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# Virtually fibered Montesinos links

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Abstract

We prove that all generalized Montesinos links in $S^3$ which are not classic $\widetilde{SL}_2$-type are virtually fibred except the trivial link of two components. We also prove the virtually fibred property for a family of infinitely many classic Montesinos links of type $\widetilde{SL}_2$. As a byproduct we find the first family of infinitely many virtually fibred hyperbolic rational homology *3*-spheres.


## 1. *Introduction*

William Thurston conjectured over 20 years ago that every compact hyperbolic 3-manifold whose boundary is a possibly empty union of tori is finitely covered by a manifold which fibers over the circle. If true, it provides a significant amount of global information about the topology of such manifolds. The first nontrivial examples supporting the conjecture were obtained by Gabai [**8**] in 1986 and Reid [**18**] in 1995. In the 1999 paper [**2**], Aitchison and Rubinstein found combinatorial conditions on certain polyhedral decompositions of 3-manifolds which guarantee the existence of a finite cover which fibers over the circle. Chris Leininger showed in 2002 that every manifold obtained by Dehn filling one component of the Whitehead link exterior is finitely covered by a surface bundle [**12**], and more recently Genevieve Walsh verified Thurston's conjecture for the exteriors of spherical Montesinos link exteriors [**22**], which includes all 2-bridge links. Jack Button [**5**] has determined many examples of nonfibered virtually fibered 3-manifolds in the Callahan–Hildebrand–Weeks and Hodgson–Weeks censuses, including over 100 closed examples. Most recently, Ian Agol [**1**] gave, in 2008, some group-theoretic criteria for a 3-manifold group which imply the virtually fibering property of the underlying manifold. Manifolds satisfying the criteria include all arithmetic hyperbolic link complements.

Call a link $K$ in a 3-manifold *virtually fibered* if its exterior has a finite cover which fibers over the circle. In this paper, we are interested in examining families of generalized Montesinos links with this property. To that end, consider a generalized Montesinos link $K = (-g; e_0; (q_1/p_1), \ldots, (q_n/p_n))$ (see § 2 for its definition) and let $(W_K, \tilde{K}) \to (S^3, K)$ be the 2-fold branched cover of $(S^3, K)$ determined by the homomorphism $\pi_1(S^3 \setminus K) \to \mathbb{Z}/2$ which sends each meridinal class to the nonzero element of $\mathbb{Z}/2$. Montesinos proved [**13**, **14**] that $W_K$ admits a Seifert fibering $f : W_K \to \mathcal{B}_K$ invariant under the covering involution $\tau : W_K \to W_K$. (It will be assumed that $W_K$ is equipped with such a structure throughout the paper.) Further, by a special case of the orbifold theorem, $W_K$ admits a geometric structure in which $\tilde{K}$ is geodesic and $\tau$ is an isometry. The type of geometric structure supported by a closed, connected Seifert fibered space $V$ with base $\mathcal{B}$ is uniquely determined by the sign of the orbifold Euler characteristic $\chi(\mathcal{B})$ and whether or not the Euler number $e(V)$ is zero (cf.

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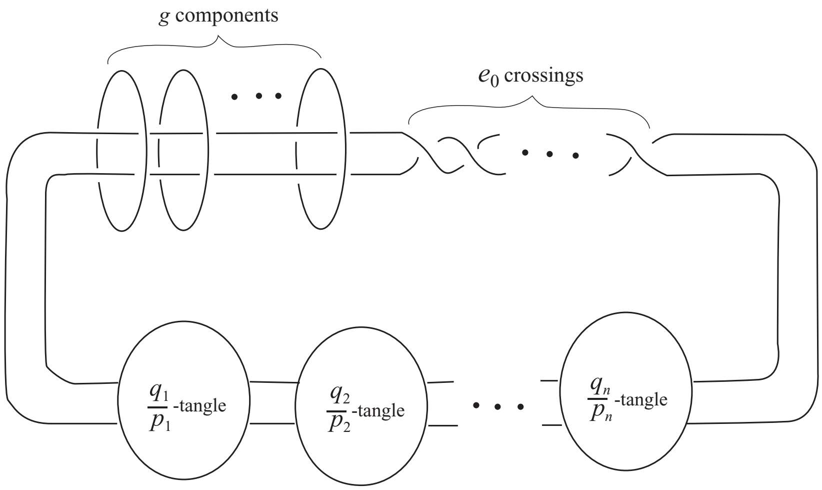



FIGURE 1. *A generalized Montesinos link* $K(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$.

[**20**, § 4]). The precise result is detailed in Table 1. We shall say that $K$ has *spherical type* if $W_K$ admits an $\mathbb{S}^3$ geometry, has *Nil* type if $W_K$ admits a Nil geometry, has $\widetilde{SL_2}$ *type* if $W_K$ admits an $\widetilde{SL_2}$ geometry, etc. When $K$ is a knot with Alexander polynomial $\Delta_K$, we have $|e(W_K)| = |H_1(W_K)|/(p_1 \dots p_n)$ [**11**, Corollary 6.2], while $|H_1(W_K)| = |\Delta_K(-1)| \neq 0$ (see [**19**, Corollary D3]), and so $K$ has either spherical, Nil, or $\widetilde{SL_2}$ type.

This geometric viewpoint is extremely useful for analyzing when generalized Montesinos links virtually fiber. For instance, Walsh observed [**22**] that if $W_K$ is spherical, it is the quotient of $S^3$ by a finite group of isometries acting freely, and hence, as $\tilde{K}$ is geodesic, there is a branched cover $(S^3, L) \to (S^3, K)$ where $L$ is a link of great circles. The exterior of any component $L_1$ of $L$ is a solid torus in which $L \setminus L_1$ is braided. Thus, the exterior of $L$ fibers over the circle with fiber a planar surface.

THEOREM 1.1 (Walsh). *Every generalized Montesinos link having spherical type virtually fibers.*

In this paper, we investigate other types of generalized Montesinos links. For instance, there are exactly two such links of type $\mathbb{S}^2 \times \mathbb{R}$ — the trivial link of two components and a connected sum of two Hopf links (see § 4.1). The first does not virtually fiber, since its exterior is reducible, while the second does, since its exterior is homeomorphic to the product of a thrice-punctured disk and $S^1$.

In order to describe our results, we must introduce some notation.

Generalized Montesinos links are specified by an ordered set of numbers $(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$ where $g, e_0, p_i, q_i \in \mathbb{Z}, g \geqslant 0, p_i > 1$, and $(q_i, p_i) = 1$ (see [**4**, **13**, **14**]). Different sets may correspond to the same link and each generalized Montesinos link with $n > 0$ has a representative set with $e_0 = 0$. We write $K(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$ to denote the link associated to $(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$. See Figure 1.

A generalized Montesinos link with $g = 0$ is called a *classic Montesinos link*.

TABLE 1. *The type of geometry supported by* $W_K$.

| | $\chi(\mathcal{B}_K) > 0$ | $\chi(\mathcal{B}_K) = 0$ | $\chi(\mathcal{B}_K) < 0$ |
|---|---|---|---|
| $e(W_K) = 0$ | $\mathbb{S}^2 \times \mathbb{R}$ | $\mathbb{E}^3$ | $\mathbb{H}^2 \times \mathbb{R}$ |
| $e(W_K) \neq 0$ | $\mathbb{S}^3$ | Nil | $\widetilde{SL_2}$ |

THEOREM 1.2. *A generalized Montesinos link which is not a classic Montesinos link of type* $\widetilde{SL_2}$ *virtually fibers if and only if it is not a trivial link of two components. In particular, it virtually fibers if either it is of type* $\mathbb{S}^3, \mathbb{E}^3, \mathrm{Nil}$, *or* $\mathbb{H}^2 \times \mathbb{R}$, *or it is a nonclassic Montesinos link of type* $\widetilde{SL_2}$.

