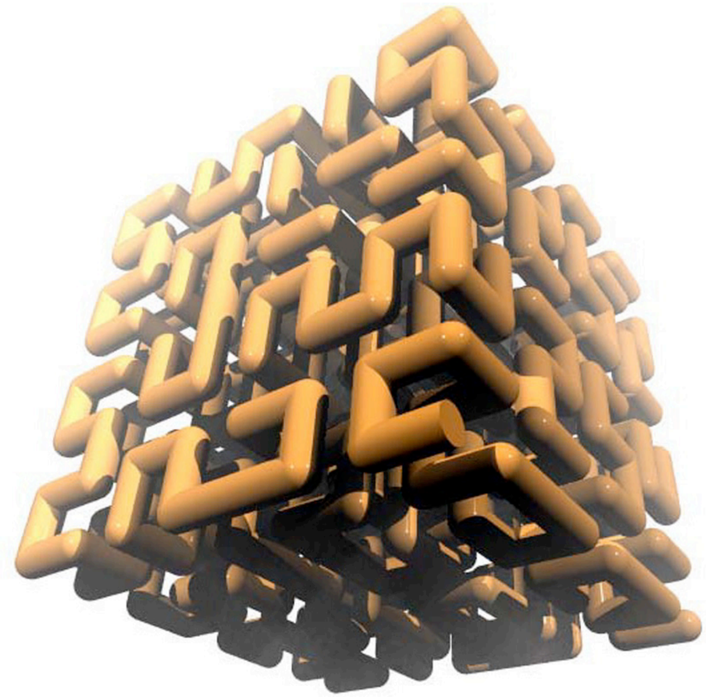


MY FAVORITE FUNCTIONS

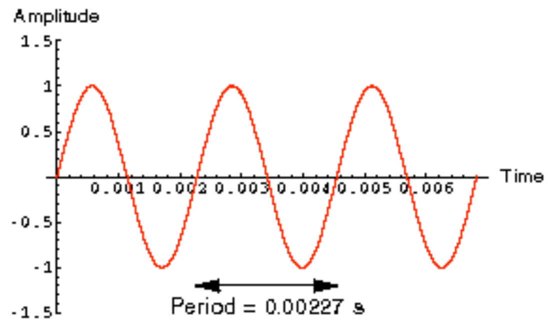
OR

Continuous from ***what*** to WHAT?!?

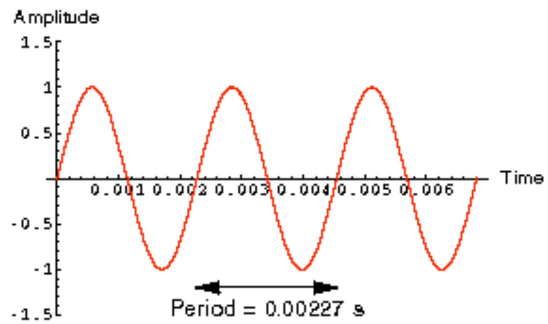
Medgar Evers College 071206

Section 1. **Accordions**

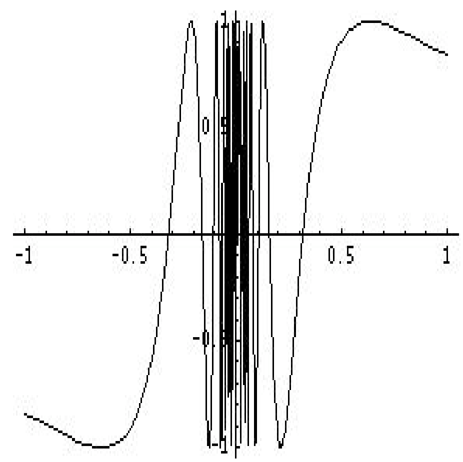
$y = \sin x$



$y = \sin x$

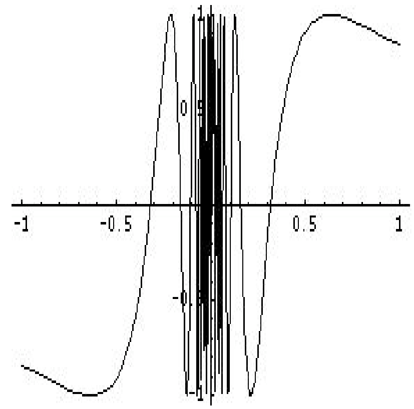


$y = \sin \frac{1}{x}$

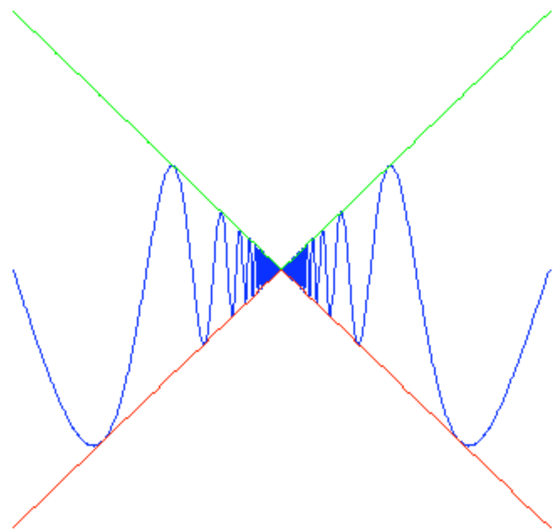


not continuous at 0

$y = \sin\frac{1}{x}$



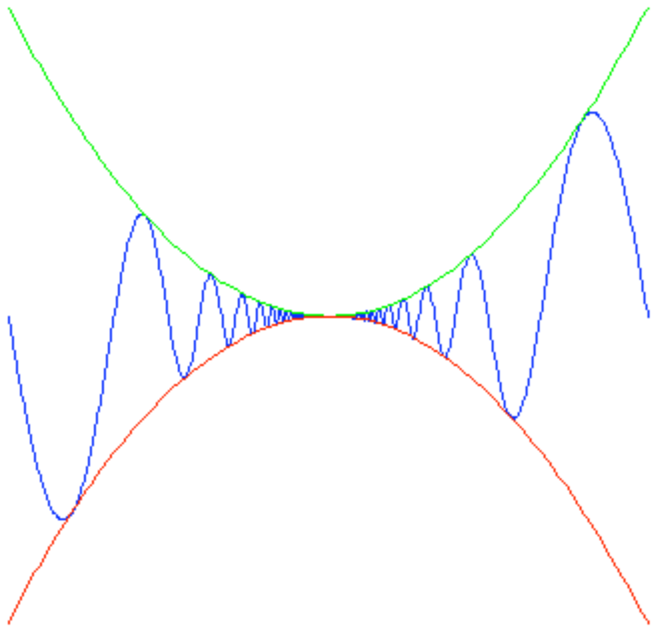
$f(x) = x\sin\frac{1}{x}$



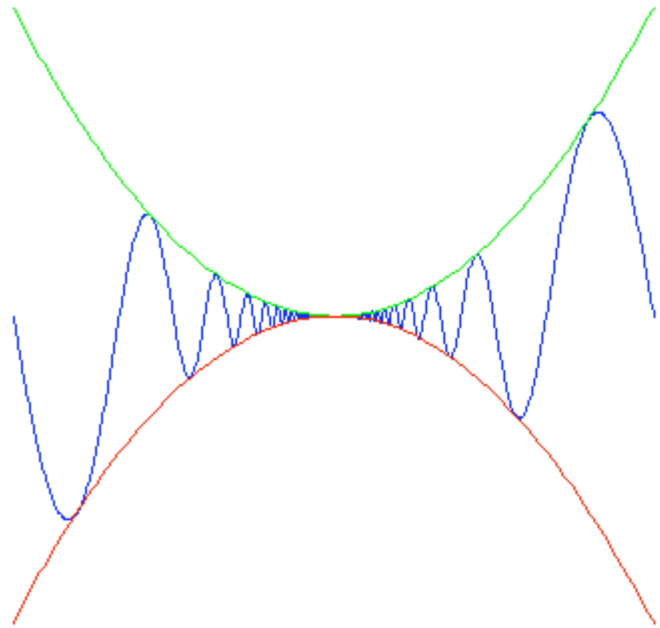
f is continuous at 0 even though there are nearly vertical slopes as you approach 0.

