COMPACT!
A Tutorial[1]

Scott W. Williams

ABSTRACT. These are the notes from a tutorial on topology presented for the students attending the fifth conference for African Americans in the mathematical sciences. A historical and intuitive approach to highlights of the subject of compact topological spaces are presented.

PREFACE.

I was invited to present a one hour tutorial on Topology for students attending the fifth annual Conference for African American Researchers in the Mathematical Sciences held June 22-26, 1999 at the University of Michigan - Ann Arbor. This five part paper is my notes for the lecture.

When organizer William Massey heard the title "Compact!" of this lecture, he said jokingly, "I hope you will not try to convince us that the finite subcover business is natural." Well, I do not believe it is natural nor do I believe it is intuitive; on the other hand, I believe Topology was invented, in part, to focus upon a few ideas, one of which is compactness and which, in its turn, was invented to expand what we know about finite sets. In this tutorial, I will switch between the intuitive and the accurate, between historical motivations, modern interest and recent results.

What is the importance of compactness? It is used in the definition of the integral; it shows that in linear programming optimal solutions exist on vertices of the feasible set; given a continuous function f from a compact set K to itself, there is an x K and an infinte sequence <n_i> of integers such that $\lim_{i\to\infty} f^{n_i}(x) = x$. So what is this property "compact?"

1. the beginnings

The Čech Mathematician Bernard Bolzano did a number of remarkable things very early. In particular, in 1817, he extracted numbers from the notion of sequence **[Kline1972]**, and gave an early formulation of finite and infinite sets **[Cantor1883, Jarnik1981]**. In the 1830's he showed that a function continuous on a closed interval is bounded, proved that a bounded sequence has a limit point **[Bolzano1841, Jarnik1981]**, and gave the first example of a continuous nowhere differentiable function (usually attributed to Weierstrass 30 years later). The proof of these essentially led to the Bolzano-Weierstrass Theorem **[Taylor1982]**:

1.1. THEOREM. Every infinite bounded subset of reals has a limit point.

A key to this theorem is an axiom implying that the real line has "no holes except at infinity":

---
[1] AMS Subject Classification 5402, 54D30

Call the set B. If B contains an infinite increasing sequence, then the least upper bound of the sequence is a limit point. The infinite decreasing sequence case is analogous. By a partitioning B, we see that a bounded set without monotone infinite sequences must be finite.

Before the 19th century, folks already knew that "small" polynomials attained their maxima and minima on closed intervals, but the Bolzano-Weierstrass Theorem led to what turns out to be one of the chief motivations for studying compactness, Weierstrass' theorem **[Taylor 1982]**:

1.2. THEOREM. Each function continuous on a closed subset of a closed interval attains its maximum.

The essence is that the continuous image of a closed and bounded set is closed and bounded. Today we know Theorem2 to be true for real-valued functions continuous on any compact space; however, theorems 1.1 and 1.2 suggest an intuitive definition:

1.3. DEFINITION. COMPACT = "NO HOLES": This should be interpreted intuitively. The "proofs" we give in 1.4 are intuitive and need either to be fleshed out with the definition given in 1.5 or the one in 5.1.

1.4. EXERCISES.

1. We recognize two possible kinds of holes - holes at infinity and holes nearby.

a. Loosely, compactness requires a kind of boundedness. ∞ is a hole of the non-negative integers $\omega = \{0,1,2,3,4 ...\}$, so ω is not compact. When the hole at ∞ is put in – considering ∞ as a point, the new object $\omega+1$ has no holes.

b. Compactness, loosely, requires a special kind of "closeness." 0 is a hole of the "convergent" sequence $\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \rangle$ and the sequence, without its limit, is not compact. With its limit, it is compact. More generally, if (0,1] with its usual open sets is declared closed in some topology on the line, then [0,1] is not compact in that topology.

2. Intutitively, "hole" implies some kind of an unending process without resolution. So intutively, finite sets are compact.

