Positive Knots And Knots With Braid Index Three Have Property-P

W. Menasco and X. Zhang

Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900
menasco@math.buffalo.edu and xinzhang@math.buffalo.edu

Abstract: We prove that positive knots and knots with braid index three in the 3-sphere satisfy the Property P conjecture.

1. Introduction

Let $K$ be a knot in the 3-sphere $S^3$ and $M = M_K$ the complement of an open regular neighborhood of $K$ in $S^3$. As usual, the set of slopes on the torus $\partial M$ (i.e. the set of isotopy classes of unoriented essential simple loops on $\partial M$) is parameterized by

$$\{m/n; \; m, n \in \mathbb{Z}, n > 0, (m, n) = 1\} \cup \{1/0\},$$

so that 1/0 is the meridian slope of $K$ and 0/1 is the longitude slope of $K$. The manifold obtained by Dehn surgery on $S^3$ along the knot $K$ (equivalently, Dehn filling on $M$ along the torus $\partial M$) with slope $m/n$, is denoted by $K(m/n)$ or $M(m/n)$. Of course $K(1/0) = S^3$, and thus the surgery with the slope 1/0 is called the trivial surgery. The celebrated Property P conjecture, introduced by Bing and Matin in 1971 [BMa], states that every nontrivial knot $K$ in $S^3$ has Property P, i.e. every nontrivial surgery on $S^3$ along $K$ produces a non-simply connected manifold. For convenience we say that a class of knots in $S^3$ have Property P if every nontrivial knot in this class has Property P. The following classes of knots were known to have Property P: torus knots [H], symmetric knots [CGLS] (the part for strongly invertible knots was proved in [BS]), satellite knots [G1], arborescent knots [W], alternating knots [DR], and small knots with no non-integral boundary slopes [D]. For a simple homological reason, to prove the conjecture for a knot $K$ one only needs to consider the surgeries of $K$ with slopes $1/n$, $n \neq 0$. A remarkable progress on the conjecture was made in [CGLS]; it was proved there that for a nontrivial knot, only one of $K(1)$ or $K(-1)$ could possibly be a simply connected manifold. Another remarkable result was given in [G1], which told us that if the Property P conjecture is false, then the Poincare conjecture is false. For some earlier progresses on the conjecture, see [K, Problem 1.15] for a summary. In this paper we prove

**Theorem 1** Positive (or negative) knots in $S^3$ satisfy the Property P conjecture.

\(^1\)Partially supported by NSF grant DMS 9971561
Recall that a knot is positive if it can be represented as the closure of a positive n-braid for some n, i.e. a braid which involves the standard elementary braid generators \( \sigma_1, \ldots, \sigma_{n-1} \) (Figure 1 gives \( \sigma_1, \sigma_2 \) when \( n = 3 \)) but not their inverses. A negative knot is similarly defined, which is actually the mirror image of a positive knot.

**Theorem 2** Knots in \( S^3 \) with braid index three satisfy the Property P conjecture.

Theorem 1 is a quick application of the Casson invariant. We refer to [AM] for the definition and basic properties of the Casson invariant. The Casson invariant is an integer valued topological invariant defined for homology 3-spheres and for knots in homology 3-spheres. If a homology 3-sphere has non-zero Casson invariant, then the manifold has an irreducible representation from its fundamental group to the group \( SU(2) \), which implies in particular that the manifold is non-simply connected. For a knot \( K \) in \( S^3 \), if \( \Delta_K(t) \) denotes the normalized Alexander polynomial of \( K \), i.e. satisfying \( \Delta_K(1) = 1 \) and \( \Delta_K(t^{-1}) = \Delta_K(t) \), then the Casson invariant \( C_K \) of \( K \) is equal to the integer \( \frac{1}{2} \Delta_K'(1) \) [AM]. Further the Casson invariant of the manifold \( K(1/n) \) is equal to \( nC_K \). So \( C_K \neq 0 \) implies in particular that the knot \( K \) has Property P. Note that the Conway polynomial of a knot in \( S^3 \) is a single variable polynomial \( \nabla_K(x) \in \mathbb{Z}[x] \) with only even powers and \( \nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t) \), from which one can easily deduce that the coefficient of \( x^2 \) in \( \nabla_K(x) \) is equal to the Casson invariant of \( K \). For a nontrivial positive knot \( K \) in \( S^3 \), it has been proved in [V] that the coefficient of \( x^2 \) in \( \nabla_K(x) \) is a positive integer. Hence such knot has Property P. A negative knot is just a mirror image of a positive knot, and obviously a knot has Property P if and only if its mirror image does. Theorem 1 now follows.

The rest of the paper is devoted to the proof of Theorem 2. The main tools we shall use are the Casson invariant and essential laminations. We refer to [GO] for the definition and basic properties of an essential lamination. In section 2 we give an outline of the proof of Theorem 2. Actually the proof of Theorem 2 is reduced there to that of two propositions, Proposition 3 and Proposition 4. These two propositions will then be proven in section 3 and section 4 respectively.

### 2. Proof of Theorem 2

Recall that the 3-braid group, \( B_3 \), has the following well known Artin presentation:

\[
B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle
\]

where \( \sigma_1 \) and \( \sigma_2 \) are elementary 3-braids as shown in Figure 1. If we let \( a_1 = \sigma_1, a_2 = \sigma_2, a_3 = \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1}, \) then \( B_3 \) also has the following presentation in generators \( a_1, a_2 \) and \( a_3 \) (see [X]):

\[
B_3 = \langle a_1, a_2, a_3 \mid a_2 a_1 = a_3 a_2 = a_1 a_3 \rangle.
\]
In this paper we shall always express a 3-braid as a word \( w(a_1, a_2, a_3) \) in letters \( a_1, a_2, a_3 \). Such a word is called positive if the power of every letter in the word is positive. A positive word \( w = a_{\tau_1} \cdots a_{\tau_k} \) is said to be in non-decreasing order \((ND-order)\) if the array of its subscripts \((\tau_1, \ldots, \tau_k)\) satisfies

\[
\tau_{j+1} = \tau_j \quad \text{or} \quad \tau_{j+1} = \tau_j + 1 \quad (\text{mod } 3 \text{ if } \tau_j + 1 = 4) \quad \text{for } j = 1, \ldots, k - 1.
\]

One can define negative word and non-increasing order \((NI-order)\) similarly. Let \( P \) be the set of positive words in \( ND \)-order, let \( N \) be the set of negative words in \( NI \)-order, and let \( \alpha = a_2 a_1 \). It is proven in [X] that for any 3-braid, there is a representative in its conjugacy class that is a shortest word in \( a_1, a_2, a_3 \) and is of the form

(i) a product of \( \alpha^k \) and a word (maybe empty) in \( P \) for some non-negative integer \( k \); or

(ii) a product of \( \alpha^k \) and a word (maybe empty) in \( N \) for some non-positive integer \( k \); or

(iii) a product of a word in \( P \) and a word in \( N \).

where the meaning of the shortest is that the length, i.e. the number of letters of the representative is minimal among all representatives in the conjugacy class of the braid. Such representative of a 3-braid is said to be in normal form. We shall only need to show that if \( K \) is a nontrivial knot in \( S^3 \) which is the closure of a 3-braid in normal form (i) or (ii) or (iii), then it has Property P.

Recall that a word \( w(a_1, a_2, a_3) \) is called freely reduced if no adjacent letters are inverse to each other, and is called cyclically reduced if it is freely reduced and the first letter and the last letter of the word are not inverse to each other. Given a word \( w(a_1, a_2, a_3) \), one can combine all adjacent letters of the same subscript into a single power of the letter, called a syllabus of the word in that subscript. A word \( w \) is called syllabus reduced if it is expressed as a word in terms of syllabuses as

\[
w = a_{r_1}^{m_1} a_{r_2}^{m_2} \cdots a_{r_k}^{m_k}
\]

such that \( a_{r_j} \neq a_{r_{j+1}} \) for \( j = 1, \ldots, k - 1 \). A word is called cyclically syllabus reduced if it is syllabus reduced and its first and last syllabuses are in different subscripts.

Let \( P^* \) denote the set of all positive words in \( a_1, a_2, a_3 \) such that between any two syllabuses in \( a_3 \) both \( a_1 \) and \( a_2 \) occurs. Obviously any 3-braid of norm form (i) is contained in \( P^* \). Let \( \beta \) be a 3-braid in \( P^* \). Suppose that \( a_3^k \) is a syllabus in \( \beta \) which is proceeded immediately by \( a_1 \). Then one can eliminate the syllabus \( a_3^k \) with the equality \( a_1 a_3^k = a_2^k a_1 \)
to get an isotopic braid which is still in $P^*$ but with one less number of syllabuses in $a_3$. Similarly if a syllabus $a_i^k$ is followed immediately by $a_2$, then one can eliminate the syllabus $a_i^k$ with the equality $a_i^k a_2 = a_3 a_i^k$ to get an isotopic braid which is still in $P^*$ but with one less number of syllabuses in $a_3$. We shall call this process index-3 reduction. So for any given $\beta \in P^*$, we can find, after a finitely many times of index-3 reduction, an equivalent braid representative $\beta'$ in $P^*$ for $\beta$ such that every syllabus in $a_3$ occurring in $\beta'$ can only possibly be proceeded immediately by $a_2$ and likewise can only possibly be followed immediately by $a_1$. We call a word $\beta$ in $P^*$ index-3 reduced if every syllabus in $a_3$ occurring in $\beta$ is neither proceeded immediately by $a_1$ nor followed immediately by $a_2$.