$$g(x) = x^2 \sin \frac{1}{x}$$

Here $\lim\limits_{h \to 0} \frac{g(h)}{h} = 0$



$$g(x) = x^2 \sin \frac{1}{x}$$

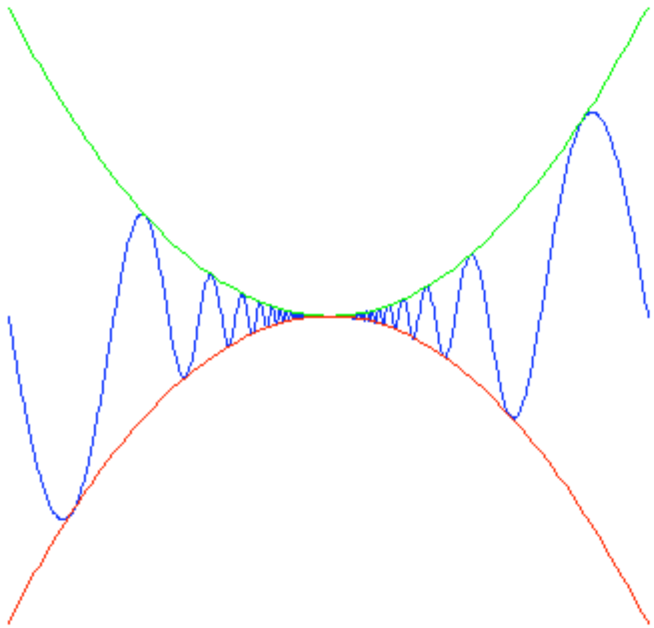


So g has a derivative at 0,

$$g'(0) = \lim_{h \to 0} \frac{g(h)}{h} = 0$$

$$g(x) = x^2 \sin \frac{1}{x}$$

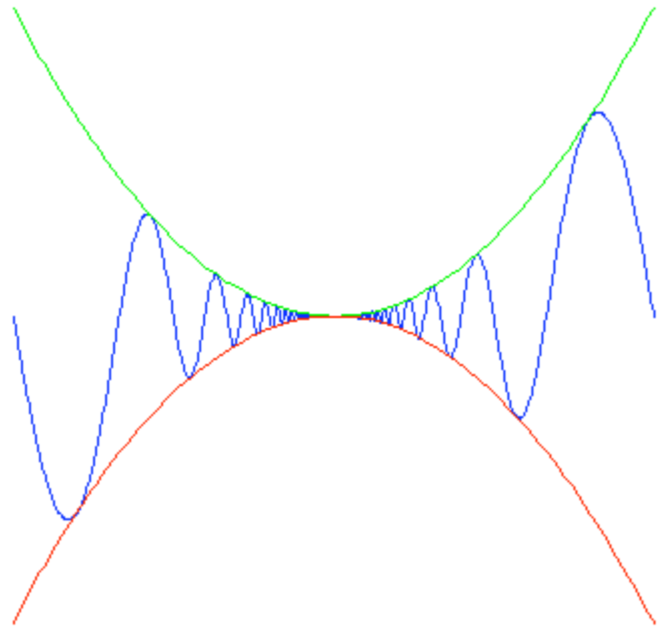
$g'(0) = 0.$



Still we have nearly vertical tangents.

$$g(x) = x^2 \sin \frac{1}{x}$$

$g'(0) = 0.$



We have nearly vertical tangents.

Further, there are sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ such that

$$\lim_{n\to\infty} b_n - a_n = 0, \text{ but } \lim_{n\to\infty} \frac{g(b_n) - g(a_n)}{b_n - a_n} = \infty.$$

Section 2. Le **Blancmange** function

Fix a non-negative integer n. Given a real number x, let k be the greatest non-negative integer such that

$a_{(x,n)} = 2^{-n}k \leq x$ and let $b_{(x,n)} = 2^{-n}(k+1)$. So $x < b_{(x,n)}$.

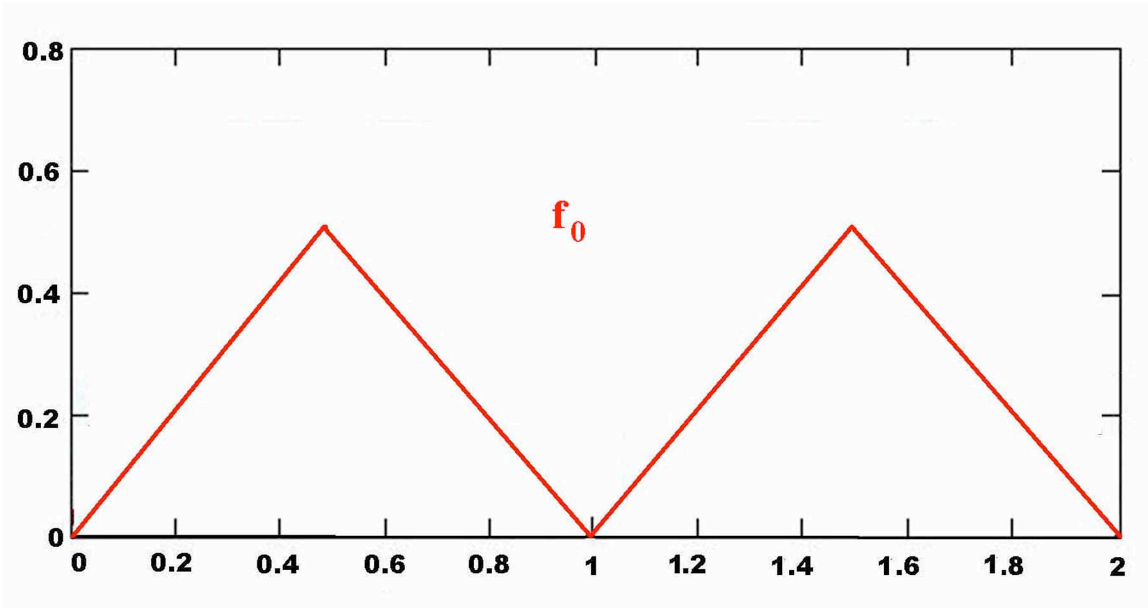
Define $f_n : \mathbf{R} \rightarrow [0,1]$ by $f_n(x) = \min\{x-a_{(x,n)}, b_{(x,n)}-x\}$.