It follows from the proof of Theorem 1.2 that a stronger statement holds for classic Montesinos links.

THEOREM 1.3. *If* $K$ *is a classic Montesinos link of type* $\mathbb{E}^3, \mathrm{Nil}$, *or* $\mathbb{H}^2 \times \mathbb{R}$, *there is a finite degree cover* $\Psi : Y \to W_K$ *such that* $Y$ *fibers over the circle in such a way that* $L = \Psi^{-1}(\tilde{K})$ *is transverse to all the surface fibers.*

This refinement of Theorem 1.2 has an interesting consequence.

COROLLARY 1.4. *Any finite-degree cover of the* 3*-sphere branched over a classic Montesinos link* $K$ *of type* $\mathbb{E}^3, \mathrm{Nil}$, *or* $\mathbb{H}^2 \times \mathbb{R}$ *which factors through the* 2*-fold cover* $(W_K, \tilde{K}) \to (S^3, K)$ *is a closed, virtually fibered* 3*-manifold.*

In independent work, Jason DeBlois has proved special cases of this corollary using a similar method (see [**6**, Theorem 1 and § 1]).

COROLLARY 1.5. *For any* $m \geqslant 1$, *the* $2m$*-fold cyclic cover of* $S^3$ *branched over a classic Montesinos link* $K$ *of type* $\mathbb{E}^3, \mathrm{Nil}$, *or* $\mathbb{H}^2 \times \mathbb{R}$ *is a closed, virtually fibered* 3*-manifold. If* $m \geqslant 2$ *and* $K$ *has type* Nil or $\mathbb{H}^2 \times \mathbb{R}$, *the set of* $2m$*-fold branched cyclic covers of* $K$ *is a family of hyperbolic manifolds which represent infinitely many different commensurability classes.*

A 3-manifold is *semifibered* if it fibers over a mirrored interval. This is equivalent to the existence of a separating surface which splits the manifold into two twisted $I$-bundles. Note that while such 3-manifolds do not fiber over the circle in general, they do admit 2-fold covers which do, and therefore provide *trivial* examples of nonfibered, virtually fibered 3-manifolds. Corollary 1.4 can be used to construct infinite families of nontrivial examples. Since the $2^m$-fold cyclic cover of $S^3$ branched over $K$ is a $\mathbb{Z}/2$-homology 3-sphere (see [**10**, § 5]), the following corollary provides a simple such family.

COROLLARY 1.6. *If* $K$ *is a classic Montesinos knot of* Nil *type and* $m \geqslant 2$, *the* $2^m$*-fold cyclic cover of* $S^3$ *branched over* $K$ *is a closed, hyperbolic, nonfibered, non-semifibered, virtually fibered* 3*-manifold. Further, these branched covers of* $K$ *represent infinitely many different commensurability classes.*

This appears to be the first infinite family of closed, nontrivially virtually fibered hyperbolic 3-manifolds known. A particular example which satisfies the hypotheses of Corollary 1.6 is given by the $(3, 3, -3)$-pretzel knot $K = K(0; 0; \frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$ (see Figure 2). In fact, all even-order branched cyclic covers of $K$ give nontrivial examples. This follows from the fact that the order of the first homology of the $n$-fold branched cover of $K$ is the resultant of $\Delta_K(t) = -2t^2 + 5t - 2$ of $K$ and the polynomial $t^n - 1$ (see [**10**, § 5]).

The proof of Theorem 1.2 depends on a result of independent interest.

THEOREM 1.7. *Let* $p : M \to S^1$ *be a fibering of a compact orientable* 3*-manifold, and let* $L$ *be a link lying in the interior of a finite union of surface fibers such that each component of* $L$ *is nontrivial in* $H_1(M; \mathbb{Q})$. *Then* $M$ *fibers over* $S^1$ *with fibers transverse to* $L$. *In particular,*

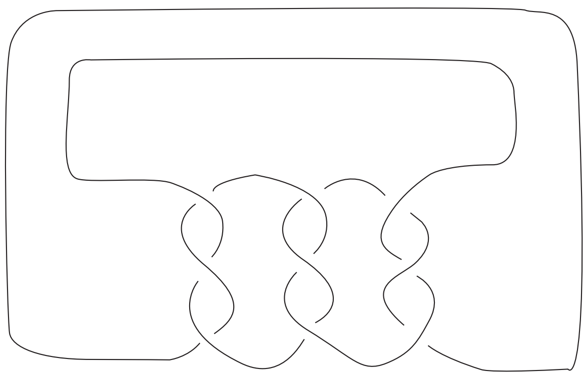


FIGURE 2. *The knot* $K(0;0;\frac{1}{3},\frac{1}{3},-\frac{1}{3})$.

*if* $M$ *is closed, there is a fibering* $M \setminus L \to S^1$ *such that the boundary of each surface fiber consists of meridians of the components of* $L$.

Here is a consequence.

COROLLARY 1.8. *If* $W$ *is a compact, connected, irreducible, orientable Seifert manifold,* $F$ *a not necessarily connected essential surface in* $W$, *and* $\emptyset \neq L \subset \mathrm{int}(F)$ *a link for which each component is homotopically essential in* $F$, *then* $W \setminus L$ *virtually fibers.*

The generic generalized Montesinos link has type $\widetilde{SL_2}$. Our next theorem constructs an infinite family of such links which virtually fiber.

THEOREM 1.9. *Let* $p \geqslant 3$ *be an odd integer. If* $n$ *is a positive multiple of* $p$, *then the generalized Montesinos link* $K(-g;e_0;(q_1/p),\dots,(q_n/p))$ *is virtually fibered.*

The links described in this result are of type $\widetilde{SL_2}$ when $pe_0 \neq -(q_1+q_2+\dots+q_n)$ and $(g,p,n) \notin \{(0,3,3),(0,p,0),(1,p,0),(2,p,0)\}$ (cf. identities (2), (3), and Table 1).

Most of the links discussed in the theorems above are not fibered. For example, see [**9**] for the classification of fibered pretzel links. Further, most classic Montesinos links are hyperbolic in the sense that their complements admit complete, finite volume hyperbolic structures. Indeed, Oertel has shown that a classic Montesinos link is either an atoroidal Seifert link (that is, its exterior is atoroidal and admits a Seifert fibered structure), or a Montesinos link of type $\mathbb{E}^3$, or a hyperbolic link (see [**16**, proof of Corollary 5]). Further, it can be shown that if $K(0;0;(q_1/p_1),\dots,(q_n/p_n))$ is a Seifert link with $n \geqslant 3$, then it is a Montesinos link of spherical type (cf. the proof of Corollary 1.4). In particular, all classic Montesinos links of type Nil, $\mathbb{H}^2 \times \mathbb{R}$, or $\widetilde{SL_2}$ are hyperbolic.

The proofs of the results discussed above use the geometric constraints on $W_K$ imposed by the hypotheses in an essential way. That of Theorem 1.9 is the most delicate and uses the strategy set out by Walsh. One shows that there is a cover of $S^3$, branched over the link in question, and a component of the branch set whose exterior admits a surface bundle structure, with the property that all the other components of the branch set are transverse to its fibers. This implies the desired conclusion. In all cases, the construction of the fiberings is explicit enough for us to deduce a strong property of the fundamental groups of the associated manifolds.