Let $P^n$ denote the set of 3-braids of the form $\beta = a_i^{-q}\delta$, where $q = 0$ or $1$ and $\delta \in P^*$ is a non-empty word, such that $\beta$ is cyclically reduced and $\delta$ is index-3 reduced. Obviously $P$ is contained in $P^n$.

**Proposition 3** Suppose that $K$ is a knot in $S^3$ which is the closure of a 3-braid $\beta = a_i^{-q}\delta$ in $P^n$ such that $\delta$ contains at least four syllabuses but $\beta$ is not one of the words in the set $E = \{ a_1^{-1}a_2a_3^2a_1a_2, a_1^{-1}a_2a_3a_1a_2a_3, a_1^{-1}a_3a_1a_2^2a_3, a_1^{-1}a_3a_1a_2a_3^2, a_2^{-1}a_3a_1a_2a_3, a_2^{-1}a_3a_1^2a_2a_3, a_2^{-1}a_1a_2a_3^2a_1, a_2^{-1}a_2a_3a_1a_2, a_3^{-1}a_1a_2a_3^2a_2, a_3^{-1}a_2a_3a_1a_2 \}$. Then $K$ has positive Casson invariant and thus has Property $P$.

Later on we shall refer the set $E$ given in Proposition 3 as the excluded set.

**Proposition 4** Suppose that $K$ is a knot in $S^3$ which is the closure of a 3-braid $\beta = \delta\eta$ which is in normal form (iii), i.e. $\delta \in P$ and $\eta \in N$. Suppose that either

1. each of $\delta$ and $\eta$ has a syllabus of power larger than one, or
2. each of $\delta$ and $\eta$ contains at least two syllabuses, or
3. one of $\delta$ and $\eta$ contains at least four syllabuses and the other has length at least two.

Then each of $K(1)$ and $K(-1)$ is a manifold which contains an essential lamination.

If a closed 3-manifold has an essential lamination, then its universal cover is $\mathbb{R}^3$ [GO] and thus in particular the manifold cannot be simply connected. Hence any knot as given in Proposition 4 has Property $P$ by [CGLS].

Given Propositions 3 and 4, we can finish the proof of Theorem 2 as follows. For a braid $\beta$, we use $\hat{\beta}$ to denote the closure of $\beta$. Let $K \subset S^3$ be a nontrivial knot with index 3. Let $\beta$ be a 3-braid in normal form such that $\hat{\beta} = K$.

First we consider the case that $\beta$ is in normal form (i), i.e. $\beta = \alpha^k \delta$ with $\delta \in P$ and $k \geq 0$. If $\beta$ contains at least four syllabuses and belongs to $P^n$, then $K = \hat{\beta}$ has positive Casson invariant by Proposition 3. So the knot $K$ has Property $P$ in this case. If $\beta$ has less than four syllabuses, then up to conjugation in $B_3$, $\beta = a_2a_1a_3$ or $\beta = a_1a_2^2a_3$ for
\(i, j, m \geq 0\). It is easy to see that in this case \(\hat{\beta}\) is an arborescent knot and thus by [W], \(K = \hat{\beta}\) has Property P. For instance if \(\beta = a_1^i a_2^j a_3^m\), then \(\hat{\beta}\) is as shown in Figure 2 which shows in fact that \(\hat{\beta}\) is a Montesinos knot. So we may assume that \(\beta\) has at least four syllabuses but is not in \(P^a\). This implies that in \(\beta = \alpha^k \delta\), we have \(k > 0\) and \(\delta\) starts with a syllabus in \(a_3\). Performing index-3 reduction on \(\beta\) once, we get an equivalent 3-braid \(\beta' \in P^r\) which is index-3 reduced, i.e. \(\beta' \in P^a\). If \(\beta'\) has less than four syllabuses, then again \(K = \hat{\beta} = \hat{\beta}'\) is an arborescent knot and thus has Property P. If \(\beta'\) contains at least four syllabuses, we may apply Proposition 3 to get Property P for the knot.

If \(\beta\) is of normal form (ii), then the mirror image of \(\beta\) is a braid of normal form (i) and thus the knot \(K = \hat{\beta}\) has Property P.

Figure 2: the closure of \(a_1^i a_2^j a_3^m\) is a Montesinos link

Suppose finally that \(\beta = \delta \eta\) is a braid of normal form (iii) (recall that \(\delta \in P\) and \(\eta \in N\)). By Proposition 4, we may assume that each of the conditions (1)-(3) in Proposition 4 does
not hold for $\beta = \delta \eta$. Then $\beta$ is a word of the form $\delta a_i^{-1}$ or $a_i \eta$, or $\beta = \delta a_i^{-k}$ where $k > 1$ and $\delta$ contains at most three syllabuses each having power 1, or $\beta = a_i^k \eta$ where $k > 1$ and $\eta$ contains at most three syllabuses each having power $-1$. If $\beta$ contains at most four syllabuses, then $K = \hat{\beta}$ is an arborescent knot and thus has Property P. (Figure 3 shows this for the case that $\beta = a_i^{-1} a_2^i a_3^{a_1^{-1}}$. Other cases can be treated similarly). So we may only consider the cases when $\beta = \delta a_i^{-1}$ or $\beta = a_i \eta$, each containing at least five syllabuses. If $\beta = \delta a_i^{-1}$, then $\beta$ is conjugate to $\beta' = a_i^{-1} \delta$ which is in $P^n$. Hence if $\beta'$ does not belong to the excluded set $E = \{ a_1^{-1} a_2 a_3 a_1 a_2 a_3, a_1^{-1} a_3 a_2 a_3 a_1^{-1} a_2 a_3 a_1 a_2^2, a_2^{-1} a_3 a_2 a_3 a_1^{-1} a_2 a_3 a_1 a_2^2, a_2^{-1} a_3 a_2 a_3 a_1^{-1} a_2 a_3 a_1 a_2 \}$, then $K = \hat{\beta}'$ has positive Casson invariant by Proposition 3 and thus has Property P. If $\beta'$ is in the excluded set $E$, then $K = \hat{\beta}'$ is an arborescent knot and thus has Property P [W]. (Figure 4 illustrates this for the case $\beta = a_1^{-1} a_2 a_3 a_1 a_2 a_3$). Other cases can be checked similarly. Finally, if $\beta = a_i \eta$, then its mirror image is a braid in $P^n$, which is a case we have just discussed. This completes the proof of Theorem 2.

Propositions 3 and 4 will be proved in subsequent two sections which constitutes the rest of the paper.

![Figure 4: the closure of the braid $a_1^{-1} a_2^3 a_1 a_2 a_3$ is a Montesinos knot](image)

3. Proof of Proposition 3

We retain all definitions and notations established earlier. For a 3-braid $\beta$, we use $n_\beta$ to denote the number of syllabuses in $a_3$ occurring in $\beta$ and use $s_\beta$ to denote the number of syllabuses of $\beta$. Obviously if $\beta'$ is the braid obtained from $\beta \in P^*$ after some non-trivial index-3 reduction, then $\hat{\beta}' = \hat{\beta}$ but $n_{\beta'} < n_\beta$. For a syllabus $a_3^{b_3} = a_1^{-1} a_2^k a_1$, we shall always assume its plane projection corresponds naturally to $a_1^{-1} a_2^k a_1$ as shown in Figure 6 (a). Hence, every 3-braid in letters $a_1, a_2, a_3$ has its canonical plane projection; namely in the projection plane we place vertically (from top to bottom) and successively the projections of letters occurring in the braid, corresponding to their natural order from left to right.
Whenever we need consider a plane projection of a 3-braid, the canonical one is always assumed unless specifically indicated otherwise.

The basic tool we are going to use to prove the proposition is the following crossing change formula (*) of the Casson invariant. If \( K_+ \), \( K_- \) are oriented knots and \( L_0 \) an oriented link with two components in \( S^3 \) such that they have identical plane projection except at one crossing they differ as shown in Figure 5, then the Casson invariant of \( K_+ \) and \( K_- \) satisfy the following relation:

\[
C_{K_+} - C_{K_-} = lk(L_0)
\]

where \( lk(L_0) \) is the linking number of \( L_0 \). This formula can be found on page 141 of [AM] (note that there was a print error there, \( K_+ \) and \( K_- \) in Figure 36 and Figure 37 of [AM] should be exchanged). The idea of proof of the proposition is repeatedly applying the

\[
\begin{array}{c}
\text{Figure 5:} \ K_+ \ (\text{the left figure}), \ K_- \ (\text{the middle figure}) \ \text{and} \ L_0 \ (\text{the right figure})
\end{array}
\]

formula (*) to a 3-braid of the type as given in the proposition to reduce the complexity of the braid and inductively prove its positivity. To do so, we first need to estimate the linking number of a two-component link which is the closure of a 3-braid of relevant type. Given an oriented link \( L \) of two components \( L_1 \) and \( L_2 \), we shall calculate the linking number of \( L \) as follows (cf. [Rn]). Take a plane projection of \( L \). A crossing of \( L \) as shown on the left of Figure 5 has positive sign 1 and the crossing in the middle of the figure has negative sign \(-1\). The linking number of \( L \) is the algebraic sum of the crossing signs at those crossings of \( L \) where \( L_1 \) goes under \( L_2 \).