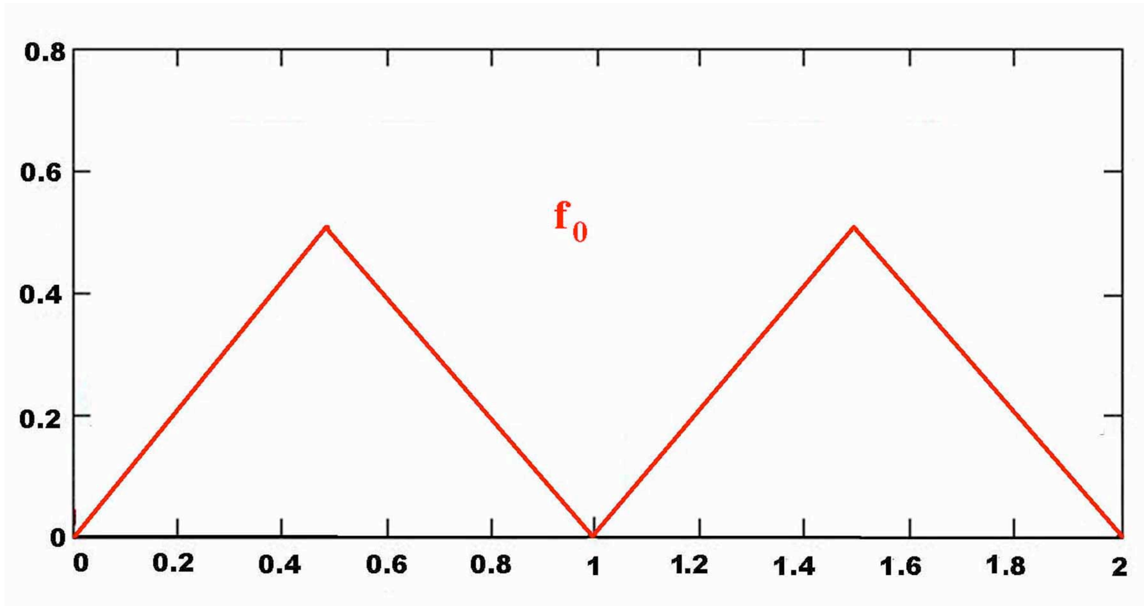
Fix a non-negative integer n. Given a real number x, let k be the greatest non-negative integer such that

$a_{(x,n)} = 2^{-n}k \leq x$ and let $b_{(x,n)} = 2^{-n}(k+1)$. So $x < b_{(x,n)}$.

Define f_n : $\mathbf{R} \to [0,1]$ by $f_n(x) = \min\{x-a_{(x,n)}, b_{(x,n)}-x\}$.

k = 0




0.8
0.6
0.4
0.2
0
f_0
0
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1
1.2
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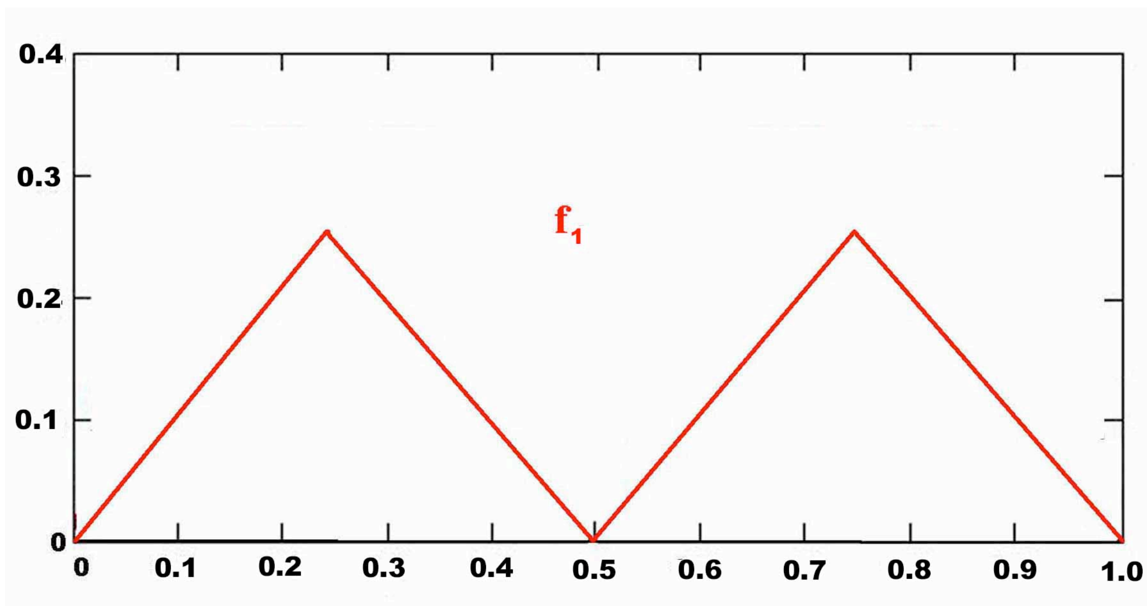
1.6
1.8
2