A group is called *biorderable* if it admits a total order which is invariant under both left and right multiplication, and *virtually biorderable* if it has a finite index biorderable subgroup. Though the fundamental groups of 3-manifolds are rarely biorderable, it is possible that they are always virtually biorderable (cf. [**3**]).

Theorem 1.10. *The fundamental groups of the exteriors of generalized Montesinos links which are not classic Montesinos links of type $\widetilde{SL_2}$ are virtually biorderable, as are the fundamental groups of the classic Montesinos links of type $\widetilde{SL_2}$ described in Theorem 1.9.*

Dale Rolfsen was independently aware of parts of this theorem.

We review several constructions and set our conventions in § 2. The virtual fibering of links contained in essential surfaces is considered in § 3 where Theorem 1.7 and Corollary 1.8 are proven. Section 4 considers the virtual fibering of generalized Montesinos links other than classic ones of type $\widetilde{SL_2}$ and in particular, Theorems 1.2 and 1.3 are dealt with there. The latter is applied in § 5 to show certain branched covers of classic Montesinos links virtually fiber. The proofs of Corollaries 1.4 and 1.5 are contained in this section. The final two sections of the paper deal with Theorems 1.9 and 1.10, respectively.

## 2. *Some constructions, conventions, and notation*

We will use the notation introduced above as well as the following constructions and conventions throughout the paper.

Let $K = K(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$ be a generalized Montesinos link and $(W_K, \tilde{K}) \to (S^3, K)$ the 2-fold branched cover described in the introductory section. We endow $W_K$ with the Seifert fibered structure $f : W_K \to \mathcal{B}_K$ described by Montesinos [**13**, **14**]. In particular, $\tilde{K}$ has exactly $g$ *vertical* components which are fibers of the given Seifert structure. Its remaining *horizontal* components intersect each Seifert fiber in either zero or two points. We use $\tilde{K}_v$ and $\tilde{K}_h$ to denote, respectively, the union of the vertical and horizontal components of $\tilde{K}$, and $K_h, K_v \subset K$ their images in $S^3$.

The orbifold $\mathcal{B}_K$ has topological form $(\#_g P^2)(p_1, \dots, p_n)$ and the image of $\tilde{K}$ in $\mathcal{B}_K$ is the disjoint union of the $g$ points $f(\tilde{K}_v)$, and a simple closed curve $K^* = f(\tilde{K}_h)$ which contains the cone points of $\mathcal{B}_K$. Number the cone points $c_1, \dots, c_n$ respecting the order with which they occur on $K^*$. This can be done so that each $c_i$ has order $p_i$.

The number of components $|K|$ of $K$ is given by

$$|K| = \begin{cases} g+1 & \text{if each } p_i \text{ is odd and } (e_0 + q_1 + \dots + q_n) \text{ is odd,} \\ g+2 & \text{if each } p_i \text{ is odd and } (e_0 + q_1 + \dots + q_n) \text{ is even,} \\ g + \#\{i : p_i \text{ is even}\} & \text{if some } p_i \text{ is even,} \end{cases} \tag{1}$$

while the Euler number of $W_K$ and Euler characteristic of $\mathcal{B}_K$ are given by

$$e(W_K) = -\left(e_0 + \sum \frac{q_j}{p_j}\right), \tag{2}$$

$$\chi(\mathcal{B}_K) = 2 - g - n + \sum \frac{1}{p_j}. \tag{3}$$

Fix a geometric structure on $W_K$ such that the covering involution $\tau$ is an isometry and the branch set $\tilde{K} = \mathrm{Fix}(\tau)$ is geodesic. There is an associated geometric structure on $\mathcal{B}_K$ such that the image of a geodesic in $W_K$ is either a point or a geodesic. In particular, $K^* = f(\tilde{K}_h)$ is geodesic. If we define $K_0^*$ to be the 1-orbifold obtained by cutting open $K^*$ at the cone points of even order and thinking of the resulting endpoints as mirror points, then the map $f| : \tilde{K}_h \to K_0^*$ is an orbifold double cover.

All coverings of $W_K$ and orbifold coverings of $\mathcal{B}_K$ will be endowed with the induced metrics. The Seifert fibers of $W_K$, or its covers, will be denoted by $\phi$.

LEMMA 2.1. *Let $g: V \to W_K$ be a smooth cover and $J_h = g^{-1}(\tilde{K}_h)$. Endow $V$ with the geometric structure and Seifert fibering induced by $g$. If $\phi$ is a Seifert fiber of $V$, then wherever $\phi$ intersects $J_h$, the intersection is perpendicular.*

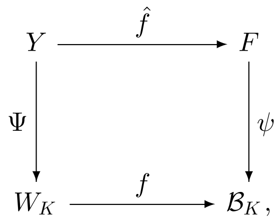
*Proof.* It suffices to verify that the lemma holds when $g = 1_{W_K}$.

Let $x \in \tilde{K}_h \cap \phi \subset W_K$ and $u, v \in T_x W_K$ be unit vectors tangent at $x$ to $\tilde{K}_h$ and $\phi$, respectively. Clearly, $d_x\tau(u) = u$ and since $\tau$ is an isometry of $W_K$ which leaves $\phi$ invariant, $d_x\tau(v) = \pm v$. Since $\tau$ is an orientation-preserving isometry of $W_K$ which is not the identity, we must have $d_x\tau(v) = -v$. Hence $\langle u, v\rangle_x = \langle d_x\tau(u), d_x\tau(v)\rangle_x = -\langle u, v\rangle_x$, which proves the lemma. □

It follows from the proof of the lemma that $\tau$ reverses the orientation of any Seifert fiber of $W_K$ which intersects some component of $\tilde{K}_h$.

If $W_K$ has an infinite fundamental group, there is a finite regular cover $\Psi : Y \to W_K$ such that the induced Seifert structure on $Y$ is a locally trivial circle bundle over an orientable surface $F$, and there is an induced orbifold cover $F \to \mathcal{B}_K$. Further, if $e(W_K) = 0$, then $Y = F \times S^1$ and $\hat{f}$ is the projection $F \times S^1 \to F$. A convenient way to construct such covers is as follows.

It is well known that 2-dimensional orbifolds with nonpositive Euler characteristic admit finite regular orbifold covers which are surfaces (that is, have no singular points). Fix such a cover $\psi : F \to \mathcal{B}_K$ and recall that there is an associated commutative diagram



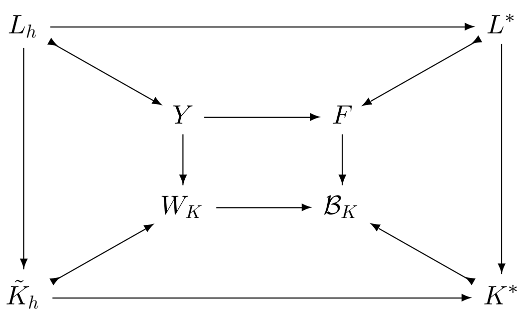

(4)

where the following hold:

(1) $\Psi$ is a regular cover of the same degree as $\psi$ (and the same group of deck transformations);
(2) the Seifert structure on $Y$ induced by $\Psi$ has base orbifold $F$, and $\hat{f}$ is the Seifert quotient map.

Note that $\hat{f} : Y \to F$ is a locally trivial circle bundle and the generic fiber of $Y$ is mapped homeomorphically to a fiber of $W_K$.

Set $L = \Psi^{-1}(\tilde{K}) \subset Y$ and $L^* = \psi^{-1}(K^*) \subset F$. By construction, $L$ is a disjoint union of $L_h = \Psi^{-1}(\tilde{K}_h)$, a geodesic link in $Y$ orthogonal to the fibers of the Seifert structure, and $L_v = \Psi^{-1}(\tilde{K}_v)$, a finite union of Seifert fibers of $Y$. Clearly, $L^* = \hat{f}(L_h)$. Hence $L^*$ is a finite union of closed geodesics in $F$ (a *geodesic* 1-*complex*).