For a 3-braid \( \beta \), we shall always orient each component of \( \hat{\beta} \) in such a way that the induced orientation on each strand of \( \beta \) in its canonical projection is pointing downward in the projection plane.

Suppose that a two component link \( L = L_1 \cup L_2 \) is the closure of a 3-braid \( \beta \). At a crossing corresponding to a letter \( a_1 \) or \( a_2 \) or \( a_1^{-1} \) or \( a_2^{-1} \) appeared in \( \beta \), if the under strand is from \( L_1 \) and the upper strand is from \( L_2 \), then this crossing contributes negative one to the linking number of \( L = \hat{\beta} \) when the crossing is corresponding to \( a_1 \) or \( a_2 \) and positive one when the crossing is corresponding to \( a_1^{-1} \) and \( a_2^{-1} \); and if the under strand is from \( L_2 \) or the upper strand is from \( L_1 \), then this crossing contributes zero to the linking number of \( L = \hat{\beta} \). The linking number of \( L = \hat{\beta} \) is the sum of the contributions from all the crossings of \( \beta \). If \( \beta' \) is a portion of \( \beta \), we use \( l(\beta') \) to denote the total contribution to the linking number of \( L \) coming from all the crossings of \( \beta' \). In particular \( l(\beta) = lk(L) \). Also if we decompose \( \beta \) into portions \( \beta = \beta_1 \beta_2 \cdots \beta_k \), then \( l(\beta) = l(\beta_1) + l(\beta_2) + \cdots + l(\beta_k) \).
Lemma 5 Let $\delta \in P^*$ be a $3$-braid satisfying: (1) $n_{\delta} = 0$; (2) $\delta$ is a link of two components; and (3) both $a_1$ and $a_2$ appear in $\delta$. Then $lk(\delta) < 0$.

Proof. Recall that $n_{\delta} = 0$ means that $\delta$ is a word in letters $a_1$ and $a_2$ only. Also, $\delta \in P^*$ is a positive word. The conclusion of the lemma is now obvious. ◊

Lemma 6 Let $\beta = a_1^{-1}\delta$ be an element in $P^a$ satisfying: (1) $n_{\delta} = 0$; (2) $\hat{\beta}$ is a link of two components; and (3) $\delta$ contains at least four syllabuses. Then $lk(\hat{\beta}) < 0$.

Proof. Again $\delta$ is a positive word without letter $a_3$. If $i = 1$, then $\delta$ must start with a syllabus in $a_2$ and also ends with a syllabus in $a_2$ since $\beta$ is cyclically reduced. Since $\delta$ contains at least four syllabuses, it follows that $\beta = a_1^{-1}a_2^ja_1^ka_2^ma_1^na_2^p\cdots$ for some $j, k, m, n, p > 0$. One can easily verify that $lk(\hat{\beta}) \leq lk(a_1^{-1}a_2^ja_1^ka_2^ma_1^na_2^p) < 0$. Similarly one can treat the $i = 2$ case. Consider now the case $i = 3$. If $\delta$ starts with $a_2$, then $\beta = a_3^{-1}a_2^ja_3^ka_2^ma_3^na_2^p\cdots$ for some $j, k, m, n > 0$, which is conjugate to $a_1^{-1}a_2^ja_1^ka_2^ma_1^na_2^p\cdots a_2$. So if $j > 1$, then we are back to the case $i = 1$ (all required conditions remain valid). If $j = 1$, then $\beta$ is conjugate to $\beta' = a_1^{-1}a_2^ja_1^na_2^p\cdots a_2$. Hence by Lemma 5 we see that $lk(\hat{\beta}) = lk(\hat{\beta'}) < 0$. Similarly one can treat the case when $\delta$ starts with $a_1$.

Lemma 7 Let $L = L_1 \cup L_2$ be a link of two components in $S^3$ which is the closure of a braid $\beta = a_1^{-1}\delta$ in $P^a$ such that $\delta$ contains at least four syllabuses. Then the linking number $lk(L)$ of $L$ is non-positive.

Proof. We may assume that $\beta$ has been chosen in its conjugacy class, subject to satisfying all conditions of the lemma, to have the minimal number $n_{\delta}$. By Lemmas 5 and 6, we may assume that $n_{\delta} > 0$.

Given the canonical plane projection of a $3$-braid $\eta$, we shall always call the strand of $\eta$ which starts at the top left corner the strand $1$ of $\eta$, call the strand which starts at the top middle place the strand $2$ of $\eta$, and call the strand which starts at the top right corner the strand $3$ of $\eta$.

Let $a_3^k = a_1^{-1}a_2^j a_1$ be a syllabus in $a_3$ occurring in $\delta$. Its crossings are shown in Figure 6 (a). Obviously the total contribution of the syllabus to the linking number is at most one; i.e. $l(a_3^k) \leq 1$. More precisely, one can easily verify that the following claim holds.

Claim 1. If $l(a_3^k) = 1$, then the strand 1 of $a_3^k$ (Figure 6 (a)) is from $L_1$, the strand 2 of $a_3^k$ is from $L_2$, $k = 1$, and the strand 3 is from $L_2$.

Claim 2. If $\delta_0 = a_3^k a_1^m a_2^j$ is a portion of $\delta$ for some $k, m, j > 0$, then $l(\delta_0) \leq 0$.

Since $l(a_3^j a_1^m a_2^j) = l(a_3^k) + l(a_1^m a_2^j)$, the claim is obviously true if $l(a_3) \leq 0$. So suppose that $l(a_3^k) = 1$. Then by Claim 1, we have that the strand 1 of $\delta_0$ (Figure 6 (b)) is from $L_1$,
Figure 6: (a) $a_3^k$  (b) $\delta_0 = a_3^k a_1^m a_2^j$

$k = 1$ and the strands 2 and 3 of $\delta_0$ are from $L_2$. So there will be a negative one contribution to the linking number at the first crossing of the syllabus in $a_2$. This proves the claim.

We also need to consider under what conditions we have $l(\delta_0) = 0$. Consider Figure 6 (b).

Case (A1). If the strand 1 of $\delta_0$ is from $L_1$ and the strands 2 and 3 of $\delta_0$ are from $L_2$, then $l(\delta_0) = 0$ happens exactly in the following conditions: $k = 1$ and $j \leq 2$;

Case (A2). If the strand 2 of $\delta_0$ is from $L_1$ and the strands 1 and 3 of $\delta_0$ are from $L_2$, then $l(\delta_0) = 0$ never happen.

Case (A3). If the strand 3 of $\delta_0$ is from $L_1$ and the strands 1 and 2 of $\delta_0$ are from $L_2$, then $l(\delta_0) = 0$ never happen.

Case (A4). If the strand 1 of $\delta_0$ is from $L_2$ and the strands 2 and 3 of $\delta_0$ are from $L_1$, then $l(\delta_0) = 0$ never happen.

Case (A5). If the strand 2 of $\delta_0$ is from $L_2$ and the strands 1 and 3 of $\delta_0$ are from $L_1$, then $l(\delta_0) = 0$ happens exactly in the following condition: $m = 1$.

Case (A6). If the strand 3 of $\delta_0$ is from $L_2$ and the strands 1 and 2 of $\delta_0$ are from $L_1$, then $l(\delta_0) = 0$ never happen.

Suppose that $\delta_1$ is a portion of the braid $\delta$ which starts with a syllabus in $a_3$ and is followed by a positive word which starts with $a_1$ and contains both $a_1$ and $a_2$, but not $a_3$.  

9
It follows from Claim 2 that

**Claim 3.** $l(\delta_1) \leq 0$.

We also need to know exactly when $l(\delta_1) = 0$. By the discussion following Claim 2, we have that the portion of $\delta_1$ which consists of the first three syllabuses of $\delta_1$ must be as in Cases A1 or A5. One can then verify that $l(\delta_1) = 0$ happens exactly in one of the following situations:

(B1) $\delta_1 = a_3 a_1^m a_2^j$, for some $m > 0, 0 < j \leq 2$, and the strand 1 of $\delta_1$ is from $L_1$ and the strands 2 and 3 from $L_2$; or

(B2) $\delta_1 = a_3 a_1^m a_2^p$, for some $m, p > 0$, and the strand 1 of $\delta_1$ is from $L_1$ and the strands 2 and 3 from $L_2$; or

(B3) $\delta_1 = a_3^k a_1 a_2^j$, for some $k, j > 0$, and the strands 1 and 3 of $\delta_1$ are from $L_1$ and the strand 2 from $L_2$.

Suppose that $\delta_2$ is a portion of the braid $\delta$ between and including two syllabuses in $a_3$.

**Claim 4.** $l(\delta_2) \leq 0$.

We have $\delta_2 = \delta_1 a_3^r$ for some $r > 0$ where $\delta_1$ is a braid as in Claim 3 and $\delta_1$ must also end with a syllabus in $a_2$ since $\delta$ is index-3 reduced. Suppose otherwise, that is $l(\delta_2) > 0$. We will get a contradiction. By Claim 3 and Claim 1, we may assume that $l(\delta_1) = 0$ and $l(a_3^r) = 1$. So $\delta_1$ is as in case (B1) or (B3) and $a_3^r$ is as in Claim 1. One can easily check that the link component assignment to the strands of $\delta_1$ and the link component assignment to the strands of $a_3^r$ never match.