3. Suppose ***** is a hole of a closed subset F of a compact space X. As a there is stuff of F "close to" *****, there is stuff of X "close to" *****. But X is compact and ***** must be a point of X. Since F is a closed subset of X, it contains all points of X close to it. So, a closed subset of a compact set must be compact.

4. A far away hole: The space ω_1 of all countable ordinal numbers has no holes approachable by a countable sequence, yet is not compact. ω_1+1 is the same object with the hole added.

1.5. DEFINITION. 1. A filter is a family Φ of non-empty sets of X which satisfy the condition: If A,B Φ, then there is a C Φ contained in A∩B.

2. Here is a correct definition of "close to": A filter Φ converges to (clusters at) the point x provided each neighborhood of x contains (intersects) a member of Φ.

Next is a 1950's definition of compactness closely related to the very first study of compact spaces **[Vietoris1921]**. Its virtues are ease in proving technical results. Its faults are in the shift from points to sets.

3. Here is a correct definition of compact: A space is said to be compact provided each filter consisting of closed sets is contained in a converent filter; or equivalently, each filter consisting of closed sets clusters at some point.

2. A SPECIAL EXAMPLE

There are some special compact sets: the convergent sequence and its limit, the unit interval and its products, and the unit circle. These tend to dominate how we think of compact objects. However, the Cantor set is a fundamentally important compact object many people believe to be an aberration - it is not. Here's one reason why **[Urysohn 1914]**:

2.1. THEOREM. Each compact metric space is the continuous image of the Cantor set.

THE CANTOR (MIDDLE-THIRDS) **SET** is the set of all real numbers the sum of an infinite series of the form $\sum_{n=1}^{\infty} \frac{2i_n}{3^n}$, where each i_n = 0 or 1. Note that if each i_n = 1 (=0), then the sum is 1 (0).

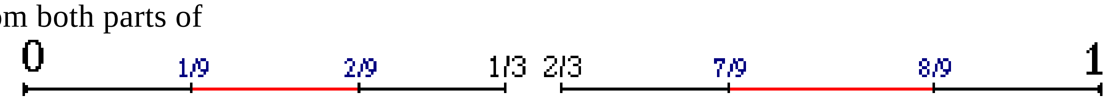
Another construction of the Cantor Middle-Thirds Set proceeds recursively via removing various intervals from [0,1]:

Step1: From [0,1]

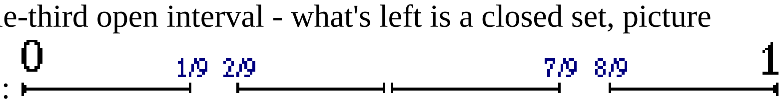
remove the middle-third open interval $\left(\frac{1}{3},\frac{2}{3}\right)$ - we get a closed set, picture

C_1: 

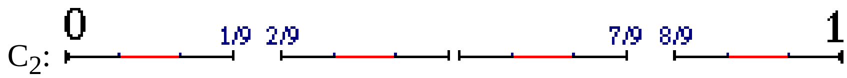
From both parts of

C_1: 

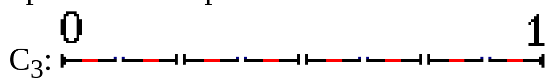
remove the middle-third open interval - what's left is a closed set, picture

C_2: 

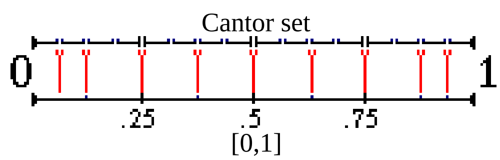
Again from each of the four parts of



remove the middle thirds open interval - picture



And continue …. The Cantor set is the intersection $\mathbf{C} = \cap_{n\ \mathbf{N}} C_n$. As an intersection of closed sets in the compact [0,1] it is closed subset and hence compact. Noticing that the adjacent pairs in **C** can be mapped in an order preserving manner onto the rationals in (0,1). We see the **C** can be also be "pictured" by considering [0,1] and replacing each rational number in (0,1) by two adjacent points. Indeed that picture gives impetus to a special case of Theorem2.1 - [0,1] is the continuous image of **C** {just send adjacent points to one}.