(5)

LEMMA 2.2. *If $K = K(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$ is a generalized Montesinos link with $e(W_K) = 0$, then $W_K$ admits the structure of a surface bundle or semibundle whose fibers are totally geodesic and everywhere orthogonal to the Seifert structure. Further, $\tilde{K}_h$ is contained in a finite union of surface fibers.*

*Proof.* First observe that $W_K$ admits an $\mathbb{X}^2 \times \mathbb{R}$ structure where $\mathbb{X}^2 = \mathbb{S}^2, \mathbb{H}^2$, or $\mathbb{E}^2$ depending on whether $\chi^{\mathrm{orb}}(S^2(p_1, \dots, p_n))$ is positive, negative, or zero. This information is contained in Table 1 when $\chi^{\mathrm{orb}}(S^2(p_1, \dots, p_n)) \neq 0$ and in the discussion following Theorem 4.3 of [**20**] when $\chi^{\mathrm{orb}}(S^2(p_1, \dots, p_n)) = 0$. The surfaces $\mathbb{X}^2 \times \{*\} \subset \mathbb{X}^2 \times \mathbb{R}$ descend to give $W$ the structure of a bundle $\mathcal{F}$ of closed totally geodesic surfaces over a 1-dimensional orbifold $\mathcal{O}^1$ whose leaves are perpendicular to the Seifert fibers (see [**20**, § 4]). Lemma 2.1 then shows that each component of $\tilde{K}_h$ is contained in some surface fiber of $\mathcal{F}$. □

REMARK 2.3. It follows from the proof that the generic surface fiber of $\mathcal{F}$ is a 2-sphere if $\chi(\mathcal{B}_K) > 0$, a torus if $\chi(\mathcal{B}_K) = 0$, and a hyperbolic surface if $\chi(\mathcal{B}_K) < 0$.

In the remainder of the paper, homology and cohomology groups will be understood to have $\mathbb{Z}$ coefficients unless otherwise specified.

## 3. *Virtual fiberings of links in essential surfaces*

We prove Theorem 1.7 and Corollary 1.8 in this section.

LEMMA 3.1. *Let $M$ be an oriented 3-manifold, $L \subset M$ an oriented link, $\alpha \in H^1(M)$ a cohomology class which is nonzero on each component of $L$. Then there exists a closed 1-form $\omega$ such that $[\omega] = \alpha$, and such that $\omega_{|TL}$ is nonzero. That is, for each $v \neq 0$ tangent to a component of $L$, $\omega(v) \neq 0$.*

*Proof.* Let $\mathcal{N}(L)$ be a regular neighborhood of $L$ diffeomorphic to $L \times D^2$ and fix a smooth map $p : M \to S^1$ such that $\alpha$ is represented by $p^*\eta$, where $\eta \in \Lambda^1(S^1)$ is a 1-form representing the fundamental class of $S^1$. Since $\alpha$ is nonzero on each component of $L$, we can homotope $p_{|L}$ to a smooth covering map $f : L \to S^1$. Let $f' : \mathcal{N}(L) \to S^1$ be the composition $\mathcal{N}(L) = L \times D^2 \xrightarrow{\mathrm{proj}} L \xrightarrow{f} S^1$. Since the inclusion $\mathcal{N}(L) \to M$ is a cofibration, $p$ is homotopic to a map $F : M \to S^1$ such that $F_{|\mathcal{N}(L)} = f'$. Finally, if we perturb $F$ rel $(L \times \frac{1}{2} D^2)$ to a smooth map $F' : M \to S^1$, then $\omega = (F')^*\eta$ is a closed 1-form on $M$ which represents $\alpha$ and which is nowhere vanishing on $TL$. □

*Proof of Theorem 1.7.* Since every component of $L$ is homologically nontrivial in $H_1(M; \mathbb{Q})$, we may find a cohomology class $\alpha \in H^1(M)$ which is nonvanishing on each such component. By Lemma 3.1, there is a closed 1-form $\omega \in \Lambda^1(M)$ representing $\alpha$ which is nowhere vanishing on the tangent vectors to $L$.

Let $\eta \in \Lambda^1(S^1)$ be a nowhere zero closed 1-form on $S^1$ representing the fundamental class of $H^1(S^1)$. Then $p^*\eta$ vanishes on the tangent space of each fiber of $p$, and so in particular, it vanishes on all tangent vectors to $L$. Note, moreover, that as $M$ is compact, there is an integer $N \gg 0$ such that $\beta = \omega + Np^*\eta$ is a nowhere vanishing closed 1-form which represents an integral cohomology class and whose restriction to $TL$ is also nowhere vanishing. Thus, kernel($\beta$) represents a plane field tangent to a fibering of $M \to S^1$ [**21**] such that the fibers are everywhere transverse to $L$. The remaining conclusions of the theorem follow immediately from this. □

We complete this section with the proof of Corollary 1.8. We begin with a lemma.

LEMMA 3.2. *Let $F$ be a compact, connected, orientable surface, and $L \subset F \times S^1$ be a link contained in the interior of a finite union of the surface fibers $F \times \{x\}$. Let $L_1, L_2, \dots, L_m$ be the components $L$, and $L_1^*, L_2^*, \dots, L_m^*$ their projection in $F$. If each $L_i^*$ is homotopically nontrivial in $F$, there is a finite connected cover $\psi : \tilde{F} \to F$ such that each component of the inverse image of each $L_i^*$ in $\tilde{F}$ is nontrivial in $H_1(\tilde{F})$.*

*Proof.* Take $p : \tilde{F} \to F$ to be the cover corresponding to the homomorphism $\pi_1(F) \to H_1(F; \mathbb{Z}/2)$. We claim that each component of the inverse image by $p$ of an essential simple closed curve $C \subset F$ is homologically nontrivial in $\tilde{F}$. To see this, first note that given $C$, it suffices to find a 2-fold cover of $F$ to which $C$ lifts to homologically nontrivial curves, since $p$ factors through any such cover. If $C$ is homologically nontrivial in $F$, any 2-fold cover will do. If $C$ is homologically trivial, we have $F = F_0 \cup F_1$ where $\partial F_0 = C \subset \partial F_1$. The essential nature of $C$ implies that $H_1(F_0; \mathbb{Z}/2), H_1(F_1; \mathbb{Z}/2) \neq 0$. If $\partial F_1 = C$, we can find a homomorphism $\pi_1(F) \to \mathbb{Z}/2$ which induces epimorphisms $\pi_1(F_j) \to \mathbb{Z}/2$ for $j = 0, 1$ and it is easy to see that the associated cover is of the desired form. On the other hand, if $\partial F_1 \neq C$, there is a homomorphism $\pi_1(F) \to \mathbb{Z}/2$ which induces an epimorphism $\pi_1(F_0) \to \mathbb{Z}/2$ and the zero homomorphism $\pi_1(F_1) \to \mathbb{Z}/2$. Then each lift of $C$ to the associated cover separates distinct boundary components of the underlying surface, so the proof is complete. □

Our next result follows immediately from Lemma 3.2 and Theorem 1.7.