Similarly one can show

**Claim 5.** Suppose that $\delta_3$ is a portion of $\delta$ satisfying (1) it ends with a syllabus in $a_3$; (2) this ending syllabus is proceeded by a positive word in $a_1$ and $a_2$ and the word contains at least two syllabuses. Then $l(\delta_3) \leq 0$ and $l(\delta_3) = 0$ happens exactly in one of the following situations:

(C1) $\delta_3 = a_1 a_2^m a_3$, for some $m > 0$, and the strand 2 of $\delta_3$ is from $L_1$ and the strands 1 and 3 from $L_2$; or

(C2) $\delta_3 = a_1^2 a_2^m a_3$, for some $m > 0$, and the strand 1 of $\delta_3$ is from $L_1$ and the strands 2 and 3 from $L_2$; or

(C3) $\delta_3 = a_2^p a_1 a_2^m a_3$, for some $m, p > 0$, and the strand 1 of $\delta_3$ is from $L_1$ and the strands 2 and 3 from $L_2$; or

(C4) $\delta_3 = a_1^k a_2 a_3^j$, for some $k, j > 0$, and the strands 1 and 2 of $\delta_3$ are from $L_1$ and the strands 3 from $L_2$. 

10
Claim 6. Suppose that $\delta_4 = \omega \delta_1$ is a portion of $\delta$ such that $\delta_1$ is a braid as in Claim 3 and $\omega$ is a positive word in $a_1$ and $a_2$ and the word contains at least three syllabuses (and ends with $a_2$). Then $l(\delta_4) < 0$.

Suppose that the claim is not true. Then we have $l(\omega) = 0$, $l(\delta_1) = 0$ and $\delta_1$ is a word as in Case (B1) or (B2) or (B3). But in each of such cases for $\delta_1$, one can easily check that $l(\omega) = 0$ cannot not hold. This contradiction proves Claim 6.

We are now ready to finish the proof of Lemma 7. We first consider the case when $q = 0$, i.e. $\beta = a_i^{-1} \delta \in P^a$ is an index-3 reduced positive word containing at least four syllabuses and $n_1 > 0$. If $n_1 = 1$, then the only syllabus in $a_3$ occurring in $\delta$ must belong to a portion of $\delta$ that looks like either in Claim 3 or in Claim 5, since $\delta$ contains at least four syllabuses. Hence, Lemma 7 holds in this case by Claim 3 and Claim 5. If $n_1 \geq 2$, then Lemma 7 follows from Claims 3 and 4.

We now consider the case when $q = 1$, i.e. $\beta = a_i^{-1} \delta \in P^a$ is cyclically reduced, where $\delta \in P^a$ is index-3 reduced with $n_1 > 0$ and contains at least four syllabuses. Suppose otherwise that $lk(L) > 0$. We will get a contradiction. We have three subcases to consider, corresponding to $i = 1, 2, 3$. Note that by Claims 3-5, we have $l(\delta) \leq 0$.

If $i = 1$ (i.e. $\beta = a_i^{-1} \delta$), then we must have $l(\delta) = 0$ and $l(a_i^{-1}) = 1$, and the first strand of $\beta$ is from $L_1$ and the second strand of $\beta$ from $L_2$. Also, $\delta$ must start with a syllabus in $a_2$ or $a_3$, and end with a syllabus in $a_2$ or $a_3$.

Suppose that $\delta$ starts with syllabus $a_2^k$ and ends with syllabus $a_3^j$. Then $\beta = a_i^{-1} a_2^k \ldots a_3^j a_2^m a_3^j$, which is conjugate to $\beta' = a_1^{-1} a_2^{k+j} \ldots a_3^m a_3^j$. The braid $\beta'$ is still in $P^a$ but has one less number of syllabuses in $a_3$. So if $\beta'$ contains at least five syllabuses, we may use induction on $n_2$ to conclude that Lemma 7 holds in this case. We may then assume that $\beta'$ contains four syllabuses and, thus, $\beta = a_1^{-1} a_2^k a_3^m a_3^j$. But by Claim 5, $\delta$ is as in case (C3); the strand 1 of $\delta$ is from $L_1$ and the strands 2 and 3 of $\delta$ are from $L_2$. But this does not match with the link component assignment already given to the strands of $a_1^{-1}$.

Suppose that $\delta$ starts with syllabus $a_2^k$ and ends with syllabus $a_2^j$. Then each syllabus in $a_3$ is followed by a word without $a_3$ in which both $a_1$ and $a_2$ occur. So we may decompose $\beta$ into portions as $\beta = a_1^{-1} \eta_0 \eta_1 \ldots \eta_n$ such that $\eta_0 \in P^a$ is a word without $a_3$ and each of $\eta_1, \ldots, \eta_n \in P^a$ is a word that starts with a syllabus in $a_3$ which is followed by a word without $a_3$ but containing both $a_1$ and $a_2$. It follows from Claim 3 we must have $l(\eta_0) = l(\eta_1) = \cdots = l(\eta_n) = 0$ and each of $\eta_1, \ldots, \eta_n$ is as in one of the situations described in (B1)-(B3). But any two cases of (B1)-(B3) will not match with their link component assignments if they are adjacent portions of $\delta$. Hence $n = 1$. By Claim 6, we have $\eta_0 = a_2^k$. One can now easily check for $\beta = a_1^{-1} a_2^k \eta_1$ that the conditions $l(a_1^{-1}) = 1$, $l(a_2^k) = 0$ and $l(\eta_1) = 0$ (so $\eta_1$ is as in one of (B1)-(B3)) imply that there is no consistent link component assignment to the strands of $a_1^{-1}$, $a_2^k$ and $\eta_1$. 11
Similarly as in the previous case, we can get a contradiction for the case when $\delta$ starts with syllabus $a_3^k$ and ends with syllabus $a_2^j$.

Suppose that $\delta$ starts with syllabus $a_3^k$ and ends with syllabus $a_2^j$. Then $\beta = a_1^{-1}a_2^{-k}a_3^ka_2^j\cdots a_1^ma_2^m$, which is conjugate to $\beta' = a_1^{-1}a_2^{-k}a_3^ka_2^j\cdots a_1^ma_2^m$. The braid $\beta'$ is still in $P^a$ but has one less number of syllabuses in $a_3$. Also $\beta'$ contains at least five syllabuses, so we may use induction to see that Lemma 7 holds in this case.

Similarly one can deal with the case that $i = 2$.

Finally, we consider the case that $i = 3$, i.e. $\beta = a_3^{-1}\delta$. Note that $\delta$ cannot start or end with $a_3$.

Suppose that $\delta$ starts with $a_1$ and ends with $a_2$. Then $\beta = a_3^{-1}a_1^ja_2^j\cdots a_1^ma_2^m$ for some $j,k,m,n > 0$. Using the identity $a_3^{-1}a_1^{-1} = a_2a_3^{-1}$, we get $\beta = a_3^{-1}a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$. So $\beta$ is conjugate to $\beta' = a_1^{-1}a_2^{-k}a_3^{-1}a_2^{-k-1}\cdots a_1^ma_2^m + 1$. Obviously $\beta' \in P^a$ unless $\beta = a_3^{-1}a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$. If $\beta = a_3^{-1}a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$, then it conjugates to $\beta'' = a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$ and thus $l(\beta) \leq 0$. Hence we may assume that $\beta \in P^a$. So if $\beta'$ contains at least five syllabuses, then we are back to the treated case $i = 1$. Note that if $k > 1$, then $s_{\beta'} > 4$. So we may assume that $k = 1$ and that $\beta' = a_3^{-1}a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$. Applying Claim 3, we see that $l(\beta) = l(\beta') \leq 0$.

Suppose that $\delta$ starts with $a_2$ and ends with $a_2$. Then $\beta = a_3^{-1}a_1^ja_2^ja_3a_2^{-1}\cdots a_1^ma_2^m$ for some $k,m,n > 0$. So $\beta$ is conjugate to $\beta' = a_1^{-1}a_2^{-k}a_3^{-1}a_2^{-k-1}\cdots a_1^ma_2^m$. Again using Claim 3, we see that $l(\beta) = l(\beta') \leq 0$.

Similarly one can deal with the case that $\delta$ starts with $a_1$ and ends with $a_1$, and the case that $\delta$ starts with $a_2$ and ends with $a_1$. The proof of Lemma 7 is now finished. 

We are now ready to prove Proposition 3, which is the content of the rest of this section. Let $\beta = a_3^{-1}\delta$ be a 3-braid in $P^a$ as given in the proposition, whose closure is the given knot $K$. If some syllabus in $a_3$ occurring in $\delta$ has power $k > 2$, we apply the crossing change formula (*) for Casson invariant to $K = \beta$ at the second crossing of the syllabus. The link $L$ (of two components) obtained by smoothing the crossing is the closure of a braid of the type as described in Lemma 7 and thus has non-positive linking number. So we get a new braid $\beta'$ which is identical with $\beta$ except with two less in power at the syllabus in $a_3$ and the Casson invariant of $\beta'$ is less than or equal to that of $\beta$. Obviously $\beta'$ is still in $P^a$. So it suffices to show that $\beta'$ has positive Casson invariant if it is not in the excluded set $E$ given in Proposition 3. If $\beta'$ is in the set $E$, then one can verify directly that the original braid $\beta$ has positive Casson invariant. Similarly we may reduce the power of a syllabus in $a_2$ or in $a_1$ to one or two, using the formula (*), so that the resulting new braid is still in $P^a$, without increasing the Casson invariant, and that if the new braid is in the set $E$, then the old braid has positive Casson invariant. Therefore we only need to show the proposition.
under the extra condition that every syllabus of \( \delta \) has power one or two. We shall prove this by induction on the number \( n_\delta \).