0.4
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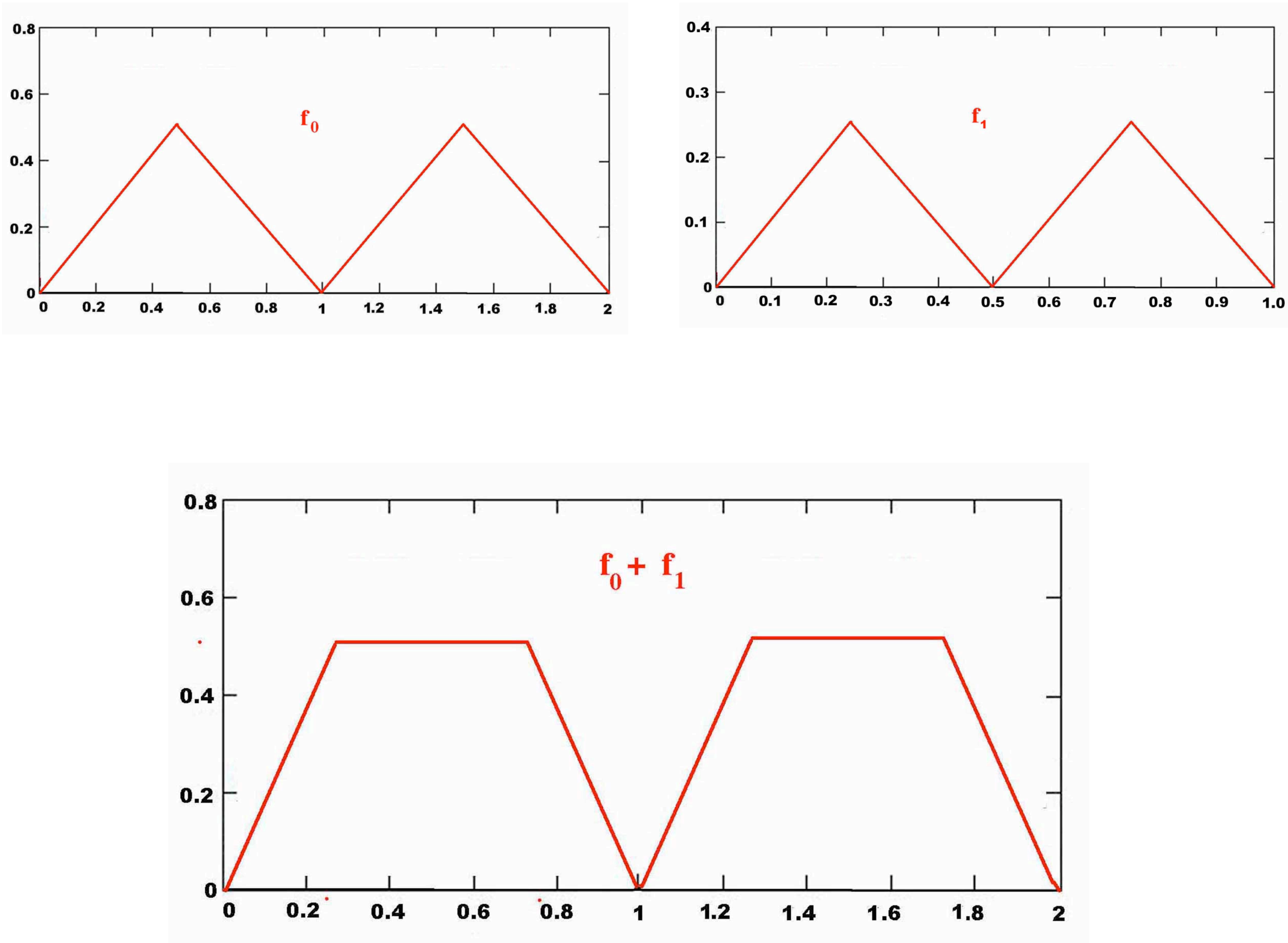
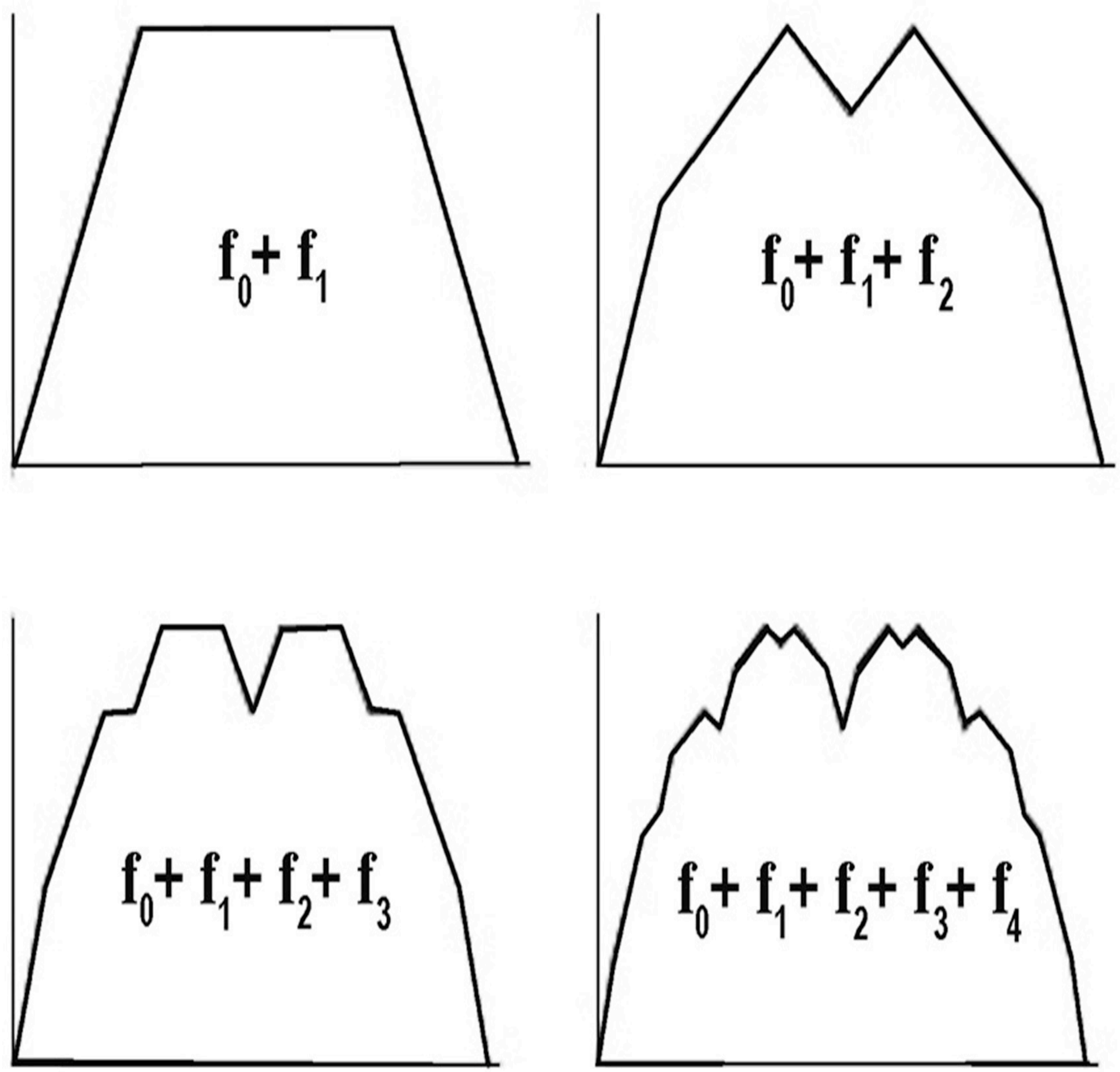
2c



2d



2e

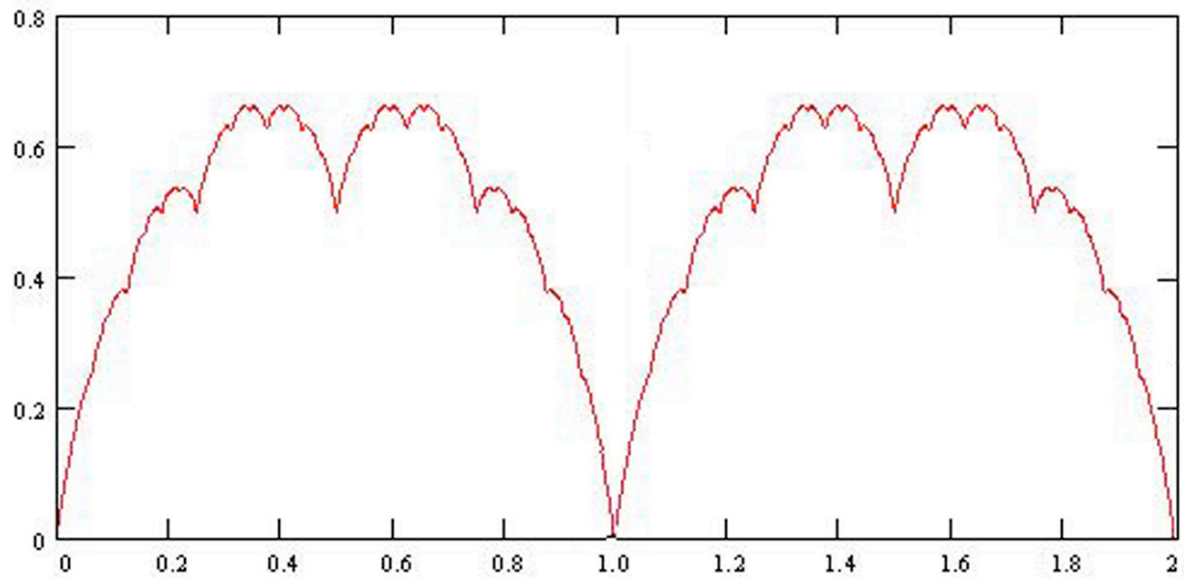
THEOREM 2. There is a function contiunous at each real x but differentiable at **no** real x.

