \alpha \Rightarrow D(\beta)^* \cap D(\alpha)^* = \varnothing$.

(4) $X = \mathrm{cl}(\bigcup \{D(\alpha)^*: \alpha \in \kappa\})$ *and each $D(\alpha)^*$ is open in $X^*$.*

PROOF. Let $E(0) = \{\{x\}: \{x\} \text{ is open in } X\}$ and choose, by local compactness, a maximal collection $E(1)$ of pairwise disjoint nonempty open compact members of $\mathfrak{R}(\mathrm{int}(X \sim \bigcup E(0)))$. For each $n \in \{0, 1\}$ let $\{e(\alpha, n): \alpha \in \kappa(n)\}$ be a listing of $E(n)$ with a cardinal $\kappa(n)$.

If $\kappa(n)$ is finite, let $I(n) = \varnothing$. If $\kappa(n)$ is infinite, then choose $I(n)$ to be a (*maximal almost-disjoint*) family of countably infinite subsets of $\kappa(n)$ maximal w.r.t. "every intersection of distinct members is finite". We may choose $|I(n)| \geqslant 2^\omega$ since it is well known [**Ru**] that $\omega$ contains a maximal almost-disjoint family of cardinal $2^\omega$.

The desired family will be

$$\{D(i, n): i \in I(n), n \in \{0, 1\}\}, \quad \text{where } D(i, n) = \bigcup \{e(\alpha, n): \alpha \in i\}.$$

Its cardinality is at least $2^\omega$ since $X$ is not compact. (1) and (2) are certainly satisfied. (3) follows since $I(n)$ is almost-disjoint or empty and $(\bigcup E(0)) \cap (\bigcup E(1)) = \varnothing$. If $C$ is clopen in $X$ and $\varnothing \neq C^*$, then $\exists n \in \{0, 1\}$ and a countably infinite set $j \subseteq \kappa(n)$ such that $C \cap e(\alpha, n) \neq \varnothing$ for each $\alpha \in j$ (otherwise, $\bigcup(E(0) \cup E(1))$ would not be dense in $X$). As $I(n)$ is infinite and a maximal almost-disjoint family, $i \cap j$ is infinite for some $i \in I(n)$. (4) follows since

$$\varnothing \neq \left(\bigcup \{C \cap e(\alpha, n): \alpha \in i \cap j\}\right)^* \subseteq C^* D(i, n)^*. \quad \square$$

3.3. LEMMA [**F.G.**, 3.1]. *If $X$ is realcompact and locally compact, then $X^*$ is an almost $P_{\omega_1}$-space.*

3.4. THEOREM. *If $X$ is a complete locally compact noncompact space each of whose nonempty open sets contains a nonempty open set of $\pi$-weight at most $2^\omega$, then $X^*$ is coabsolute with a linearly ordered space having a dense set of P-points.*

PROOF. From 3.1 we may assume $X$ is extremally disconnected, locally compact, noncompact with the same $\pi$-weight conditions. Thus, in 3.2 we may assume $\pi w(e(\alpha, n)) \leqslant 2^\omega$, and hence, from extremal disconnectedness $2^\omega \leqslant \pi w(D(\alpha, i)^*) \leqslant (2^\omega)^\omega = 2^\omega$. Since each $D(\alpha, i)$ is $\sigma$-compact, each $D(\alpha, i)^*$ is an almost $P_{\omega_1}$-space. From 1.3(2) and 1.12, each $\mathfrak{R}(D(\alpha, i)^*)$, and hence, $\mathfrak{R}(X^*)$ has a cofinal tree $T$ whose branches fail to have countable cofinality. The desired space is the Dedekind completion (with endpoints) of the space $L$ in 2.1(2) $\Rightarrow$ (3). $\square$

As each locally compact metric space is the free union of $\sigma$-compact spaces which must have $\pi$-weight at most $\omega$, and as every locally compact metric space admits a complete uniformity, we have shown

3.5. COROLLARY. *A locally compact noncompact metric space $X$ has $X^*$ coabsolute with a linearly ordered space having a dense set of P-points.*

3.6. LEMMA. *If $X$ is a locally compact, extremally disconnected, noncompact space with $\pi w(X) = \omega$, then $X^*$ is homeomorphic to one of $\mathbf{N}^*$, $(\mathfrak{C}(\mathbf{R}))^*$, or $\mathbf{N}^* + \mathfrak{C}(\mathbf{R})^*$.*

PROOF. Following the proof of 3.2, $\pi w(X) = \omega$ yields $X = E(0) \cup E(1)$, where $E(0)$ is the closure of all of the at most $\omega$ isolated points of $X$ and $E(1) = X \sim E(0)$. As $X$ is extremally disconnected $X^* = E(0)^* \cup E(1)^*$.