Concerning the Cantor set, one must be careful with intuition. First it is very thin, because the sum of the lengths of the deleted intervals is $\sum_{n=1}^{\infty} \frac{1}{3^n} = 1$; i.e., its measure is zero. On the other hand it has the same size as the entire interval [0,1]. This is strengthened by the problem which appears in W. Rudin's textbook, and on some Ph.D. Qualifying Exams:

2.2. EXERCISE. Each real in the interval [0,2] is the sum of two members of the Cantor set: For $0 \leq b \leq 2$. Consider the graph L_b of the intersection of line x+y=b with subsets of the square $[0,1]^2$. Indeed, each $C_n{}^2 \cap L_b$ is a non-empty closed set identical to some C_k. Thus, 1.5(3) shows $\mathbf{C}^2 \cap L_b = \cap_{n\ \mathbf{N}} C_n{}^2 \cap L_b \neq$. So there are x and y in **C** such that x+y=b.

Recently, the great topologist Mary Ellen Rudin **[Rudin 1999]** solved an outstanding problem generalizing 2.1, which asked for a kind of "triangular inequality" extension of metric known as "monotonically normal." She proved "sufficient" in:

2.3. THEOREM. In order for a compact space X to be the continuous image of a compact linear ordered space it is necessary and sufficient that for each pair consisting of a point x X and its neighborhood G, there exists an open set G_x satisfying two conditions:

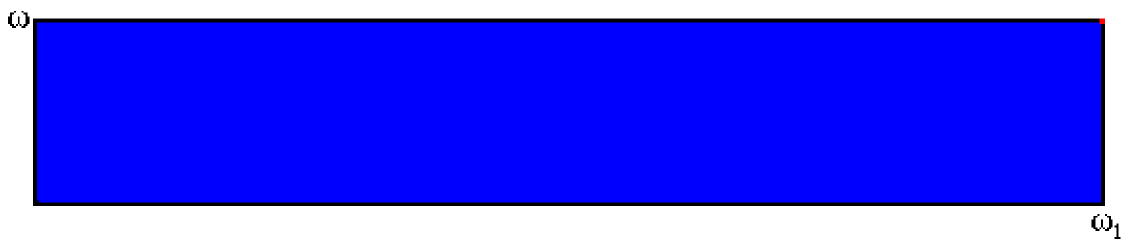
1. x G_x G. 2. If $G_x \cap H_y \neq$, then either y G or x H.

A pre-print of Rudin's paper is available at the web site TOPOLOGY ATLAS http://at.yorku.ca/topology/ {The "trianguar inequality" comment is motivated by observing that in a metric space when B is the open ball about x of radius r, we may take B_x to be the open ball about x of radius $\frac{r}{3}$. Then the triangular inequality proves condition (2)}

3. FUNCTIONAL SEPARATION

Disjoint or non-intersecting closed sets one of which is compact in a (Hausdorff) topological space can be expanded to disjoint open sets - this is called separating them. On the other hand, the distance, in the plane, between the disjoint closed sets, graphs of y=0 and xy=1 is 0. Neither of these sets is compact and ∞ is necessary for this process.

Suppose H and K are disjoint closed sets of a space X. If we can not expand these two to disjoint open sets then there must be some kind of "hole" present. Recall ω and $\omega+1$ from 1.4(1). and ω_1 and ω_1+1 from 1.4(4). Consider the Tychonov Plank, the space X is the product $\omega_1+1\times\ \omega+1$ with the upper right hand corner removed; i.e., $\omega_1+1\times\ \omega+1\setminus\{<\infty,\infty>\}$. The closed sets are the top A = $\omega_1\times\{\infty\}$ and the right hand side B=$\{\infty\}\times\ \omega$.