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Example:

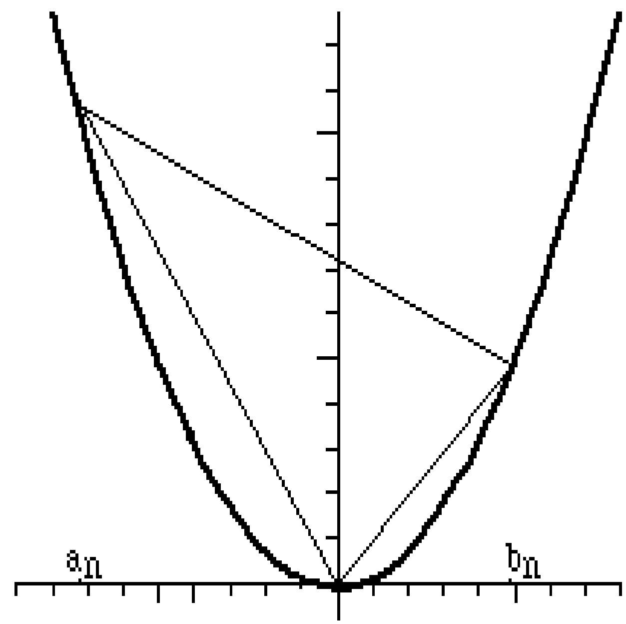
$$f_0\left(\frac{7}{16}\right) = \frac{7}{16}; \quad f_1\left(\frac{7}{16}\right) = \frac{7}{16}; \quad f_2\left(\frac{7}{16}\right) = \frac{1}{16}; \quad \mathbf{f}\left(\frac{7}{16}\right) = \frac{1}{16}.$$

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \leq \sum_{n=1}^{\infty} 2^{-n}$$



Lemma2: Suppose a function $h : \mathbf{R} \rightarrow \mathbf{R}$ is differentiable at x. If a_n and if b_n are such that $\forall n$, $a_n \leq x \leq b_n$, then

$$h'(x) = \lim_{n \to \infty} \frac{h(b_n) - h(a_n)}{b_n - a_n}$$



Section 3. **Stretching zero to one.**

Cantor's Middle Third Set **C** is a subset of [0,1] formed inductively

by deleting middle third open intervals.

Say $({}^{1}/_{3},{}^{2}/_{3})$ in step one.

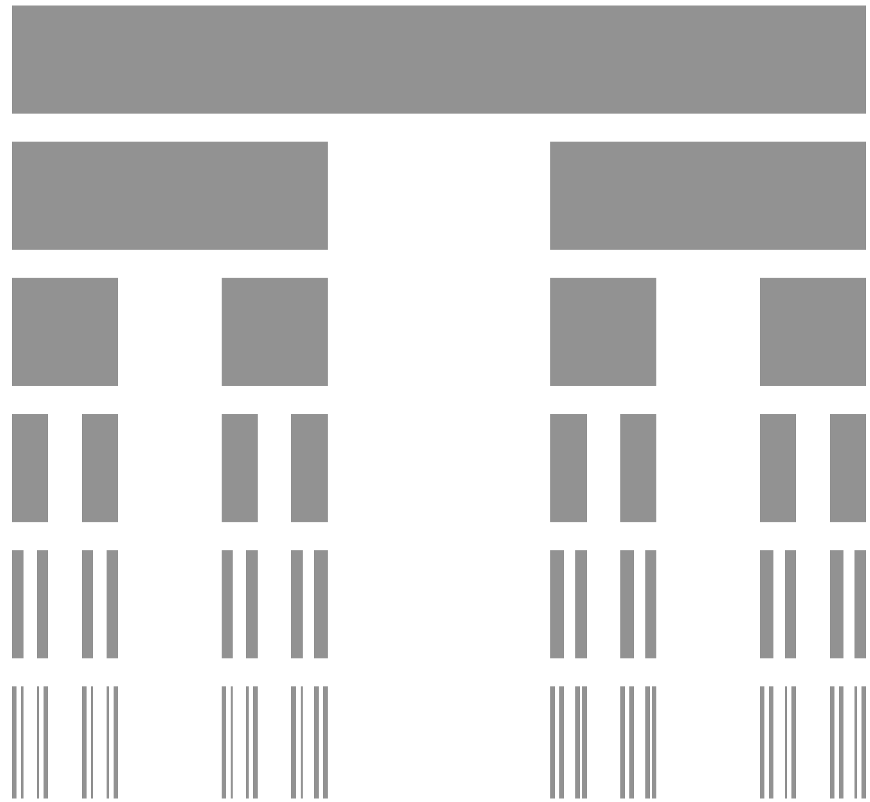
In step two, remove the middle-thirds of the remaining two intervals of step one, they are $(^{1}/_{9},^{2}/_{9})$ and $(^{7}/_{9},^{8}/_{9})$.

In step three, remove the middle thirds of the remaining four intervals.

and so on for infinitely many steps.

What we get is **C**, *Cantor's Middle Thirds Set*.

C is very "thin" and a "spread out" set whose measure is 0 (since the sum of the lengths of intervals removed from [0,1] is 1.

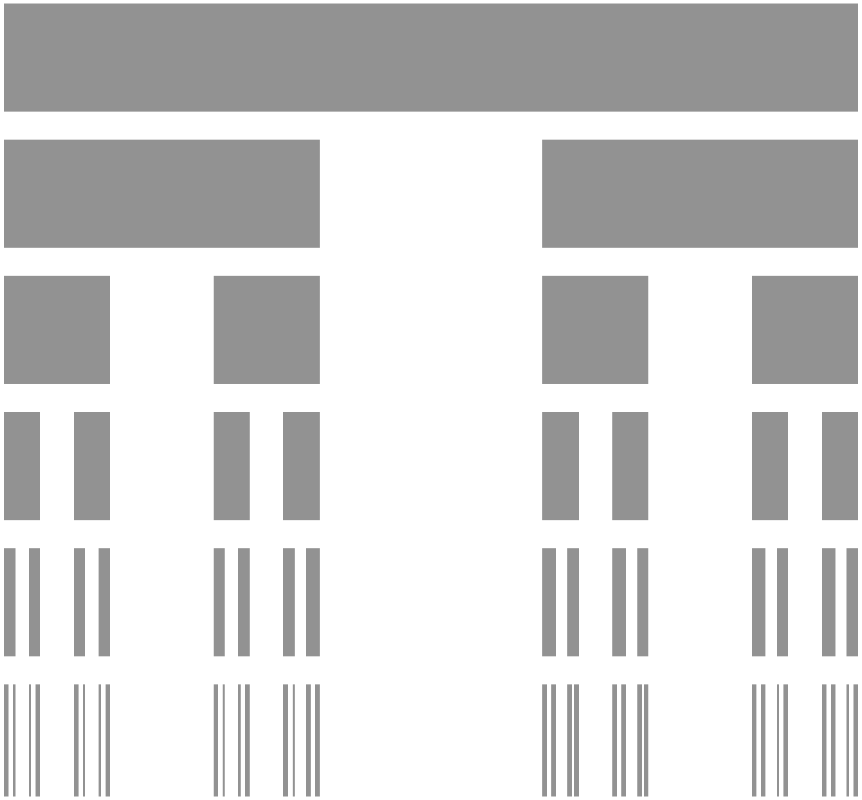


Cantor's Middle Thirds set

As [0,1] is thick and as
C is a thin subset of [0,1],
the following is surprising:

THEOREM 3.

There is a continuous function from **C** onto [0,1].



Cantor's Middle Thirds set

THEOREM 3. There is a continuous function from **C** onto [0,1].