When $E(0)$ is not compact, each clopen set free ultrafilter on $E(0)$ traces to an ultrafilter on $\operatorname{int}(E(0)) \cong \mathbf{N}$; therefore, $E(0)^* \cong \mathbf{N}^*$. When $E(1)$ is not compact, it is the disjoint union of $\omega$ many (since $\pi w(X) = \omega$) compact spaces of $\pi$-weight $\omega$ without isolated points. From [**Si**, 9c] each of the spaces is homeomorphic to $\mathfrak{C}$(Cantor set). Applying the same argument to $\mathbf{R}$, we have $E(1) \cong \mathfrak{C}(\mathbf{R})$. □

3.7. LEMMA. *If $K$ is a compact space and if, for each $n \in \{0, 1\}$, $X(n)$ is the free union of a cardinal $\kappa(n)$ many copies of $K$, where $\omega \leqslant \kappa(n) \leqslant 2^\omega$, then $X(0)^*$ and $X(1)^*$ contain homeomorphic dense open subspaces.*

PROOF. Write $X(n) = \Sigma\{K(\alpha): \alpha \in \kappa(n)\}$, where $K(\alpha) \cong K$ for each $\alpha$. Since $\kappa(n) \leqslant 2^\omega$, $\kappa(n)$ contains precisely $2^\omega$ countably infinite subsets. Hence, following the argument in the proof of 3.2, we may choose a maximal almost-disjoint family $I(n)$ of countably infinite subsets of $\kappa(n)$ such that $|I(n)| = 2^\omega$. Then

$$\left\{\left\{x \in X(n)^*: \left(\left(\sum\{K(\alpha): \alpha \in i\}\right) \sim K(\gamma)\right) \in x \,\forall \gamma \in i\right\}: i \in I(n)\right\}$$

is a pairwise-disjoint family of clopen subsets of $X(n)^*$ whose union is dense in $X(n)^*$ and whose members are homeomorphic to $(\Sigma\{K(\alpha): \alpha \in \omega\})^*$. □

3.8. THEOREM. *If $X$ is a complete locally compact noncompact space of $\pi$-weight at most $2^\omega$, and if every nonempty open set of $X$ contains a nonempty open set of countable $\pi$-weight, then $X^*$ is coabsolute with one of $\mathbf{N}^*$, $\mathbf{R}^*$, or $\mathbf{N}^* + \mathbf{R}^*$.*

PROOF. Following the proof of 3.4, we observe that each $D(\alpha, i)$ may also be assumed here to have countable $\pi$-weight, and from their construction in 3.2 there are precisely $2^\omega$ of the sets $D^*(\alpha, i)$ each of which is homeomorphic to one of the three spaces above by 3.6. Now apply 3.7 with $K = \mathbf{N}^*$ and $K = \mathbf{R}^*$. □

3.9. COROLLARY. *Suppose $X$ is a locally compact noncompact metric space of density at most $2^\omega$; then $X^*$ is coabsolute with*

(1) $\mathbf{N}^*$, *if $X$ has a dense discrete subspace,*

(2) $\mathbf{R}^*$, *if the set of isolated points of $X$ has compact closure,*

(3) $\mathbf{N}^* + \mathbf{R}^*$, *otherwise.*

In [**Wo1, Wo2**] it is shown that if **CH** is assumed $X^*$ is coabsolute with $\mathbf{N}^*$ whenever $X$ is locally compact, noncompact, and either metric of density at most $2^\omega$

or with $|\mathcal{R}(X)| = 2^{\omega}$; however, this follows from 1.14 which shows **CH** *implies* $\mathcal{R}(\mathbf{N}^*) \equiv \mathcal{R}(Y)$ *whenever* $Y$ *is an almost P-space with* $\pi w(Y) = 2^{\omega}$ *and no isolated points*. We end this section with an example which shows $2^{\omega}$ is essential in 3.4 and which allows us to remove **CH** from the hypothesis of Woods' results.