COROLLARY 3.3. *Let $F$ be a compact, connected, orientable surface, and $L \subset F \times S^1$ a link contained in the interior of a finite union of the surface fibers $F \times \{x\}$. If the projection of each component of $L$ is homotopically nontrivial in $F$, there is a finite degree cover $\Psi : Y = \tilde{F} \times S^1 \to F \times S^1$ and a surface bundle structure on $Y$ which is everywhere transverse to $\tilde{L} = \Psi^{-1}(L)$. In particular, if $F$ is closed, the boundary of any fiber in the induced bundle structure on the exterior of $\tilde{L}$ consists of meridians of $\tilde{L}$.*

*Proof of Corollary* 1.8. Let $W$ be a connected, irreducible, orientable Seifert manifold, $F$ a not necessarily connected essential surface in $W$, and $\emptyset \neq L \subset \mathrm{int}(F)$ a link each component of which is homotopically essential in $F$. Then no component of $F$ is a sphere or a disk. Since $W$ is irreducible, we can suppose, after an isotopy, that the components of $F$ are either all horizontal or all vertical. In the former case, there is a finite cyclic cover $Y = F_0 \times S^1 \to W$ where $F_0$ is a component of $F$. The inverse image of $F$ in $Y$ is a union of fibers $F_0 \times \{*\}$ which contains the inverse image $\hat{L}$ of $L$. Since each component of $\hat{L}$ is homotopically nontrivial in the surface fiber that contains it, Lemma 3.2 implies that there is a finite connected cover $\tilde{F}_0 \to F_0$ such that each component of the inverse image $\tilde{L}$ of $\hat{L}$ in $\tilde{Y} = \tilde{F}_0 \times S^1$ (under the natural covering map $\tilde{Y} \to Y$) is homologically nontrivial in the surface fiber $\tilde{F}_0 \times \{*\}$ that contains it. Theorem 1.7 then shows that there is a product structure on $\tilde{Y}$ such that $\tilde{L}$ is everywhere transverse to the surface fibers, which implies the desired result.

Now suppose that $F$ is a finite union of vertical surfaces. Then $L \subset F_1 \cup \ldots \cup F_n$, where each $F_i$ is a vertical torus or annulus. We can suppose that for each $i$, $L \cap F_i \neq \emptyset$ and so is a finite collection of parallel curves in $F_i$. Let $\mathcal{N}(F) = \mathcal{N}(F_1) \cup \mathcal{N}(F_2) \cup \ldots \cup \mathcal{N}(F_n)$ be a fibered tubular neighborhood of $F$. That is, $\mathcal{N}(F)$ is a union of Seifert fibers of $W$. The Seifert structure on each $\mathcal{N}(F_i)$ can be altered so that $L \cap F_i$ is a union of fibers. Since $\overline{W \setminus \mathcal{N}(F)}$ inherits a Seifert structure from $W$, we see that $W \setminus L = (\overline{W \setminus \mathcal{N}(F)}) \cup (\mathcal{N}(F_1) \setminus (L \cap F_1)) \cup \ldots \cup (\mathcal{N}(F_n) \setminus (L \cap F_n))$ is a graph manifold with nonempty boundary. The main result of [**23**] states that such a manifold virtually fibers, which completes the proof of the corollary. □

## 4. *Virtual fiberings of Montesinos links other than classic ones of type* $\widetilde{SL_2}$

In this section we prove Theorems 1.2 and 1.3. To that end, let $K = K(-g; e_0; (q_1/p_1), \dots, (q_n/p_n))$ be a generalized Montesinos link which is not of type $\widetilde{SL_2}$. Given Walsh's result (Theorem 1.1), we need only consider the following cases:

(1) $K$ has type $\mathbb{S}^2 \times \mathbb{R}$;
(2) $K$ has type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$;
(3) $K$ has type Nil;
(4) $K$ is a nonclassic generalized Montesinos link of type $\widetilde{SL_2}$.

These will be dealt with individually in the following four subsections.

Remark 4.1. If $n = 0$ but $(g, e_0) \neq (0, 0)$, the exterior of $K$ is Seifert fibered for $g = 1$ and a graph manifold in general (cf. Figure 1). Thus it is virtually fibered by [**23**].

### 4.1. *$K$ has type* $\mathbb{S}^2 \times \mathbb{R}$

In this case we must prove Theorem 1.2.

We can assume that $e_0 = 0$ when $n > 0$. Then since $\chi(\mathcal{B}_K) > 0$ and $e(W_K) = 0$, identities (2) and (3) imply that either (a) $g = e_0 = n = 0$ or (b) $g = 0, n = 2$, and $(q_1/p_1) = -(q_2/p_2)$, or (c) $g = 1$ and $n = 0$. The latter case can be dealt with using Remark 4.1, but it is simpler to note that $K$ is the connected sum of two Hopf links, and so its exterior fibers over the circle with fiber a pair of pants. In case (a), $K$ is a trivial link of two components. Since the exterior of this link is reducible, it cannot be virtually fibered.

Finally consider case (b). Identity (1) shows that $K$ has two components. By applying Lemma 2.2 and the remark which follows its proof, we see that $W_K$ admits the structure of a bundle or a semibundle $\mathcal{F}$ whose generic fiber is a totally geodesic 2-sphere everywhere orthogonal to the Seifert structure. Further, $\tilde{K} = \tilde{K}_h$ is contained in a finite union of surface fibers.

Since $\mathcal{B}_K = S^2(p_1, p_1)$ is orientable, the Seifert fibers of $W_K$ can be coherently oriented, and therefore so can the surface fibers of $W_K$. Hence $W_K$ fibers over $S^1$. We noted after the proof of Lemma 2.1 that the covering involution $\tau$ reverses the orientation of the Seifert fibers which intersect $\tilde{K}_h = \tilde{K}$, and so it induces an orientation-reversing involution $\bar{\tau} : S^1 \to S^1$. Hence there are precisely two $\tau$-invariant leaves $F_1, F_2$ of $\mathcal{F}$. Clearly, $\tilde{K} = \mathrm{Fix}(\tau) \subset F_1 \cup F_2$. Since $\tau | F_j$ reverses orientation, $\mathrm{Fix}(\tau) \cap F_j$ is a circle and $F_0/\tau, F_1/\tau \subset S^3$ are disjoint 2-disks whose union has boundary $K$. Thus $K$ is a trivial link of two components, and so as above, it cannot virtually fiber.

### 4.2. *$K$ has type* $\mathbb{E}^3$ *or* $\mathbb{H}^2 \times \mathbb{R}$

We must prove Theorem 1.2 in this case and Theorem 1.3 under the additional assumption that $K$ is a classic Montesinos link.

If $K$ is a generalized Montesinos of type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$, then $W_K$ admits the structure of a surface bundle or a semibundle whose fibers are totally geodesic surfaces everywhere orthogonal to the Seifert structure (Lemma 2.2). Further, $\tilde{K}_h$ is contained in a finite union of surface fibers (or semifibers). Since $e(W_K) = 0$, there is a finite cover $\Psi : Y = F \times S^1 \to W_K$ such that the circles $\{x\} \times S^1$ are Seifert fibers of the induced Seifert structure on $Y$, and $L_h = \Psi^{-1}(\tilde{K}_h)$ is contained in a finite union of surface fibers of the form $F \times \{x\}$. Since each component of $L_h$ is geodesic in $Y$, it is homotopically essential in the surface fiber that contains it. By construction, $L_v = \Psi^{-1}(\tilde{K}_v)$ is a finite union of Seifert fibers of $Y$, and hence its exterior admits an induced product structure $F_0 \times S^1$ in which each component of $L_h$ is a homotopically essential loop in a surface fiber $F_0 \times \{x\}$. Corollary 1.8 can now be applied to complete the proof of Theorem 1.2 in the case we are considering. Further, when $K$ is a classic Montesinos link, $L_v = \emptyset$ and so

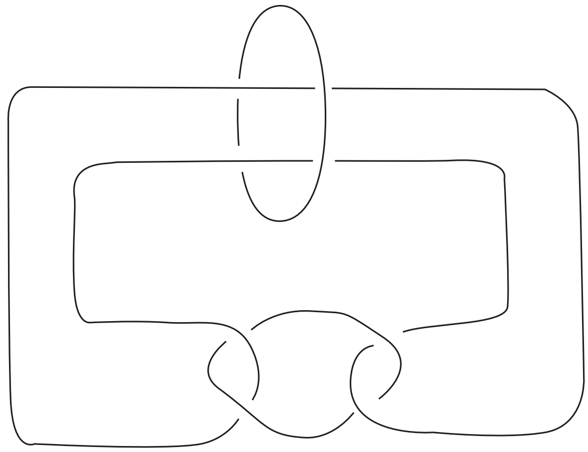


FIGURE 3. *The link* $K(-1;0;\frac{1}{2},\frac{1}{2})$.