The points of **C** are the points equal to the sums of infinite series of form

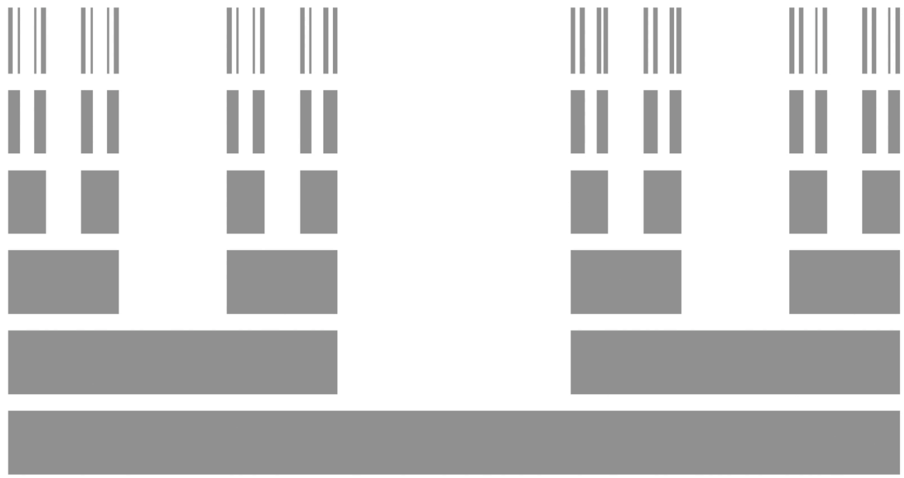
$$\sum_{n=1}^{\infty} 2s(n)3^{-n} \quad \text{where } s(n) \in \{0,1\}.$$

$$F\left(\sum_{n=1}^{\infty} 2s(n)3^{-n}\right) = \sum_{n=1}^{\infty} s(n)2^{-n}$$

defines a continuous surjective function whose domain is **C** and whose range is [0,1].

Example. The two geometric series show $F(^1/_3)=F(^2/_3)=^1/_2$.

Picturing the proof.



Stretch the two halves of step 1 until they join at 1/2.

Now stretch the two halves of each pair of step 2
Until they join at 1/4 and 3/4…
Each point is moved to the sum of an infinite series.

Section 4. Advancing Dimension

$\mathbf{N} \leftrightarrow \mathbf{N}^2$

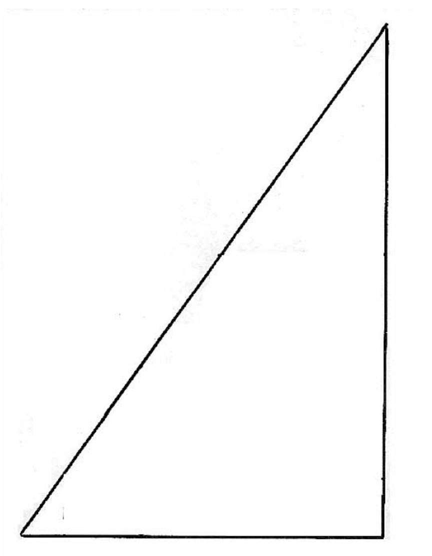
$2^{n-1}(2m-1) \leftrightarrow \langle m,n \rangle$

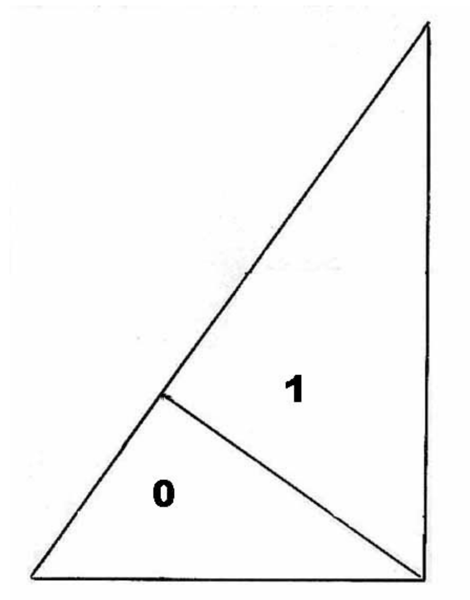
	1	2	3	4
1	1	2	4	8
2	3	6	12	24
3	5	10	20	40
4	7	14	28	56

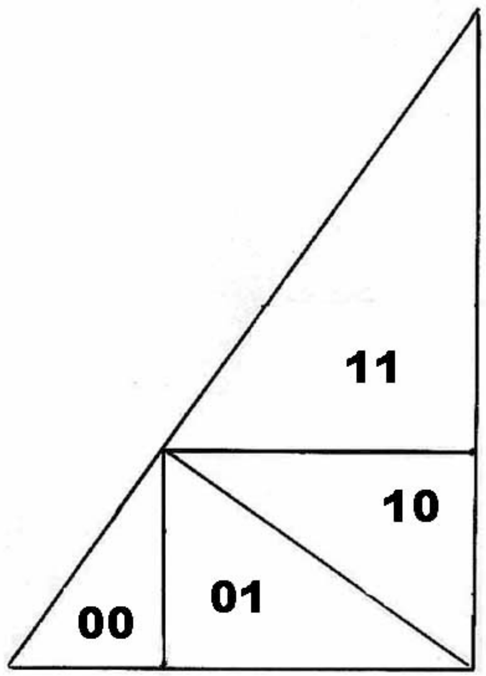
	1	2	3	4
1	<1,1>	<1,2>	<1,3>	<1,4>
2	<2,1>	<2,2>	<2,3>	<2,4>
3	<3,1>	<3,2>	<3,3>	<3,4>
4	<4,1>			

Theorem4. There is a continuous function from [0,1] onto the square.

We'll cheat and do it with the triangle.