3.10. EXAMPLE. Suppose $\kappa > \omega$ is a cardinal and $\mathcal{D}(\kappa) = \Sigma\{\mathcal{D}(\kappa, n): n \in \omega\}$, where each $\mathcal{D}(\kappa, n) \cong {}^{\kappa}2$ given the Tychonov product topology. If $K$ denotes the linearly ordered space obtained from ordering ${}^{\omega_1}2$ lexicographically, then the following are equivalent:

(1) $\kappa \leqslant 2^{\omega}$.

(2) $\mathcal{D}(\kappa)^*$ and $K$ are coabsolute.

(3) $\mathcal{D}(\kappa)^*$ is coabsolute with a linearly ordered space.

PROOF. For each ordinal $\alpha < \omega_1$, we set

$$A(\alpha) = \{Z(f): \operatorname{dom}(f) = \alpha\}, \quad \text{where } Z(f) = \{g \in {}^{\kappa}2: g \mid \alpha = f\},$$

and $L(\alpha) = \{\operatorname{int}((\bigcup\{Z(f, n): n \in \omega\})^*): \forall n \in \omega\, Z(f, n) \in A(\alpha)$ and $Z(f, n) \subseteq D(\kappa, n)\}$. Then $T = \{L(\alpha): \alpha \in \omega_1\}$ is a tree in $\mathcal{R}(\mathcal{D}(\kappa)^*)$ whose $\alpha$th level is $L(\alpha)$. We claim that $T$ is an unbounded tree.

First we observe that $\mathcal{D}(\kappa)^*$ has a $\pi$-base $P(\kappa)$ of sets of the form $\operatorname{int}(Z^*)$ such that for each $n \in \omega\, \exists h_n \in {}^{\kappa}2$ and a countable set $C_n \subseteq \kappa$ such that $g \in Z \cap \mathcal{D}(\kappa, n)$ iff $g \restriction C_n = h_n$. So if $\alpha < \omega_1$ is the first ordinal with $\omega_1 \cap (\bigcup\{C_n: n \in \omega\}) \subseteq \alpha$, then $\operatorname{int}(Z^*)$ intersects two elements of $L(\alpha + 1)$. From 1.2(1), $T$ is unbounded.

(1) $\Rightarrow$ (2). From 1.9 and 1.12 $\mathcal{R}(\mathcal{D}(\kappa)^*)$ has a cofinal tree of height $\omega_1$ since $\pi w(\mathcal{D}(\kappa)^*) = 2^{\omega}$. From 3.3 it has a cofinal tree order isomorphic to the union of the limit ordinal levels of $\mathrm{TREE}(\omega_1)$. If $L$ is constructed as in 2.1(2) $\Rightarrow$ (3), then $L$ is homeomorphic to a dense subspace of $K$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). As $\mathcal{D}(\kappa)$ has caliber $\omega_1$, it follows that every pairwise-disjoint family of $P(\kappa)$ has cardinality at most $2^{\omega}$. On the other hand, $\pi w(\mathcal{D}(\kappa)^*)$ is at least $2^{\omega} + \kappa$. So if $\mathcal{D}(\kappa)^*$ is coabsolute with a linearly ordered space, then $\mathcal{R}(\mathcal{D}(\kappa)^*)$ has a cofinal tree $S$ of height $\omega_1$. Since $|S| = 2^{\omega} \cdot \omega_1 = 2^{\omega}$ and $S$ is a $\pi$-base of $\mathcal{D}(\kappa)^*$, $\kappa \leqslant 2^{\omega}$. □

3.11. LEMMA. *If* $X$ *is a locally compact complete noncompact space, then* $\#(\mathcal{R}(X^*)) \leqslant \#(\mathcal{R}(\mathbf{N}^*))$.