We first consider the initial step of the induction, i.e. the case when \( n_\delta = 0 \). If \( q = 0 \), then \( K = \beta = \hat{\delta} \) is a positive knot. So by the proof of Theorem 1, Proposition 3 holds in this case. Actually in this case one can easily give a quick self-contained proof as follows. First apply the formula (*) to any syllabus of \( \beta \) which has power larger than one so that \( \beta \) becomes a new braid with two less crossings and the Casson invariant of \( K \) is equal to the Casson invariant of the closure of the new braid minus some negative integer (by Lemma 5). So after several such steps, the braid \( \beta \) has simplified to a braid \( \beta_0 \) in which every syllabus has power one such that the Casson invariant of \( K = \beta \) is larger than that of \( \beta_0 \) unless \( \beta = \beta_0 \). But \( \beta_0 \) is a \((3, n)\) torus knot. The normalized Alexander polynomial of a \((3, n)\)-torus knot \( T(3, n) \) is

\[
\Delta_{T(3, n)}(t) = \frac{(t^{3n} - 1)(t - 1)}{(t^3 - 1)(t^n - 1)t^{n-1}}.
\]

Pure calculation of the second derivative of \( \Delta_{T(3, n)}(t) \) valued at \( t = 1 \) gives

\[
\frac{1}{2} \Delta''_{T(3, n)}(1) = \frac{n^2 - 1}{3}.
\]

The conclusion of Proposition 3 follows in this case.

Suppose then that \( q = 1 \) (and \( n_\delta = 0 \)). If \( \beta \) contains exactly five or six syllabuses and at least one of them has power 2 then one can verify directly that \( \hat{\beta} \) has positive Casson invariant (the checking is pretty quick since every syllabus of \( \beta \) has power at most two and also note that \( \beta \) is cyclically reduced and \( \hat{\beta} \) is a knot). If \( \beta \) has more than six syllabuses and one of them has power 2, then we apply the formula (*) at such syllabus to reduce the number of syllabuses of \( \beta \). If the new braid \( \beta' \) is not in \( P^n \), then a cancellation must occur between \( a_i^{-1} \) and an \( a_j \) in \( \beta' \). After the cancellation, we get a braid which is a positive word in \( a_1 \) and \( a_2 \) and thus its closure has positive Casson invariant unless it is a trivial knot. So Lemma 6 implies that the Casson invariant of the old knot \( \hat{\beta} \) is positive. If the new braid \( \beta' \) contains at least seven syllabuses, then we may continue to do such simplification. Note that \( s_{\beta'} \) is one or two less than \( s_{\beta} \) when \( \beta' \) is still in \( P^n \). So we may assume that every syllabus of \( \delta \) has power one. Hence \( \delta \) looks like \( \delta = (a_1 a_2)^p \) or \( \delta = (a_1 a_2)^p a_1 \) or \( \delta = (a_2 a_1)^p \) or \( \delta = (a_2 a_1)^p a_2 \) for some \( p \geq 2 \). The case \( \delta = (a_1 a_2)^p \) or the case \( \delta = (a_2 a_1)^p \) cannot happen since in such case we must have \( i = 3 \) and the closure of \( \beta \) is then not a knot. Consider the case \( \delta = (a_1 a_2)^p a_1 \). We have \( i = 2 \) or \( 3 \). If \( i = 2 \), then to be a knot, \( \beta = a_2^{-1} (a_1 a_2)^p a_1 \) for \( p > 2 \) and \( p \not\equiv 2 \) (mod 3). Now one can verify directly that \( C_\beta > 0 \). If \( i = 3 \), then \( \beta = a_3^{-1} (a_1 a_2)^p a_1 \) is conjugate to \( a_1 (a_1 a_2)^p a_1 \) which obviously has positive Casson invariant. The case \( \delta = (a_2 a_1)^p a_2 \) can be treated similarly. The proof of Proposition 3 for the initial step \( n_\delta = 0 \) is complete.

Now we may assume that \( n_\delta > 0 \). We warn the reader that the rest of the proof will
involve more patient case counting. Nevertheless the guiding idea will still be more or less as follows: apply the crossing change formula (\( \ast \)) for the Casson invariant and Lemma 7 at a suitable chosen crossing of the given braid \( \beta = a_i^{-q} \epsilon \) to reduce its complexity, simplify the new braid (if necessary) to get a braid \( \beta' = a_i^{-q} \delta' \) in \( P^a \), use the induction if \( s' \geq 4 \) and \( \beta' \) is not in the excluded set

\[
E = \left\{ a_i^{-1} a_2 a_3^2 a_1 a_2, \quad a_i^{-1} a_3^2 a_1 a_2 a_3, \quad a_i^{-1} a_3 a_1 a_2 a_3, \quad a_i^{-1} a_2 a_3 a_1 a_2^2, \right. \\
\left. a_i^{-1} a_3 a_1 a_2 a_3^2, \quad a_i^{-1} a_3^2 a_1 a_2 a_3, \quad a_i^{-1} a_3 a_1 a_2 a_3^2, \quad a_i^{-1} a_3^2 a_1 a_2 a_3^2, \right. \\
\left. a_i^{-1} a_2 a_3 a_1 a_2, \quad a_i^{-1} a_2 a_3 a_1 a_2^2, \quad a_i^{-1} a_2 a_3 a_1 a_2, \quad a_i^{-1} a_2 a_3 a_1 a_2^2 \right\}
\]

otherwise verify directly that the original braid \( \beta \) has positive Casson invariant. Besides we shall often make use of the conditions such as \( \hat{\beta} \) is a knot, \( \beta \) is cyclically reduced and \( \delta \) is index-3 reduced.

Claim D1. If \( a_3^2 \) is a syllabus of \( \delta \), then the Proposition holds.

Consider the first such syllabus occurring in \( \delta \). Applying the formula (\( \ast \)) to \( \hat{\beta} \) at the second crossing of the syllabus \( a_3^2 \), we get a new braid \( \beta' = a_i^{-q} \delta' \), where \( \delta' \) is the braid obtained from \( \delta \) by deleting the syllabus \( a_3^2 \), such that \( \delta' \in P^* \), \( n' = n - 1 \), and \( C_{\hat{\beta}} \geq C_{\beta'} \) (by Lemma 7). If \( \beta' \) is still in \( P^a \) and \( \delta' \) contains at least four syllabuses but \( \beta' \) is not a word in the excluded set \( E \), then we may use induction.

Suppose that \( \beta' \in E \). Since the syllabus we are considering is the first such occurring in \( \beta \) and since \( \delta \) is index-3 reduced, \( \beta \) can only be the word \( a_i^{-1} a_2 a_3 a_1 a_2 a_3 a_1 \) or \( a_i^{-1} a_3 a_1 a_2 a_3 a_1 a_2 a_3 a_1 \) or \( a_i^{-1} a_2 a_3 a_1 a_2 a_3 a_1 a_2 a_3 a_1 \). In such a case one can verify directly that \( C_{\hat{\beta}} \) is positive.

Hence, we may assume that either \( \beta' \) is still in \( P^a \) with \( \delta' \) containing exactly three syllabuses, or \( \beta' \) is no longer in \( P^a \). In the former case, \( \beta \) is a word in the set

\[
\left\{ a_1 a_2^2 a_3 a_1^m, \quad a_2 a_1 a_2 a_3^2, \quad a_1 a_2 a_3 a_1^m, \quad a_3 a_2 a_1 a_2^m, \quad a_2 a_1 a_2 a_3 a_1, \quad a_3 a_2 a_1 a_2 a_3 a_1, \right. \\
\left. a_2 a_3 a_1 a_2 a_3, \quad a_3 a_2 a_1 a_2 a_3, \quad a_2 a_3 a_1 a_2 a_3, \quad a_3 a_2 a_1 a_2 a_3, \quad a_2 a_3 a_1 a_2 a_3, \quad a_3 a_2 a_1 a_2 a_3 \right\}
\]

for some \( j, k, m \in \{1, 2\} \). When \( \beta \) is one of words in this set but is not in the excluded set \( E \), one can verify directly using the formula (\( \ast \)) that \( \hat{\beta} \) has positive Casson invariant. For instance, when \( \beta = a_1^{-1} a_2 a_3 a_1^2 a_2^m a_3^2, \) it is conjugate to \( a_2 a_3 a_1^2 a_2^m a_3^2 a_1^2 = a_2 a_3 a_1^2 a_2^m a_3^2 a_1^2 \) which in turn is conjugate to \( a_1^{-1} a_2 a_3 a_1^2 a_2^m a_3^2 \). So we need to show that the Casson invariant of the closure of the braid \( \eta = a_1^{-1} a_2 a_3 a_1^2 a_2^m a_3^2 \) is positive. We have eight possible cases for \( \eta \) corresponding to various possible values of \( j, k \) and \( m \). But only in case \( j = k = m = 1 \) or case \( j = 1, k = m = 2 \) or case \( j = k = 2, m = 1 \), the closure of the involved braid is a knot, and in such a case one can verify directly using formula (\( \ast \)) that \( \hat{\beta} = \hat{\eta} \) has positive Casson invariant. In a similar way one can verify the proposition for each of the other words in the above set. (We did the checking!)