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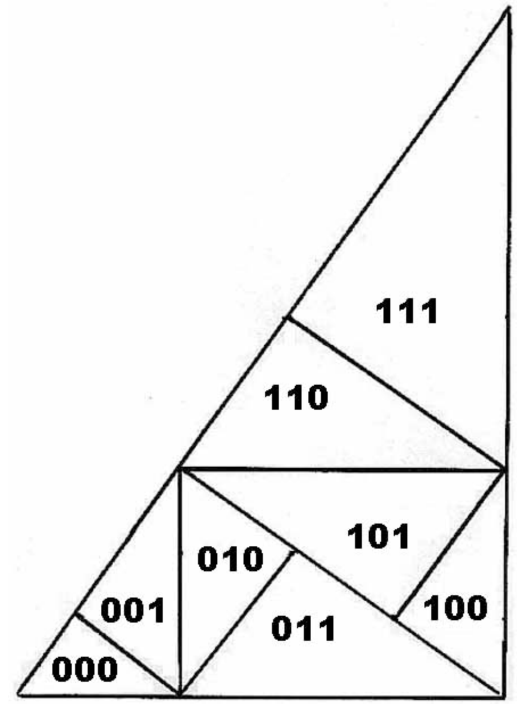
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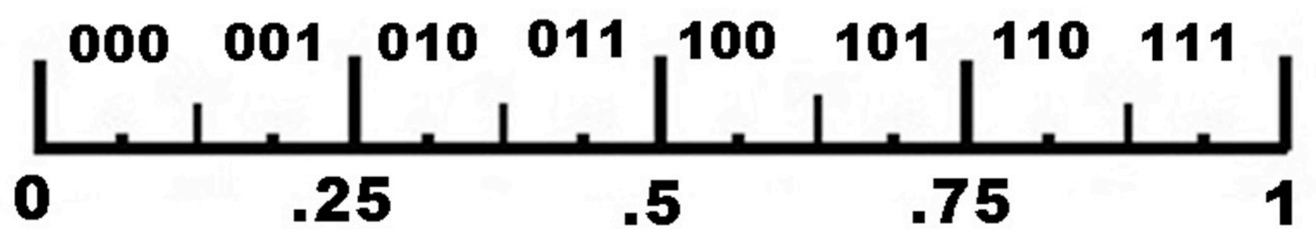
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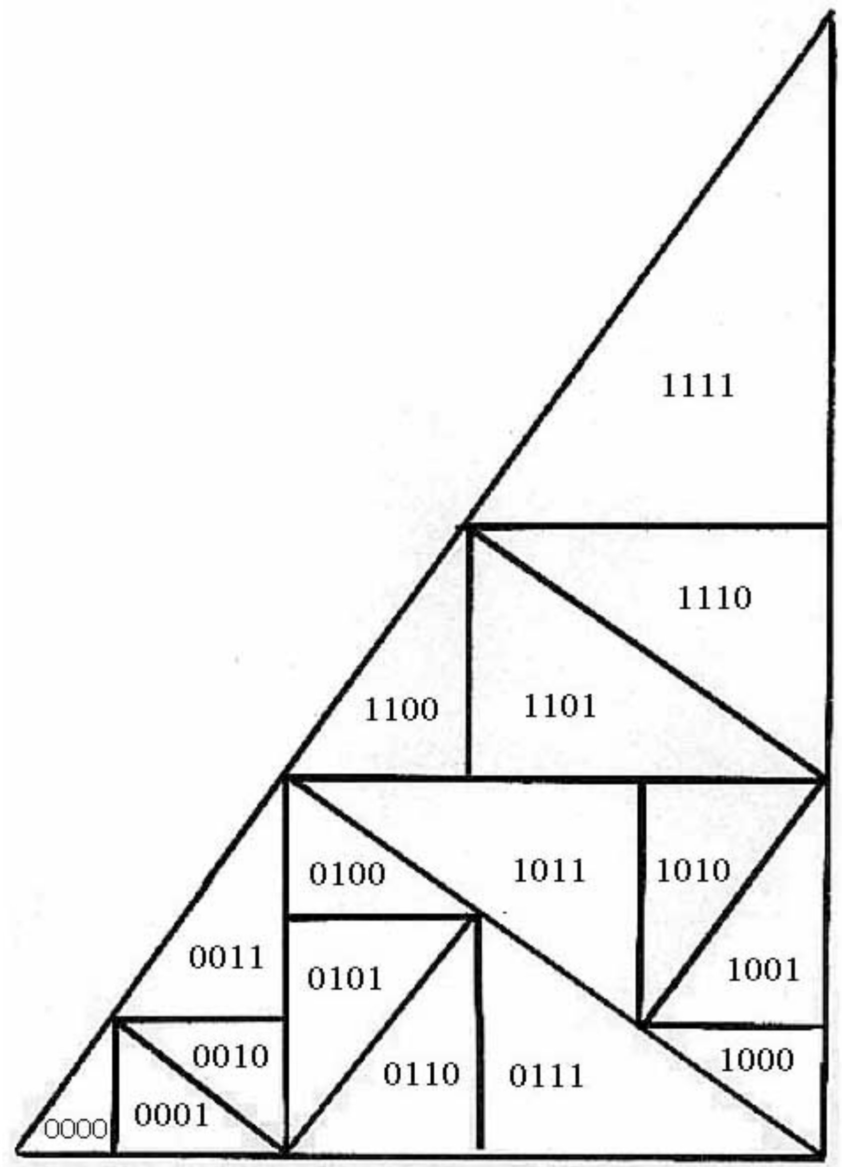
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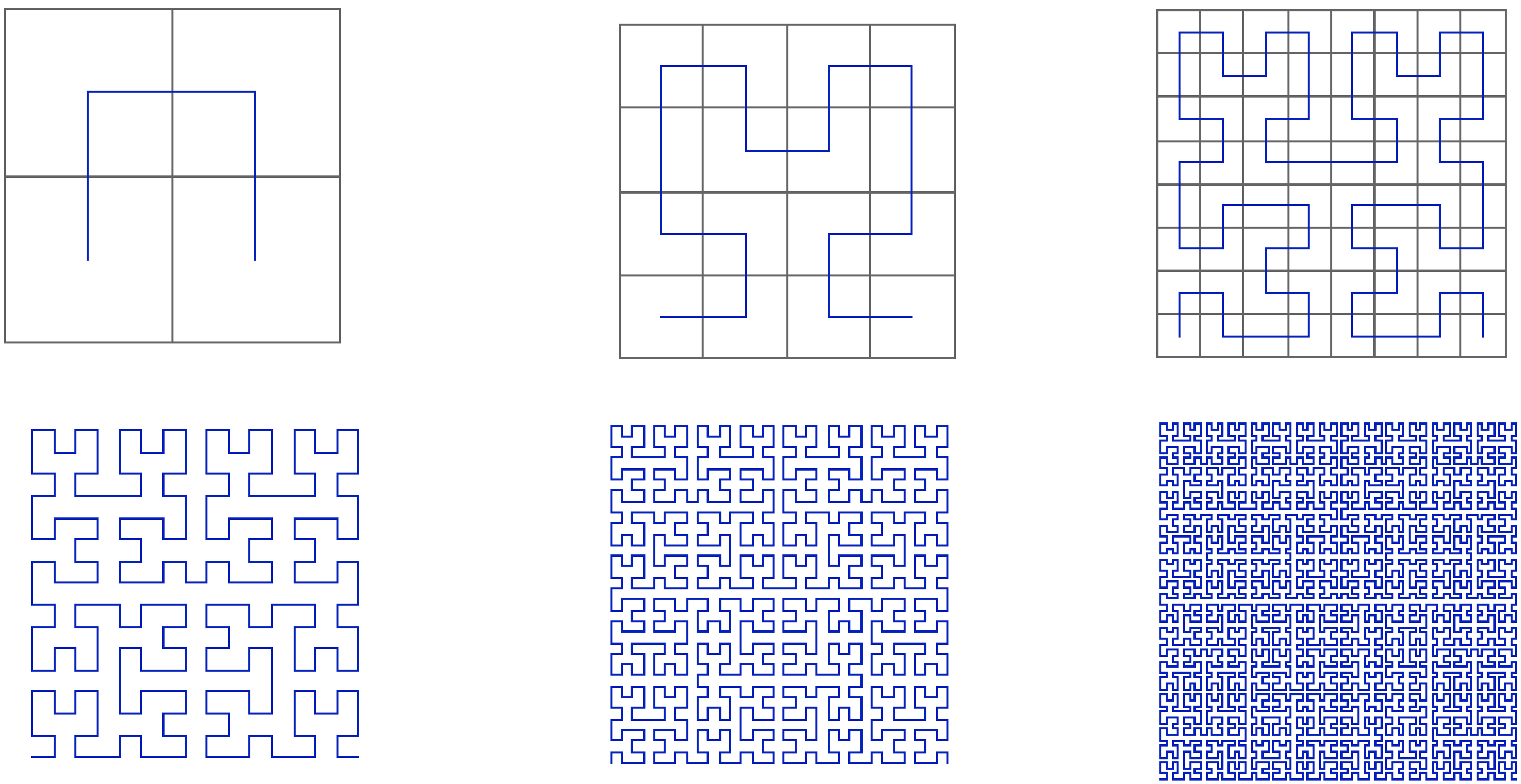
4d

An java animated version of a different Space Filling Curve can be found at

http://www.geom.uiuc.edu/~dpvc/CVM/1998/01/vsfcf/article/sect2/brief_history.html



4e

Section 5. An addition for the irrationals

By an *addition* for those objects $X \in [0,\infty)$ we mean a continuous function $s : X \times X \to X$ (write $x+y$ instead of $s(\langle x,y \rangle)$) such that for $x+y$ the following three rules hold:

(1). $x+y = y+x$ (the commutative law) and

(2). $(x+y)+z = x+(y+z)$ (the associative law).