$F_0 = F$ is closed. Hence Corollary 3.3 implies that Theorem 1.3 holds for classic Montesinos links of type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$.

### 4.3. *K has type* Nil

We prove Theorem 1.2 in this case and Theorem 1.3 under the additional assumption that $K$ is a classic Montesinos link. This will complete the proof of the latter theorem.

When $K$ has type Nil, $\mathcal{B}_K$ is Euclidean, so if $F \to \mathcal{B}_K$ is the cover discussed in § 2 (see Diagram 4), then $F$ is a torus. In particular, its closed geodesics are simple. Choose a geodesic $\alpha$ on $F$ whose slope is distinct from the slopes of the closed geodesics whose union is $L^*$, and note that $F$ fibers over the circle with every fiber a geodesic parallel to $\alpha$. Then $L^*$ is transverse to these fibers and since the restriction of $\hat{f} : L \to L^*$ is a cover away from the nonmanifold points of $L^*$, $L_h$ is transverse to the fibers of the induced torus bundle structure on $Y$. Hence the exterior of $L_h$ in $Y$ has an induced surface bundle structure $\mathcal{F}$ whose fibers have genus 1, and boundary components which are meridians of the components of $L_h$. Note that this completes the proof of Theorem 1.3 when $K$ is a classic Montesinos link of type Nil, for then $L = L_h$. It also reproves this theorem when $K$ is a classic Montesinos link of type $\mathbb{E}^3$.

To complete the proof of Theorem 1.2, suppose that $K$ is a generalized Montesinos link of type Nil with $g > 0$. We can suppose that $n > 0$ by Remark 4.1, and therefore we can arrange for $e_0 = 0$. Identity (3) then implies that $\mathcal{B}_K = P^2(2,2)$, so $g = 1$. Thus $K = K(-1;0;\frac{a}{2},\frac{b}{2})$ for some odd integers $a, b$. On the other hand, since $g = 1$, we can twist along a meridian disk in the exterior of the unknot $K_v$ to construct a homeomorphism from the exterior of $K$ to that of either $K(-1;0;\frac{1}{2},\frac{1}{2})$ or $K(-1;0;\frac{1}{2},\frac{-1}{2})$. In the first case, the exterior of $K$ is homeomorphic to a 2-fold cover of the Whitehead link exterior (see Figure 3) and so fibers over $S^1$, while in the second, its exterior is homeomorphic to the exterior of a generalized Montesinos link of type $\mathbb{E}^3$ (cf. identities (2), (3), and Table 1), and therefore virtually fibers.

### 4.4. *K is a nonclassic generalized Montesinos link of type* $\widetilde{SL_2}$

We prove Theorem 1.2 in this case. Let $K = K(-g;e_0;(q_1/p_1),\dots,(q_n/p_n))$ where $g > 0$. The $g$ components of $K_v$ are Seifert fibers in the (orbifold) Seifert structure on $S^3$, with singular set the classic Montesinos link $K_h$ and cone angle $\pi$ [**13**, **14**]. Thus the 2-fold cover $W^0_{K_h}$ of the exterior of $K_v$ branched over $K_h$ is Seifert fibered with nonempty boundary and base orbifold $\mathcal{B}^0_{K_h}$ whose underlying space is a $2g$-punctured 2-sphere and $n$ cone points of order $p_1,\dots,p_n$.

Hence $\chi(\mathcal{B}^0_{K_h}) = \chi(\mathcal{B}_K) - g < 0$, and so as $\partial W^0_{K_h} \neq \emptyset$, $W^0_{K_h}$ admits an $\mathbb{H}^2 \times \mathbb{R}$ structure. An argument like that used in the proof of Lemma 2.2 shows that $W^0_{K_h}$ admits a fibering over a 1-dimensional orbifold whose fibers are perpendicular to the Seifert structure. Further, $\tilde{K}_h$ is contained in a finite union of surface fiberes. Thus there is a finite cover $F \times S^1 \to W^0_{K_h}$ such that the inverse image $L$ of $\tilde{K}_h$ in $F \times S^1$ is contained in a finite union of sets of the form $F \times \{x\}$. Further, each component of $L$ is homotopically essential in the surface $F \times \{x\}$ that contains it. Corollary 3.3 implies that $K$ virtually fibers. This completes the proof of Theorem 1.2.

Remark 4.2. It follows from the arguments above and Remark 2.3 that for the constructed virtual fibering of a generalized Montesinos link exterior, the genus of a surface fiber is 0 when $K$ has type $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$, and 1 when $K$ has type $\mathbb{E}^3$ or Nil.

## 5. *Virtually fibered branched covers*

In this section we prove Corollary 1.4 and Corollary 1.5.

*Proof of Corollary* 1.4. Let $K = K(0; e_0; (q_1/p_1), \ldots, (q_n/p_n))$ be a classic Montesinos link of type $\mathbb{E}^3$, Nil, or $\mathbb{H}^2 \times \mathbb{R}$, and $Z \to W_K$ a cover branched over $\tilde{K}$. According to Theorem 1.3, there is a cover $Y \to W_K$, where $Y$ is a surface bundle over the circle whose fibers are transverse to the inverse image $L$ of $\tilde{K}$. Hence, the inverse image of this surface fibration in any cover of $Y$, branched over $L$, is also a surface bundle. Since the cover $\tilde{Z}$ of $Z$ induced by $Y \to W_K$ admits such a branched cover to $Y$, $Z$ virtually fibers over the circle. □

*Proof of Corollary* 1.5. It is well known that for $m \geqslant 3$, the $2m$-fold cyclic covers $\Sigma_{2m}(K) \to S^3$ branched over a hyperbolic knot $K$ are hyperbolic [**7**] and represent infinitely many different commensurability classes. To see the latter, let $M_K$ denote the exterior of $K$ and $\mu$ its meridinal slope. The orbifold $M_K(\mu^{2m})$ obtained by $\mu^{2m}$ Dehn surgery on $K$ lies in the same commensurability class as $\Sigma_{2m}(K)$, and so has the same invariant trace field. Since $\pi_1(M_K(\mu^{2m}))$ has elements of order $2m$ and its invariant trace field is generated by the traces of the squares of its elements, this number field contains $\cos(2\pi/m)$. Since the degree of the latter over $\mathbb{Q}$ tends to infinity with $m$, there are infinitely many different invariant trace fields associated to the $\Sigma_{2m}(K)$, which implies the desired conclusion.

Given Corollary 1.4, the proof of Corollary 1.5 will be completed if we show that classic Montesinos links of type Nil and $\mathbb{H}^2 \times \mathbb{R}$ are hyperbolic, for in this case the associated branched covers are hyperbolic manifolds by [**7**].