We now consider the latter case when \( \beta' \) is not in \( P^a \). It follows that the syllabus \( a_3^2 \) is
either the first or the last syllabus of δ and q = 1. We consider the case when the syllabus a_3^2 is the first syllabus of δ. The case when the syllabus a_3^2 is the last syllabus of δ can be treated similarly. It follows that β = a_1^{-1}a_2^j a_3^k a_4^l \ldots, for some j, k \in \{1, 2\}, and β does not end with a_1. If j = 2, then β' = a_1^{-1}a_2^j a_3^k \ldots which is isotopic to β'' = a_1 a_2^k \ldots which is in P_n. So if β'' contains at least four syllabuses, we may use the induction. We may then assume that β'' has less than four syllabuses. It follows that β'' = a_1 a_2^k a_3^\nu which is a knot only when k = 2, m = 1 or k = 1, m = 2 and in these two cases C_β ≥ C_β'' > 0.

So we may assume that j = 1. In this case β' is isotopic to β'' = a_2^k \ldots which is in P_n. Hence if β'' contains at least four syllabuses, we may use the induction. If β'' has less than four syllabuses, then we must either have β = a_1^{-1}a_2^2 a_1 a_2^k a_3^m or β = a_1^{-1}a_3^2 a_1 a_2^k a_3^\nu. In the former case, β'' has positive Casson invariant (a nontrivial positive knot). In the latter case, only when m = 1, j = 1, β is a knot. But this braid is in the set E. The proof of Claim D1 is now complete.

By Claim D1, we may now assume that every syllabus in a_3 occurring in δ has power equal to one.

**Claim D2.** We may assume that every syllabus in a_1 occurring in δ has power equal to one.

Suppose that δ contains syllabuses in a_1 of power two. Consider the first such syllabus occurring in δ. Applying the formula (1) to β at the first crossing of the syllabus a_1^2, we get a new braid β' = a_i^{-1}d' such that the Casson invariant of β' is less than or equal to that of β by Lemma 7. Obviously β' is still a positive word in a_1, a_2, a_3 but may not be in P_n or P_n'. We have several possibilities for δ around the given syllabus a_1^2: δ = \ldots a_2^j a_1^2 a_2^k \ldots or δ = \ldots a_2^j a_3 a_2^k \ldots or δ = \ldots a_2^j a_3 \ldots or δ = \ldots a_2^j a_2 a_3 \ldots for some j, k \in \{1, 2\}.

Case (D2.1). δ = \ldots a_2^j a_1^2 a_2^k \ldots.

Then β' = \ldots a_2^j a_2^k \ldots and β' = a_i^{-1}d' is still in P_n. Also if j \neq k \neq 2, we may apply the formula (1) one more time to bring it down to one or two. Let β'' = a_i^{-1}k' be the resulting braid. Then if β'' contains at least four syllabuses and β'' is not in E, then we have eliminated one a_1^2. If β'' is in E, then one can verify directly that the original braid β has positive Casson invariant. Note that s_{i'} = s_i - 2. Suppose that s_{i'} < 4. Then s_4 is four or five and β = a_i^{-1}a_1 a_2 a_3 or β = a_i^{-1}a_1 a_2 a_3 or β = a_i^{-1}a_1 a_2 a_3. Easy to check that when q = 0, any knot from these cases has positive Casson invariant. So assume that q = 1 in these cases.

If β = a_i^{-1}a_1 a_2 a_3, then i = 1 since β is cyclically reduced. So β = a_1^{-1}a_1 a_2 a_3. To be a knot, we have k = j = 1 or k = j = 2. In each of the two cases, one can verify directly that the Casson invariant of the knot is positive.

If β = a_i^{-1}a_1 a_2 a_3, then i = 2. That is β = a_2^{-1}a_2 a_3 a_2 a_3. One can also directly
verify that $C_\beta > 0$ (for those values of $k, j \in \{1, 2\}$ which make $\hat{\beta}$ a knot).

The case that $\beta = a_i^{-q}a_3a_1a_2^j a_2^k$ can be treated similarly.

Case (D2.2). $\delta = \cdots a_3a_1^2a_2^k \cdots$.

Then $\delta' = \cdots a_3a_2^k \cdots = \cdots a_2a_1a_2^{k-1} \cdots$.

Case (D2.2.1.) $k = 2$.

If $\delta$ does not start with $a_3$, then $\beta' = a_i^{-q}\delta' = a_i^{-q} \cdots a_2^{j+1} a_1a_2 \cdots$ is still in $P^n$ and is not in the set $E$. In such case if $\delta'$ contains at least four syllabuses, we may apply induction since $n_{\delta'} = n_\delta - 1$. If $s_{\delta'} < 4$, then $\beta = a_i^{-q}a_3^2a_1^2a_2^2$. To be a knot, we have $q = 0, j = 1$ or $q = 1, i = 1, j = 2$. In each of the two cases we have $C_\beta > 0$ by direct calculation.

If $\delta$ starts with $a_3$ but $q = 0$, then $\beta' = \delta'$ is still in $P^n$ but not in $E$. Also $s_\beta = s_\delta = s_{\delta'} = s_{\delta'}$ and $n_{\delta'} = n_\delta - 1$. So we may apply induction.

If $\delta$ starts with $a_3$ and $q = 1$, then $i = 1$ or 2. If $i = 1$, then $\beta' = a_1^{-1}a_2a_1a_2 \cdots$ is still in $P^n$ but not in $E$ and contains at least five syllabuses. So we may use induction since $s_{\delta'} = s_{\delta}$ but $n_{\delta'} = n_\delta - 1$. If $i = 2$, then $\beta' = a_1a_2 \cdots$ is in $P^n$ and is not in $E$. Also $s_{\delta'} = s_{\delta} - 1$. Hence if $s_{\delta} > 4$, we may use induction. So suppose that $s_{\delta} = 4$. Then $\beta = a_2^{-1}a_3^2a_2^2a_3$ or $\beta = a_2^{-1}a_3a_1^2a_2^2a_3$. But the former is not a knot. The latter is a knot when $j = 2$, in which case $C_\beta > 0$.

Case (D2.2.2). $k = 1$.

Then $\delta = \cdots a_3a_1^2a_3^j \cdots$ or $\delta = \cdots a_3a_2^2a_2^j \cdots$ or $\delta = \cdots a_3a_1^2a_2$. Correspondingly, we have $\delta' = \cdots a_2a_3a_3^j \cdots$ or $\delta = \cdots a_2a_1^2 \cdots$ or $\delta = \cdots a_2a_1$. If $\beta'$ is in $P^n - E$ and $s_{\delta'} \geq 4$, we may use induction since $n_{\delta'} < n_\delta$. If $\beta' \in E$, then one can check that the old braid $\beta$ has positive Casson invariant. If $\beta' \in P^n$ but $s_{\delta'} < 4$, then one can also verify that $\beta$ always has positive Casson invariant, applying the conditions that (1) $\beta \in P^n - E$, (2) $s_{\delta} \geq 4$ and (3) $\hat{\beta}$ is a knot.

Case (D2.3). $\delta = a_i^2a_2^k \cdots$.

Then $\delta = a_1^2a_2a_3^j \cdots$ or $\delta = a_2^2a_3^k a_1^j \cdots$. And $\delta' = a_2^2a_1^j a_2^m \cdots$ or $\delta' = a_2^2 a_3 a_1^j \cdots$. So $\beta'$ is in $P^n$ unless $\beta$ starts with $a_2^{-1}$. If $\beta$ starts with $a_2^{-1}$, then $\beta' = a_2^{-k}a_3^2a_1^m \cdots$ or $\beta' = a_2^{-k}a_3a_1^j \cdots$ which is in $P^n$. Again in each of these cases, if $\beta'$ is the set $E$ or $s_{\delta'} < 4$, one can calculate directly that the old braid $\beta$ has positive Casson invariant. Otherwise one can use the induction.

Case (D2.4). $\delta = \cdots a_1^2$.

This case can be treated similarly as in the previous case.

So by Claims D1 and D2, we now assume that every syllabus in $a_3$ and in $a_1$ occurring
in $\delta$ have power equal to one.

**Claim D3.** We may assume that every syllabus in $a_2$ occurring in $\delta$ has power equal to one.

This can be proved similarly as Claim D2.

So we now assume that every syllabus occurring in $\delta$ has power one.