PROOF. From 3.1 and 3.2 $X = \Sigma\{X(n): n \in \omega\}$, where each $X(n)$ is compact and extremally disconnected, can be assumed. One readily observes that the map $X(n) \to n$ induces a function from $\mathcal{R}(\mathbf{N}^*)$ to $\mathcal{R}(X^*)$ which takes unbounded trees to unbounded trees isomorphic to their pre-images. □

3.12. THEOREM. *The following are equivalent statements in* **ZFC**:

(1) $\mathbf{N}^*$ *and* $D(\omega_1)^*$ *are coabsolute.*

(2) $\#(\mathcal{R}(\mathbf{N}^*)) = \omega_1$.

(3) *If* $X$ *is a locally compact complete noncompact space and if* $\pi w(X) \leqslant 2^{\omega}$, *then* $X^*$ *and* $\mathbf{N}^*$ *are coabsolute.*

PROOF. Since $\#(\mathfrak{R}(\mathfrak{D}(\omega_1)^*)) = \omega_1$, (1) $\Rightarrow$ (2). From 3.11 $\#(\mathfrak{R}(X^*)) = \omega_1$ so (2) $\Rightarrow$ (3) is similar to 3.10 (3) $\Rightarrow$ (1). (3) $\Rightarrow$ (1) is obvious. □

As we have observed before **CH** implies $\#(\mathfrak{R}(\mathbf{N}^*)) = \omega_1$. However, as remarked in [**B.P.S.**] there are numerous models of **ZFC** + $\neg$**CH** in which $\#(\mathfrak{R}(\mathbf{N}^*)) = \omega_1$, for example in any model $\mathfrak{M}$ for which $\mathfrak{M} \models {}^{\omega}\omega$ to have an $\omega_1$-scale [**He**] and $\mathfrak{M} \models \omega_1 < 2^{\omega}$. On the other hand, **MA** + $\neg$**CH** implies $\mathfrak{R}(\mathbf{N}^*)$ is cofinally $\kappa$-closed $\forall \kappa < 2^{\omega}$ [**Ru**], and hence, $\#(\mathfrak{R}(\mathbf{N}^*)) = 2^{\omega} > \omega_1$. Therefore, we have

3.13. COROLLARY. *The following statement is implied by* **CH**, *and is consistent with and independent of* $\neg$**CH**: *If $X$ is a locally compact complete noncompact space with $\pi w(X) \leqslant 2^{\omega}$, then $X^*$ and* $\mathbf{N}^*$ *are coabsolute.*

3.14. REMARKS. (1) 3.7 was communicated verbally to the author by S. Broverman for the case $|K| = 1$. The proof he gave is similar. R. G. Woods has informed us that 3.9(2) can also be proved using a recent result of E. K. van Douwen on remote points and a theorem in [**Wo3**].

(2) *Are* $\mathbf{N}^*$ *and* $\mathbf{R}^*$ *coabsolute*? As $\mathbf{N}^*$ is an almost $P_{\kappa}$-space iff $\mathbf{R}^*$ is an almost $P_{\kappa}$-space [**vD1**], we observe that a proof similar to that of 3.12 (2) $\Rightarrow$ (3) shows $\mathbf{N}^*$ and $\mathbf{R}^*$ are coabsolute whenever $\mathbf{N}^*$ is an almost $P_{\kappa}$-space $\forall \kappa < \#(\mathfrak{R}(N^*))$. The latter is true in a variety of situations (including **MA**). However, we do not know whether a negative answer is consistent.

(3) 3.10 was motivated by an example in [**vD.vM.**] called here $\mathfrak{D}(2^{\omega})$.

(4) In the first draft of this paper we used a result in [**vD2**] to show 3.11. Originally we obtained 3.13 prior to 1.12 (and independently of [**B.P.S.**]); however, our proof was longer while a key lemma to our short proof in the first draft of this paper was false.

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ADDED IN PROOF. The answer to the questions in 2.12(2) is "yes" (Todorčévic); in 2.12(4) is "not $\mathfrak{D}(\omega_2)$" (myself) and it is consistent that $X^*$ has no dense linearly ordered subspace for every non-pseudo-compact space $X$.

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