Suppose that $K$ is a classic Montesinos link of type Nil or $\mathbb{H}^2 \times \mathbb{R}$. Then $\mathcal{B}_K = S^2(p_1, \ldots, p_n)$, where $n \geqslant 3$ and $(p_1, \ldots, p_n)$ is not a Platonic triple. In this case, the fundamental group of $W_K$ is infinite, non-Abelian, and its center is infinite cyclic and generated by the class $\zeta$ represented by a regular fiber $\phi$.

We remarked in the introductory section that a classic Montesinos link is either a hyperbolic link, an $\mathbb{E}^3$ classic Montesinos link, or an atoroidal Seifert link. In the latter case, the reader will verify that $K$ is the union of at most three fibers of some Seifert structure on $S^3$ and further, the sum of the number of components of $K$ and the number of exceptional fibers in the induced Seifert structure on the exterior of $K$ is at most three. The latter Seifert structure induces a $\tau$-invariant one on $W_K$ having base orbifold a 2-sphere with at most three cone points. Further, $\tau$ preserves the orientation of the associated Seifert fibers. Since $\pi_1(W_K)$ is non-Abelian, there are precisely three cone points in this structure, and so, as above, the center of $\pi_1(W_K)$ is generated by the class $\zeta'$ represented by a regular fiber $\phi'$ of this structure. We can assume that the orientation of $\phi'$ is chosen, so that $\zeta' = \zeta$. On the other hand, $\tau$ reverses

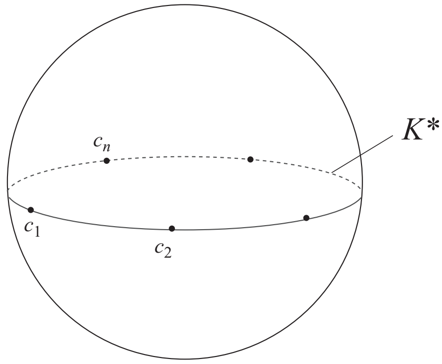



FIGURE 4. *The orbifold $S^2(p,\dots,p)$ with $n$ cone points $c_1,\dots,c_n$, each with order $p$.*

the orientation of the Montesinos Seifert fibers $\phi$, and therefore

$$\zeta^{-1} = \tau_\#(\zeta) = \tau_\#(\zeta') = \zeta' = \zeta.$$

Thus $\zeta^2 = 1$ and so as $W_K$ is irreducible, $\pi_1(W_K)$ is finite. In other words, $K$ has spherical type, contrary to our assumption. Thus Nil and $\mathbb{H}^2 \times \mathbb{R}$ classic Montesinos links are hyperbolic. This completes the proof of Corollary 1.5. □

## 6. *An infinite family of virtually fibered generalized Montesinos links of type $\widetilde{SL_2}$*

The goal of this section is to prove Theorem 1.9. To that end, let $K = K(-g; e_0; (q_1/p), \dots, (q_n/p))$ be a generalized Montesinos link where $p \geqslant 3$ is odd and $n$ is a non-negative multiple of $p$. By Remark 4.1 we can suppose that $n > 0$, and therefore that $e_0 = 0$. We also assume that we are not in the situation $p = n = 3$ or $e(W_K) = 0$ or $g > 0$ as those cases have been handled in Theorem 1.2. Thus $K$ is a classic Montesinos link of type $\widetilde{SL_2}$. It follows from identity (1) that $K$ has at most two components. We shall prove Theorem 1.9 in detail in the case where it is a knot (§ 6.1) and indicate afterwards what changes are necessary to deal with the case that it has two components (§ 6.2).

### 6.1. *$K$ is a knot*

In this case $K$ is a classic Montesinos knot, so $K = K_h$. The Seifert structure is $f : W_K \to S^2(p,\dots,p)$ and the branched cover $(W_K, \tilde{K}) \to (S^3, K)$, where $K^* = f(\tilde{K})$ is a geodesic equator of $S^2(p,p,\dots,p)$ which passes through each of the cone points (cf. Figure 4). Since $p$ is odd, $f| : \tilde{K} \to K^*$ is a double cover.

The group $\Gamma = \pi_1(S^2(p,\dots,p))$ has a presentation

$$\Gamma = \langle x_1,\dots,x_n \mid x_1^p = \dots = x_n^p = x_1x_2\dots x_n = 1\rangle,$$

where $x_i$ is represented by a small circular loop in $S^2(p,\dots,p)$ centered at $c_i$. Let $\psi : F \to S^2(p,\dots,p)$ be the $p$-fold cyclic orbifold cover of $S^2(p,\dots,p)$ corresponding to the homomorphism

$$h : \Gamma \longrightarrow \mathbb{Z}/p, x_k \longmapsto \bar{1} \text{ for each } k.$$

This homomorphism takes each $x_k$ to an element of order $p$, and therefore $F$ is a closed orientable surface without cone points. Further, $\psi^{-1}(c_i)$ is a point $\hat{c}_i \in F$ for $1 \leqslant i \leqslant n$.

Let $\Psi : Y \to W_K$ be the associated $p$-fold cyclic cover (cf. § 2) and set $L = \Psi^{-1}(\tilde{K})$, a geodesic link in $Y$. The composition $\pi_1(W_K) \to \pi_1(S^2(p,\dots,p)) \xrightarrow{h} \mathbb{Z}/p$ factors through

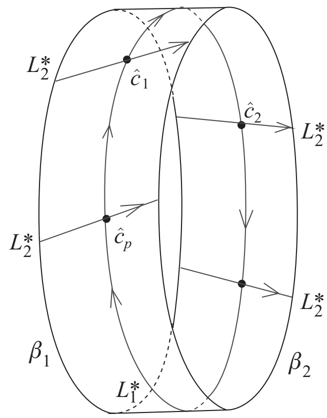



FIGURE 5. *A neighborhood of $L_1^*$ in the surface $F$.*

$H_1(W_K)$, and since $\tilde{K}$ is homologically trivial in $W_K$ (it bounds the lift of a Seifert surface for $K$), the image of a fundamental class of $\tilde{K}$ is sent to 0 under the composition. Thus $L$ has $p$ components,

$$L = L_1 \cup L_2 \cup \ldots \cup L_p.$$

Recall $L^* = \hat{f}(L)(= \psi^{-1}(K^*))$, so that

$$L^* = L_1^* \cup \ldots \cup L_p^*,$$

where $L_i^* = \hat{f}(L_i)$ is a geodesic in $F$. Since $p$ is odd, each $L_j^*$ contains $\{\hat{c}_1, \ldots, \hat{c}_n\}$. Further, $L_i^*$ is simple. To see this, first note that as $f| : \tilde{K} \to K^*$ is a 2-fold cover, $2[K^*] = 0 \in H_1(\Gamma) \cong (\mathbb{Z}/p)^{n-1}$. Since $p$ is odd, it follows that $[K^*] = 0 \in H_1(\Gamma)$. Thus $\psi| : L_j^* \to K^*$ is a homeomorphism, and thus $f| : L_j \to L_j^*$ is a 2-fold cover. Now $L_j^*$ passes successively through the $\hat{c}_i$. If the first return to $\hat{c}_1$ along $L_j^*$ is at an angle $2\pi j/p$, then following twice around you come in at an angle of $4\pi j/p$. But going twice around brings you back to the beginning, so $4\pi j/p = 2k\pi$ for some integer $k$. Thus $kp = 2j$, so that $2\pi j/p$ is either $\pi$ or $2\pi$. The first case is impossible, so $j = p$, which yields the desired conclusion.