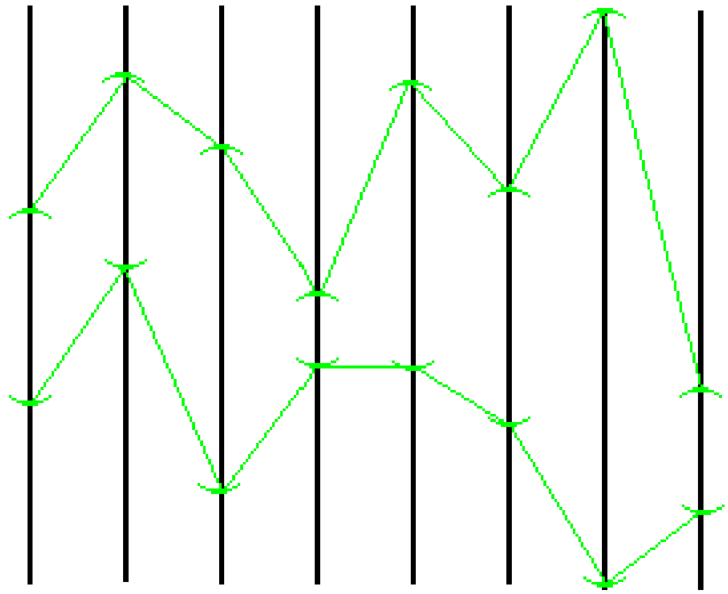
{Key to the proof is that the hole at the end of A (see 1.4.5) is so far away that any open set containing B contains countable sequences converging to A; i.e., limit points belonging to A. Thus, there can be no disjoint open sets containing A and B.}

Even stronger (superficially) than "expansion of disjoint closed sets to disjoint open sets" is Urysohn’s "separation of closed sets by a continuous function."

3.1. THEOREM. If A and B are disjoint closed subsets of a compact space X there is a continuous $f:X\rightarrow[0,1]$ such that f(A)=0 and f(B)=1.

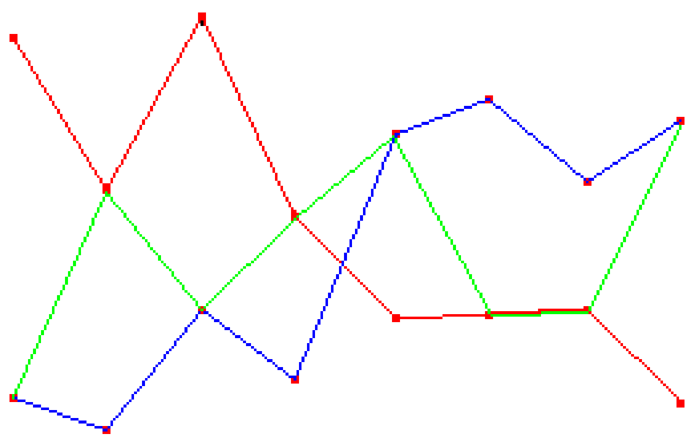
Let us return to products: Fréchet was the first to define a finite product of topological spaces **[Fréchet1910]**. That the product of two compact spaces are compact is intuitively clear: A "hole" in the product of X and Y ought to imply it in a factor. Induction shows "two factors" can be replaced by "finitely many factors." But what about "infinitely many?"

Compact!



Tietze's product topology

In 1923 Tietze first gave for the general definition of the product of spaces, a topology, now called the box topology, on a product of infinitely many spaces to be that which is generated by the product of open sets **[Tietze1923**] (pictured above). This definition is the "right" way to define product in many areas of mathematics (example, coordinate wise addition in Algebra). However, even the product of countably many two element sets is not compact.