If $a_1a_2a_1$ appears immediately after an $a_3$, then $\beta = \cdots a_3a_1a_2a_1 \cdots = \cdots a_3a_1a_3a_2 \cdots = \cdots a_2a_1a_2 \cdots$ and so we get an isotopic braid in $P^n$ with less number of syllabuses in $a_3$. Hence we may apply induction unless $\beta$ is in the set \{ $a_3a_1a_2a_1$, $a_2^{-1}a_3a_1a_2a_1$, $a_3^{-1}a_3a_1a_2a_1$, $a_2^{-1}a_3a_1a_2a_1$, $a_3^{-1}a_3a_1a_2a_1$, $a_3^{-1}a_3a_1a_2a_1$ \}. If $\beta$ is a word in this set, then either $\beta$ is not a knot or $\beta$ has positive Casson invariant. So we may assume that no $a_3a_1a_2a_1$ occurs in $\beta$. A similar argument shows that we may assume that no $a_2a_1a_2a_3$ occurs in $\beta$. Hence we may assume that $\beta$ is one of the words in the set

$$
\begin{align*}
\{ &a_1a_2a_3, \quad (a_3a_1a_2)^m, \quad (a_3a_1a_2)^m a_3, \quad (a_3a_1a_2)^m a_3 a_1, \\
& a_2(a_3a_1a_2)^m, \quad a_1a_2(a_3a_1a_2)^m, \quad a_1a_2(a_3a_1a_2)^m a_3, \quad a_2(a_3a_1a_2)^m a_3, \\
& a_1^{-1}a_3a_1a_2, \quad a_1^{-1}a_3a_1a_2 a_3, \quad a_1^{-1}a_3a_1a_2 a_3 a_1, \quad (a_3a_1a_2)^m a_3 a_1, \\
& a_1^{-1}a_3a_1a_2 a_3 a_1, \quad a_1^{-1}a_3a_1a_2 a_3 a_1 a_3 a_1, \quad a_1^{-1}a_3a_1a_2 a_3 a_1 a_3 a_1 a_3 a_1
\end{align*}
$$

where $m > 0$.

If $\beta = a_1a_2a_3a_1$, then it has positive Casson invariant. The case $\beta = (a_3a_1a_2)^m$ cannot occur since its closure is not a knot for all $m > 0$. Similarly each of the cases $a_2(a_3a_1a_2)^m a_3 a_1, a_1a_2(a_3a_1a_2)^m a_3, a_2^{-1}a_1a_2a_3 a_1, a_3 a_1a_2a_3 a_1 a_3 a_1$ and $a_1^{-1}a_3a_1a_2 a_3 a_1$ cannot occur as $\beta$. If $\beta = (a_3a_1a_2)^m a_3$, then it is conjugate to $\beta' = a_3^3a_3^3a_1a_2a_3a_1 a_3 a_1 a_3 a_1 a_3 a_1$. So we may apply Claim D1 (note that $n_{\beta'} < n_{\beta}$) unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_3a_1a_2a_3$ has positive Casson invariant. If $\beta = (a_3a_1a_2)^m a_3 a_1$, then it is conjugate to $\beta' = a_2a_3^3a_2(a_3a_1a_2)^m a_3 a_1$ and $a_2a_3^3a_2(a_3a_1a_2)^m a_3 a_1$ cannot occur as $\beta$. If $\beta = (a_3a_1a_2)^m a_3 a_1 a_3 a_1$, then it is conjugate to $\beta' = (a_3a_1a_2)^m a_3 a_1 a_3 a_1 a_3 a_1$. So we may apply Claim D2 (note that $n_{\beta'} = n_{\beta}$) unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_2a_3a_1a_2$ has positive Casson invariant. Similarly one can deal with the cases $a_1 a_2(a_3a_1a_2)^m a_3 a_1, a_2(a_3a_1a_2)^m a_3 a_1$ and $a_1 a_2(a_3a_1a_2)^m a_3 a_1 a_3 a_1$.

Suppose that $\beta = a_1^{-1}(a_3a_1a_2)^m a_3$. To be a knot, we have $m \geq 3$ and $m = 3(mod2)$. Also $\beta$ is conjugate to $\beta' = a_1^{-1}a_3a_1a_2 a_3$ which has less number of syllabuses in $a_3$ and thus we may apply the induction. Similarly one deal with the case when $\beta = a_2^{-1}(a_3a_1a_2)^m a_3$.

If $\beta = a_2^{-1}a_3a_1a_2 a_3$, then to be a knot $m$ must be even. Applying the formula (*) at the first crossing of $\beta$, we get $C_{\beta} = C_{\beta_1} + lk(\lambda_1)$ where $\beta_1 = a_1a_2(a_3a_1a_2)^m a_3$ and $\lambda_1 = a_2(a_3a_1a_2)^m a_3$. One can easily deduce from the projection of $\lambda_1$ that $lk(\lambda_1) = -m/2$. 

17
Hence, we have $C_{\hat{\beta}} = C_{\hat{\beta}_1} - m/2$. In such case it suffices to show the following.

**Claim D4.** $C_{\hat{\beta}_1} > m/2$.

We knew that $m = 2p$ with $p > 0$. We shall prove the claim by induction on the number $p$. Write $\beta_1$ as $\beta_1 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} (a_3 a_1 a_2)^{m-2}$. Applying formula (*) to $\beta_1$ at the first crossing of the first syllabus $a_1^2$, we get $C_{\hat{\beta}_1} = C_{\hat{\beta}_2} - lk(\hat{\lambda}_2)$ where $\beta_2 = a_1 a_2 a_3^{-1} a_3 a_1^{-1} a_2 a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$ and $\lambda_2 = a_1 a_2 a_3^{-1} a_3 a_1^{-1} a_2 a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$. One can easily deduce from the projection of $\lambda_2$ that $lk(\hat{\lambda}_2) = -2(m - 2) - 3 = -4p + 1$. Hence, we have $C_{\beta_2} = C_{\beta_3} + 4p - 1$. We then apply formula (*) to $\beta_2$ at the first crossing of the first syllabus $a_2^2$, we get $C_{\hat{\beta}_2} = C_{\hat{\beta}_3} - lk(\hat{\lambda}_3)$ where $\beta_3 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} (a_3 a_1 a_2)^{m-2}$ and $\lambda_3 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$. One can calculate to see that $lk(\hat{\lambda}_3) = -p - 1$. Hence, we have $C_{\hat{\beta}_3} = C_{\hat{\beta}_4} + p + 1$. We then apply formula (*) to $\beta_3$ at the first crossing of the syllabus $a_3^2$, we get $C_{\hat{\beta}_3} = C_{\hat{\beta}_4} + lk(\hat{\lambda}_4)$ where $\beta_4 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} (a_3 a_1 a_2)^{m-2}$ and $\lambda_4 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$. Also one can calculate to see that $lk(\hat{\lambda}_4) = -p$. Hence, we have $C_{\hat{\beta}_4} = C_{\hat{\beta}_5} - p$. We then apply formula (*) to $\beta_4$ at the first crossing of the syllabus $a_2^2$, we get $C_{\hat{\beta}_4} = C_{\hat{\beta}_5} - lk(\hat{\lambda}_5)$ where $\beta_5 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} (a_3 a_1 a_2)^{m-2}$ and $\lambda_5 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$. We also have $lk(\hat{\lambda}_5) = -(p - 1) - 2$. Hence we have $C_{\hat{\beta}_5} = C_{\hat{\beta}_6} + 4(p - 1) + 2$. Applying formula (*) to $\beta_5$ at the first crossing of the syllabus $a_1^2$, we get $C_{\hat{\beta}_5} = C_{\hat{\beta}_6} - lk(\hat{\lambda}_6)$ where $\beta_6 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} (a_3 a_1 a_2)^{m-2}$ and $\lambda_6 = a_1 a_2 a_3^{-1} a_2 a_1^{-1} a_3 a_2^{-1} a_2 a_3 a_1 a_2 (a_3 a_1 a_2)^{m-2}$. We also have $lk(\hat{\lambda}_6) = -(p - 1) - 3 = -p$. Hence we have $C_{\hat{\beta}_6} = C_{\hat{\beta}_7} + p$. In summary, we get

$$C_{\hat{\beta}_7} = C_{\hat{\beta}_8} + 9p - 2.$$  

Now one can easily see that the claim follows.

Similarly we can treat the rest of cases. The proof of Proposition 3 is now complete.

### 4. Proof of Proposition 4

Given a 3-braid $\beta$ in letters $a_1, a_2, a_3$, whose closure is a knot, there is a canonical way to construct a Seifert surface of $\hat{\beta}$ as follows: in the projection plane we have the braid diagram in its canonical form, place three rectangular disks in the space so that disk 1 lies in the projection plane and is on the left hand side of the braid, disk 2 also lies in the projection plane but on the right hand side of the braid, disk 2 lies perpendicularly above the projection plane, each disk having one side running parallel to the braid from top to the bottom, then to each letter $a_1 \ (a_1^{-1})$ occurring in $\beta$ use a half negatively (positively) twisted band connecting disks 1 and 2, to each letter $a_2 \ (a_2^{-1})$ use a half negatively (positively) twisted band connecting disks 2 and 3, and to each letter $a_3 \ (a_3^{-1})$ use a half negatively (positively) twisted band connecting disks 1 and 3 (behind disk 2). Figure 7 illustrates such construction for $\beta = a_1 a_2 a_3 a_1^{-1} a_3^{-1} a_2^{-1}$. We call the Seifert surface of $\hat{\beta}$ so constructed canonical Seifert surface of $\hat{\beta}$. In [X], it was proved that if $\beta$ is a 3-braid of norm form (whose
Lemma 8 Let $\beta$ be a 3-braid such that $\hat{\beta}$ is a knot. Let $S$ be the canonical Seifert surface of $\beta$ and suppose that it has minimal genus.

1. If $S$ contains two half twisted bands corresponding to the same letter $a_1$ or $a_2$ or $a_3$, then the $-1$-surgery on $\hat{\beta}$ is a manifold with essential lamination.