With sets like **Q**, the set of positive rationals, the addition inherited from the reals **R** works, but with the set **P** of positive irrationals it does not work: $(3+\sqrt{2}\,)+(3-\sqrt{2}) = 6$.

THEOREM5.

The set **P** of positive irrationals has an addition.

Our aim is to consider another object which has an addition
And also “looks like” **P**.

Continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Given an irrational x, the sequence $<a_n>$ is computed as follows:

Let $G(x)$ denote the greatest integer $\leq x$. Let $a_0 = G(x)$.

If $a_0,,,a_n$ have been found as below, let $a_{n+1} = G(1/r)$.

$$x = a_0 + \cfrac{1}{a_1 + \cdot \cdot \cfrac{1}{a_n + r}}$$

Continuing in this fashion we get a sequence which converges to x. Often the result is denoted by

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

However, here we denote it by $CF(x) = < a_0, a_1, a_2, a_3, \ldots >$.

We let <2> denote the constant < 2, 2, 2, 2,…>.
Note <2> = CF(1+ √2) since

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\,..}}}}}} = 1+\sqrt{2}$$

Hint: A quick way to prove the above is to solve for x in $x = 2+\frac{1}{x}$ or $x^2 - 2x - 1 = 0$.

Prove $\langle \underline{1} \rangle = \mathrm{CF}\left(\frac{1+\sqrt{5}}{2}\right)$ and $\langle \underline{1,2} \rangle = \mathrm{CF}\left(\frac{2+\sqrt{3}}{2}\right)$

We add two continued fractions “pointwise,” so
$\langle \underline{1} \rangle + \langle \underline{1,2} \rangle = \langle \underline{2,3} \rangle$ or <2,3,2,3,2,3,2,3,…>.

Here are the first few terms for π, <3,7,15,1,…>.
No wonder your grade school teacher told you
$\pi = 3 + \frac{1}{7}$. The first four terms of CF(π), <3,7,15,1>
approximate π to 5 decimals.

Here are Euler’s first few terms for e,
<2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,….>

Lemma. Two irrationals x and y are "close" as real numbers iff the "first few" partial continued fractions of CF(x) and CF(y) are identical.

For example <2,2,2,2,2,1,1,1,…> and <$\underline{2}$> are close,
but <2,2,2,2,2,2,2,2,2,91,5,5,…> and <$\underline{2}$> are closer.

Here is the “addition:” We define $x \oplus y = z$ if
CF(z) = CF(x) + CF(y).
Then the lemma shows $\oplus$ is continuous.

However, strange things happen:

$$\frac{1+\sqrt{5}}{2} \oplus \frac{1+\sqrt{5}}{2} = 1+\sqrt{2}$$

Problems

1. How many derivatives has

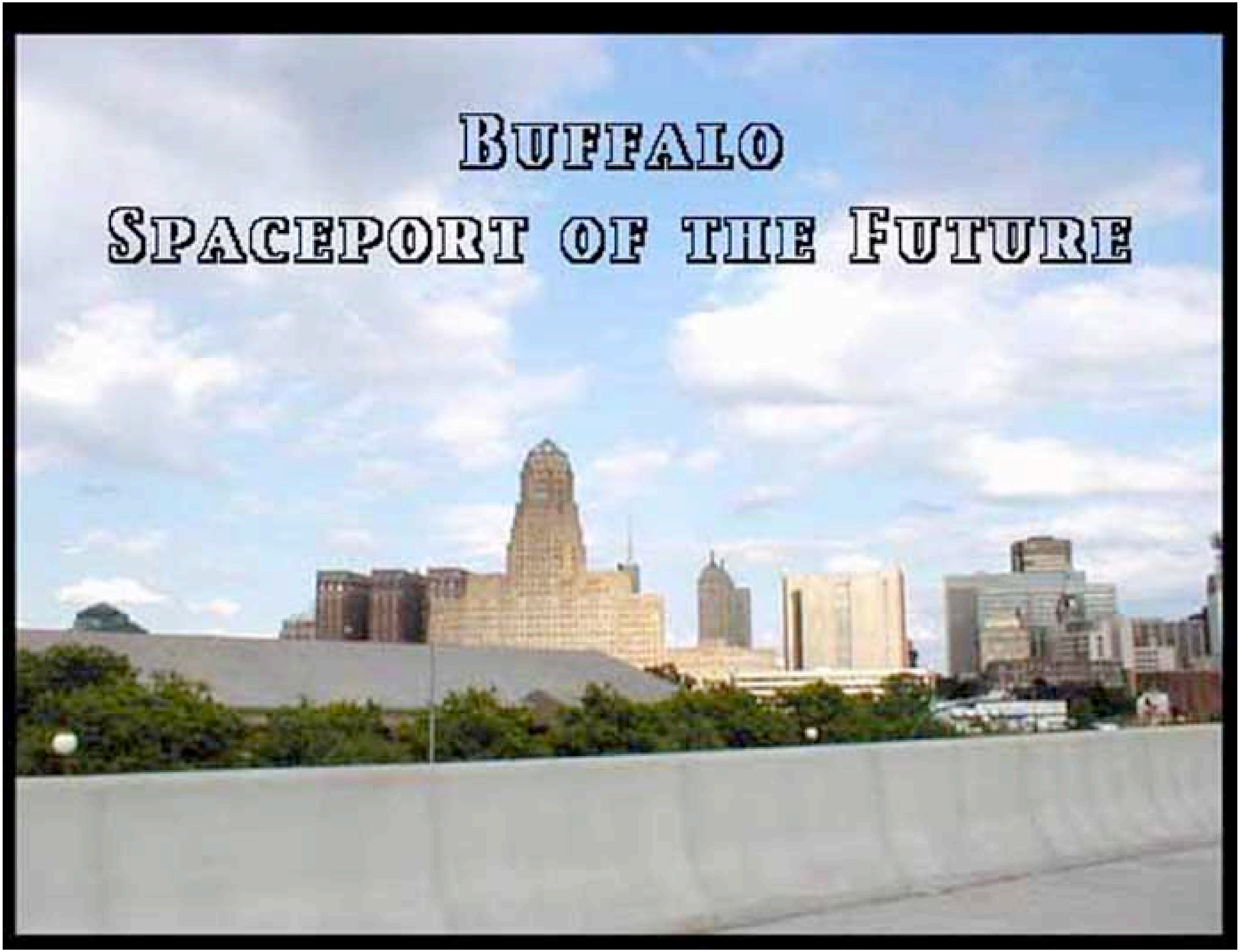
$$g(x) = x^2 \sin \frac{1}{x}$$

2. Prove that each number in [0,2] is the sum of two members of the Cantor set.

Problems

3. Prove there is no distance non-increasing function whose domain is a closed interval in **N** and whose range is the unit square **[0,1]** $\times$ **[0,1]** .

4. Determine $\sqrt{2} \oplus \frac{2+\sqrt{3}}{2}$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BUFFALO
SPACEPORT OF THE FUTURE


Buffalo
Spaceport of the Future


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