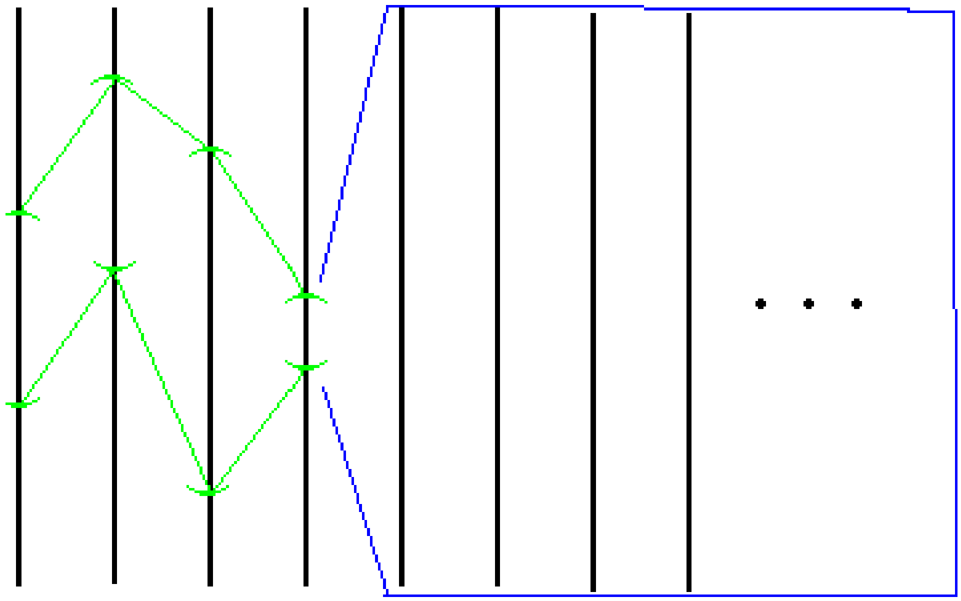


the product of two element sets

Further , an important old and major unsolved problem **[Williams1984]** in topology asks,

Does the product (with the box topology) of countably many copies of [0,1] satisfy the conclusion of Theorem 3.1 ?
The answer to large products is "NO!" in general; i.e., there are disjoint closed subsets A and B of the product $\prod$ of uncountably many copies of [0,1] for which no continuous function f: $\prod \to [0,1]$ satisfies both f(A)=0 and f(B)=1 **[Lawrence1994]**.

Clearly, Tietze's topology is not good for proving theorems about infinite products (e.g., the preservation of compactness, connectedness, metric etc.), and thus we use a product topology **[Tychonov1930]** which, like compactness, extends finite delicately - the topology is generated by a product of open sets which, only finitely often, may be different from the entire factor:



Tychonov product topology

This guarantees that a hole in the product must come from a hole in at least one factor and the standard:

3.2. THEOREM. The product of arbitrarily many compact spaces is compact.

The product of countably many two element spaces is topologically the same as the Cantor set.

4. THE UNIVERSE IN A BOX

Given its derivation from the study of continuity, topology concerns itself with things which are close together while disregarding those which are far apart. Thus, it should be no surprise that we can bound the metric of a space while keeping the topology unaffected by changing the metric to the minimum of 1 and the old distance between two points. In this metric space with a fixed closed set F - the distance d(x,F) between a point x and F forms a continuous function from X $\to$ [0,1] with value 0 on F.

In general, a reasonable axiom for separating points and closed sets in a space X is: given a closed set F   X and x   X\F there is a continuous g:X $\to$ [0,1] such that g(x)=1 and g(F)=0. Using the set **F**, of all continuous f:X $\to$ [0,1], as an index, we build the space P as the product of **F** many copies of [0,1]. The space P is compact and a copy of the space X "sits" in P.

When a "copy" of X sits in P we say X is embedded in P. The above embedding is denoted by ev, for evaluation, and is defined so that the f'th coordinate of ev(x) in the product is f(x).