2. If $S$ contains two half twisted bands corresponding to the same letter $a_1^{-1}$ or $a_2^{-1}$ or $a_3^{-1}$, then the 1-surgery on $\hat{\beta}$ is a manifold with essential lamination.

Proof. We shall only prove part (1) when $S$ contains two half twisted bands corresponding to the same letter $a_1$. All other cases can be proved similarly.

Let $M$ be the knot exterior of $\hat{\beta}$ in $S^3$. We shall also use $M(-1)$ to denote the manifold obtained by Dehn surgery on the knot $\hat{\beta}$ with the slope $-1$. Let $V$ be the solid torus filled in $M$ to obtain the manifold $M(-1)$. We first construct an essential branched surface $B$ in the exterior $M$ and then prove that $B$ (which has boundary on $\partial M$) can be capped off by a branched surface in $V$ to yield an essential branched surface $\hat{B}$ in $M(-1)$. The construction of $B$ is similar to that given in [Rs].

Since $S$ contains two half negatively twisted bands corresponding to $a_1$, there is a disk $D$ in $M$ as shown in Figure 8 (1) whose boundary lies in $S \cup \partial M$ and whose interior is disjoint from $S \cup \partial M$. With more detail, the boundary of $\partial D$ intersects $S$ in two disjoint arcs and intersects $\partial M$ in two disjoint arcs with the latter happening around the places corresponding to the two bands of $a_1$. (Similar disks were used in [BMe] for a different purpose). The branched surface $B$ is the union of the Seifert surface $S$ and the disk $D$ with their intersection locus smoothed as shown in Figure 8 (2). The arrows in the figure indicate the cusp direction of branched locus. An argument as in [Rs] shows that the branched surface fully carries a lamanition with no compact leaves and each negative slope.
can be realized as the boundary slope of a lamination fully carried by $B$. Since the disk $D$ intersects the knot exactly twice, the branched surface $B$ is essential in $M$ by [G2, 3.12]. The branched locus of $B$ is a set of two disjoint arcs properly embedded in $S$, each being non-separating. The branched surface $B$ meets $\partial M$ yielding a train track in $\partial M$ as shown in Figure 9 (1). Let $\mathcal{L}$ be a lamination fully carried by $B$ whose boundary slope is $-1$. Then $\partial \mathcal{L}$ must look like as shown in Figure 9 (2).

![Figure 8: construction of $B$](image)

![Figure 9: (1) $\partial B$ on $\partial M$ (2) the curve of slope $-1$ fully carried by $\partial B$](image)

We now construct a branched surface $B_{\nu}$ in the sewn solid torus $V$ such that the train track $B_{\nu} \cap \partial V$ is equivalent to the train track $B \cap \partial M$ and $B_{\nu}$ fully carries a lamination which is a set of meridian disks of $V$. Hence, $B$ and $B_{\nu}$ match together and form a branched surface $\hat{B}$ in $M(-1)$. Take a meridian disk $D_0$ of $V$ and push part of it near and around $\partial V$ as shown in Figure 10 (1) and then identify two disjoint sub-disks of $D_0$ as shown in Figure 10 (2). This gives a branched surface $B_1$ with the cusp direction along its singular locus (an arc) as shown in Figure 10 (2). Then we split $B_1$ locally at a place around a point of $\partial B_1$ as shown in Figure 10 (3) and then we start pinch the resulting branched surface along part of its boundary as shown in Figure 10 (4). The pinching continues as shown in Figure 10 (5) until we get the branch surface whose boundary is as shown in Figure 10 (6). The resulting surface is the branched surface $B_{\nu}$. Obviously the train track $\partial B_{\nu}$ on $\partial V$ is
equivalent to the train track \( \partial B \) in \( \partial M \) and they can be matched in \( \partial V = \partial M \).

To see that \( \hat{B} \) is essential we have five things to check by [GO, Definition 2.1]:

(i) \( \hat{B} \) has no discs of contact;
(ii) The horizontal surface \( \partial_h N(\hat{B}) \) is incompressible and \( \partial \)-incompressible in \( M(\partial M) - \hat{N} \) (\( \hat{B} \)), there are no monogons in \( M(\partial M) - \hat{N} \) (\( \hat{B} \)) and no component of \( \partial_h N(\hat{B}) \) is a 2-sphere;
(iii) \( M(\partial M) - \hat{N} \) (\( \hat{B} \)) is irreducible;
(iv) \( \hat{B} \) contains no Reeb branched surface;
(v) \( \hat{B} \) fully carries a lamination.

Condition (v) follows automatically by the construction since leaves of a lamination fully carried by \( B \) with boundary slope \(-1\) match on \( \partial M = \partial V \) with (disk) leaves of a lamination fully carried by \( B_v \). It also follows that \( \hat{B} \) does not carry any compact surface since \( B \) does not. Hence in particular condition (iv) holds also for \( \hat{B} \). By the construction, one can easily see that \( V - \hat{N}(B_v) \) has two components, each of which topologically looks like as shown in Figure 11. It follows that \( M(\partial M) - \hat{N}(\hat{B}) \) is topologically the same as \( M - \hat{N}(B) \), with the same horizontal surface. From this we get conditions (ii) and (iii) for \( \hat{B} \).

We now show that \( \hat{B} \) has no disk of contact. Note that \( \partial_c(N(\hat{B})) \) is a set of two annuli and each of the annuli is obtained from matching a component of \( \partial_c(N(B_v)) \) (a vertical disk) and a component of \( \partial_c(N(B_v)) \) (a vertical disk). Hence, if \( D_c \) were a contact disk in \( N(\hat{B}) \), then its boundary would have to intersect a component of \( \partial_c(N(B_v)) \). It follows then that the interior of \( D_c \) must enter into the region of \( N(B) \) correspond to a branch of (S-
the singular locus of \( B \)). But one can easily see from Figure 8 that the complement of the singular locus of \( B \) in \( S \) is a connected surface. It follows that \( D_c \) has to intersect every \( I \)-fiber of \( B \) since \( D_c \) is transverse to \( I \)-fibers of \( B \). In particular \( \partial D_c \) has to intersect both of the vertical annuli of \( \partial_c(\hat{B}_v) \), which gives a contradiction.

\[
\delta V = \delta M
\]

\[
\partial h N(B_v)
\]

\[
\partial N(B_v)
\]

\[
\partial h N(B_v)
\]

Figure 11: a component of \( V - \overset{\circ}{N} (B_v) \)

We now prove Proposition 4. By [X], the canonical Seifert surface of \( \hat{\beta} \) has minimal genus. If the condition (1) of Proposition 4 holds, then the conclusion of Proposition 4 follows obviously from Lemma 8. Suppose that the condition (2) of Proposition 4 holds. To show that the 1-surgery on \( \hat{\beta} \) gives a manifold with an essential lamination, we may assume, by Lemma 8, that \( \eta \) contains at most three syllabus, and they have different subscripts and all have power \(-1\). But \( \eta \) contains at least two syllabuses. Suppose that the first and the second syllabuses of \( \eta \) are \( a_3^{-1} \) and \( a_2^{-1} \), i.e. \( \eta = a_3^{-1}a_2^{-1}\cdots \). Then since \( \beta \) is a shortest word, the word \( \delta \) does not end with a syllabus in \( a_3 \). Suppose that \( \delta \) ends with a syllabus in \( a_1 \). Then we have \( \beta = \cdots a_1 a_3^{-1}a_2^{-1}\cdots \). By a band move isotopy of the Seifert surface as shown in Figure 12 (1) we get an isotopic 3-braid \( \beta' \) which contains two \( a_2^{-1} \). (Algebraically, \( \beta = \cdots a_1 a_3^{-1}a_2^{-1}\cdots = \cdots a_2^{-1}a_1 a_3^{-1}\cdots = \beta' \)). Further the canonical Seifert surface of \( \hat{\beta}' \) is isotopic to that of \( \hat{\beta} \) and thus has minimal genus. So we may apply Lemma 8 to see that for the knot \( \hat{\beta}' = \hat{\beta} \), the 1-surgery gives a manifold with essential lamination. Suppose then that \( \delta \) ends with a syllabus in \( a_2 \). Since \( \delta \) is assumed to contain at least two syllabuses, \( \beta = \cdots a_1 a_2 a_3^{-1}a_2^{-1}\cdots \). Again we may first use the band-isotopy as shown in Figure 12 (2) and then use the band isotopy of Figure 12 (1) to get an isotopic 3-braid whose canonical Seifert surface contains two bands corresponding to \( a_2^{-1} \). Hence Proposition 4 follows from Lemma 8 in this case as well. Similarly one can treat the cases when \( \eta \) starts with \( a_2^{-1}a_1^{-1} \) or with \( a_1^{-1}a_3^{-1} \). The case when \( \delta \) contains at most three syllabuses, each having power at most one, can be proved similarly. This proves Proposition 4 under its condition (2). Finally if the condition (3) of Proposition 4 holds then either condition (2) of Proposition 4 holds or one can get directly two letters \( a_i \) of the same subscript in \( \delta \) and two letters \( a_j^{-1} \) of the same subscript in \( \eta \). So again the Proposition follows from Lemma 8.

\[ \diamond \]
Figure 12: the band move isotopies

References


[H] J. Hempel, A simply connected 3-manifold is $S^3$ if it is the sum of a solid torus and the complement of a torus knot. Proc. Amer. Math. Soc. 15 (1964) 154–158.