This tells us when we can consider our space as part of a compact universe:

4.1. THEOREM. In order for a space X to be contained (or embedded) in a compact space it is necessary and sufficient that for each pair consisting of a closed set F   X and x   X\F there is a continuous f:X → [0,1] such that f(x)=0 and f(F)=1.

4.2. DEFINITION. Suppose a space X is contained (embedded) in a compact space K. Its closure (the set of all points in or close to X) is compact (see 1.3). This closure of a copy of X in a compact space is called a compactification of X. We think of a compactification as filling in the holes of X because we are allowing certain non-convergent filters in X to converge "outside of X."

4.3. EXERCISES. 1. [0,1] is a compactficiation of (0,1) and hence the reals.

2. The map t → <cos2πt,sin2πt> : (0,1) → unit circle establishes that the unit circle is also a compactficiation of (0,1) copy of [0,1] with end points identified; i.e., [0,1] and the unit circle, are both compactifications of (0,1) and the reals. Another compactification of (0,1) is the figure 8.
3. There is no continuous function f : [0,1] → [-1,1] such that   x, $0<x\leq 1$, f(x)=sin(1/x); i.e., sin(1/x) : (0,1] → [-1,1] has no extension to [0,1]. But by identifying (0,1] with a copy, its graph G in the plane (map each x → <x,sin(1/x)>), we do see that (0,1] is embedded into its compacification K=G   ({0}× [-1,1]) for which there is a continuous function f : K → [-1,1] which extends sin(1/x), namely, project each <x,y>   K to y. K is a compactification of (0,1].
4. Give an example of a compactification K of the integers and continuous function f : K → {0,1,2} which extends g(n) = n mod3.

From the view of compactifications, whether one takes the universe to be $\mathbf{R}^3$, $\mathbf{R}^{11}$, $\mathbf{R}^\infty$,or one of its subspaces, I say, philosophically, that **the universe is contained in a box** - for example, the compact space P described in the beginning of this section.

4.4. DEFINITION. Prior to Theorem 4.1 above we have nearly described a construction of what is called βX, the Stone-Čech compactification of a non-compact space X. βX is the closure of the "copy" of X and it is called the "largest" compactification **[Willard]** of X because it is characterized by the property that each continuous function f from X to a compact space Y can be extended to a continuous function βf from βX to the same compact space. It is the rule that βX is big - for example, when X is the space of positive integers βX has more points in it than there are reals. For the same reason β(0,1] is also quite large. On the other hand, $\beta\omega_1 = \omega_1+1$.

If the universe is contained in a box, an interesting question to consider is "What is left in the box, when we remove the universe?" In other words, what is the nature of $\beta X - X$, a research area with I was involved in the early 1980s.

5. OPEN COVERS

Borel proved the following in his 1894 thesis: A countable covering of a closed interval by open intervals has a finite subcover **[Hildebrandt 1924]**. It turns out that Borel's approach was similar to the approach Heine used to prove in 1872 that a continuous function on a closed interval was uniformly continuous (actually first proved, but unpublished for 60 years, by Dirichlet in 1852).
In 1898, Lebesgue (and apparently someone named Cousins in 1895) removed "countable" from the hypothesis of Borel's result. Thus, we have the generalized theorem, which is now commonly called the Heine-Borel theorem, and with modern notation, is:

5.1. THEOREM. A subset of $\mathbf{R}^n$ is compact iff it is closed and bounded.

Unfortunately, this notion of "bounded" does not generalize the theorem for metric spaces, and in topology "metric" need not be present. Vietoris' **[Vietoris1921]** seems to have first seriously considered abstract compact spaces and he proved 1.4(3) and proved "expansion of disjoint closed sets to disjoint open sets" (see the notes before theorem 3.1), but, independently, Alexandrov and Urysohn **[Alexandrov and Urysohn1923]** first gave it the modern definition (though the Russians called the notion "bicompact" for many years):

5.2. DEFINITION. A space X is compact if each covering by open sets contains finitely many open sets which cover.

5.3. EXAMPLES. With the sequence $< \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots > \quad \{0\}$ or with $\mathbf{N} \quad \{\infty\}$, we see that an open set containing 0 (or ∞}) contains all but finitely many of the points. Thus, an open cover has a finite subcover. To see that [0,1] is compact, use the least upper bound property.

There is an important consequence of compactness which at first appeared to be a property of completeness. It is now known as the Baire Category Theorem due to Baire (1889) for the reals and Hausdorff (1914) for complete metric spaces. It was E. Čech who saw the earlier results were all a consequence of covering properties of certain subsets of a compact spaces **[Čech1937]**:

5.4. THEOREM. Suppose $\{G_n : n \quad \mathbf{N}\}$ is a countable family of open dense sets in a compact space. Then the intersection $\cap_{n \quad \mathbf{N}} G_n$ is dense.

Note that the reals can be embedded as (0,1) into [0,1]. A complete metric space can be embedded as a such an intersection in a compact space. Using 5.4, Banach gave, in 1931, an elegant proof of the Bolzano 1833 **[Jarnik1981]** result usually attributed to Weierstrass (1852):

5.5. COROLLARY. There is a continuous nowhere differentiable function from the reals to the reals.

REFERENCES

Alexandrov, P. and Urysohn, P. (1923) Sur les espaces topologiques compacts, Bull. Intern. Acad. Pol. Sci. Sér. A., 5-8.

Bolzano, B. Functionlehre, 1833-1841 (from Bolzano's manuscripts in Spisy B. Bolzana 1, Prague 1930.

Bolzano, B. Zahlenlehre, 1842 (from Bolzano's manuscripts in Spisy B. Bolzana 2, Prague 1931.

Cantor, G. (1883) Über unendliche lineare Punktmannigfaltigkiet, Math. Annalen 21, 51-58, 545-591.

Čech, E. (1937) On bicompact spaces, Annals of Math. 38, 823-844.

Fréchet, M. (1910) Les dimensions d'un ensemble abstrait, Math. Ann. 68, 145-168.

Hildebrandt, T. H. (1925) The Borel Theorem and its Generalizations. In J. C. Abbott (Ed.), The Chauvenet Papers: A collection of Prize-Winning Expository Papers in Mathematics. Mathematical Association of America.

Jarnik, V. (1981) Bolzano and the foundations of mathematical analysis, Society of Czechoslovak Mathematicians and Physicists.

Kline, M. (1972) Mathematical Thought: From Ancient to Modern Times. Oxford University Press.

Lawrence, L. B. (1996) Failure of normality in the box product of uncountably many real lines. Trans. Amer. Math. Soc. 348, no. 1, 187--203.

Rudin, M.E. (1972) The box product of countably many compact metric spaces, Gen. Top. Appl. 2, 293-248.

Rudin, M. E. (1999) Nikiel's Conjecture, to appear.

Taylor, A. (1982). A Study of Maurice Fréchet: I. His Early Work on Point Set Theory and the Theory of Functionals. Archive for History of Exact Sciences, 27 (3), 233-295.

Taylor, A. (1985). A Study of Maurice Fréchet: II. Mainly about his Work on General Topology 1909-1928. Archive for History of Exact Sciences 34 (3), 279-380.

Tietze, H. (1923). Beitrage zur allgemeinen topologie I, Math. Ann. 88, 280-312.

Tychonov, A. (1930). Über die topologische Erweiterung von Räume, Math. Ann. 102, 544-561.

Vietoris, L. (1921). Stetige Mengen, Monatsh. für Math. und Phys. 31, 173-204.

Willard, S. (1968). Generally

Williams, S. (1984), Box products, Handbook of Set-Theoretic Topology (K. Kunen and J.E.Vaughan ed.), North-Holland, 169-200.

State University of New York at Buffalo
Buffalo, New York 14214 U.S.A.

sww@buffalo.edu
http://www.acsu.buffalo.edu/~sww/