Denote by $\gamma$ the deck transformation of $\Psi$ corresponding to $\bar{1} \in \mathbb{Z}/p$, and let $\gamma^*$ be the corresponding action on $F$. By construction, $\mathrm{Fix}(\gamma^*) = \{\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\}$ and $F$ admits an orientation such that $\gamma^*$ is rotation by $2\pi/p$ near each $\hat{c}_i$. We may suppose that $\gamma(L_j) = L_{j+1}$ and $\gamma^*(L_j^*) = L_{j+1}^*$ for each $j \pmod p$. If we fix an orientation for $L_1^*$ and give $L_j^* = (\gamma^*)^{j-1}(L_1^*)$ the induced orientation, then $L_j^*$ intersects $L_1^*$ in an angle of $(2\pi j/p)$ at each of the points $\hat{c}_i$. Thus each $L_j^*$ is a nonseparating curve in $F$, and $L_j$ is transverse to $T = \hat{f}^{-1}(L_1^*) \subset Y$ for $2 \leqslant j \leqslant p$. The situation near $L_1^*$ in $F$ is depicted in Figure 5.

By construction, $T$ is a totally geodesic torus which inherits a Euclidean structure from the $\widetilde{SL_2}$ structure on $Y$. In particular, given any pair of closed, simple geodesics $\gamma_1, \gamma_2 \subset T$ which carry a basis for $H_1(T)$, there is an identification of $T$ with $S^1 \times S^1$ where each $S^1 \times \{*\}$ is a geodesic isotopic to $\gamma_1$ and each $\{*\} \times S^1$ is a geodesic isotopic to $\gamma_2$. With respect to the product metric on $S^1 \times S^1$, the identification scales the $j$th factor by the length of $\gamma_j$.

PROPOSITION 6.1. *The exterior of $L_1$ in $Y$ is a surface semibundle.*

*Proof.* To prove this, we apply [26].

Let $R_2$ be the closed $\epsilon$-neighborhood of $L_1^*$ in $F$ ($\epsilon$ small) and $R_1$ the closure of the complement of $R_2$ in $F$. Denote by $\beta_1, \beta_2$ the two boundary components of $R_2$ (cf. Figure 5), and define $T_1$ and $T_2$ to be the vertical tori in $Y$ lying over $\beta_1$ and $\beta_2$, respectively. Let $Y_i$ be the submanifold of $Y$ lying over $R_i$ $(i = 1, 2)$ and set

$$Y_0 = Y_1 \cup_{T_1} Y_2,$$
$$R_0 = R_1 \cup_{\beta_1} R_2.$$

The restriction of the Seifert fibering of $Y$ to each $Y_i$ is a trivial circle bundle over $R_i$. We denote the fibers of any of these bundles by $\phi$.

Choose a horizontal section $B_0$ of $Y_0 \to R_0$, so that $B_0 \cap T$ is geodesic and set $B_i = B_0 \cap Y_i$ $(i = 1, 2)$. Since $B_2 \cap T$ and $\phi$, a fiber of $Y$ contained in $T$, carry a basis for $H_1(T)$, there is an identification $T = S^1 \times S^1$ (cf. the paragraph preceding this proposition) such that $S^1 \times \{1\} = B_2 \cap T$ and for each $* \in S^1$, $S^1 \times \{*\}$ is a geodesic isotopic to $B_2 \cap T$, and $\{*\} \times S^1$ is geodesic fiber of $Y \to F$.

By construction, $Y_2$ is an $\epsilon$-neighborhood of $T$ in $Y$. Identify it with $T \times [-\epsilon, \epsilon]$ where $T = T \times \{0\}$ and for $x \in T$, $\{x\} \times [-\epsilon, \epsilon]$ gives the unit speed parameterization of the normal geodesic to $T$ at $x$. Without loss of generality, we suppose that $B_2 = (B_2 \cap T) \times [-\epsilon, \epsilon]$.

We shall call an arc $\sigma \times \{*\} \subset T \times \{*\}$ *straight* if $\sigma$ is geodesic in $T$. Each fiber of $Y_2 \to R_2$ is straight.

Orient $R_0$ and let $R_1$ and $R_2$ and their boundaries have the induced orientations. Equip $B_0, B_1, B_2$ and their boundaries with the induced orientations. Next, orient the circle fibers of $Y_0$ consistently and fix two such fibers $\phi_1, \phi_2$ on $T_1, T_2$, respectively. We shall write $T_{i,j}$ for $T_j$ when we think of it as a boundary component of $Y_i$. Similarly, we write $\phi_{i.j}$ for $\phi_j$ when we think of it as lying in $T_{i,j}$. Set $\alpha_{i,j} = B_i \cap T_{i,j}$, the boundary component of $B_i$ on $T_{i,j}$. Note that (abusing notation) $\{\alpha_{i,j}, \phi_{i,j}\}$ forms a basis of $H_1(T_{i,j})$. As *homology classes in* $Y_i$, we have $\alpha_{i,1} = -\alpha_{i,2}$ and $\phi_{i,1} = \phi_{i,2}$.

The Seifert manifold $Y$ is obtained from $Y_1$ and $Y_2$ by gluing $T_{1,1}$ to $T_{2,1}$ and $T_{1,2}$ to $T_{2,2}$ using the maps $g_j : T_{2,j} \to T_{1,j}$ determined by the conditions

$$\begin{aligned} (g_1)_*(\alpha_{2,1}) &= -\alpha_{1,1} & \qquad (g_1)_*(\phi_{2,1}) &= \phi_{1,1} \\ (g_2)_*(\alpha_{2,2}) &= -\alpha_{1,2} + e\phi_{1,2} & \qquad (g_2)_*(\phi_{2,2}) &= \phi_{1,2}, \end{aligned}$$

where $e \in \mathbb{Z}$ is the Euler number of the oriented circle bundle $Y \to F$. The naturality properties of Euler numbers imply that

$$|e| = p|e(W_K)| = p\left|\sum_j \frac{q_j}{p_j}\right| = |q_1 + q_2 + \ldots + q_n|,$$

and therefore as $p$ is odd and $K$ has only one component, $e$ is an odd integer (cf. identity (1)).

Note that $T$ is fibered by geodesics isotopic to $L_1$, and this determines a fibering of $Y_2 = T \times [-\epsilon, \epsilon]$ by circles. Since $L_1$ has slope in $T$ different from that of the fibers of $Y$, this leads to a new circle fibration $Y_2 \to \bar{R}_2$ with $\bar{R}_2$ an annulus. We call the fibering of $Y$ coming from its Seifert structure the *original fibration* and its fibers, denoted $\phi$, *original fibers*. In the same vein, $Y_2 \to \bar{R}_2$ will be called the *new fibration* and its fibers, denoted $\bar{\phi}$, *new fibers*. Subsets of $Y$ that are vertical in the original fibration will be called $\phi$-vertical. Subsets of $Y_2$ that are vertical in the new fibration will be called $\bar{\phi}$-vertical.

Choose a section of $Y_2 \to \bar{R}_2$ whose image $\bar{B}_2$ intersects $T$ in a geodesic. This gives a new product decomposition $T = S^1 \times S^1$ where $S^1 \times \{1\} = \bar{B}_2 \cap T$, and $\{*\} \times S^1$ is a new fiber for each $* \in S^1$ (cf. the remarks prior to the statement of this proposition). It also gives a new product decomposition $Y_2 = \bar{R}_2 \times S^1$.

Let $N$ be a regular neighborhood of $L_1$ in $Y_2$, disjoint from $L_2 \cup \ldots \cup L_p$, which consists of new fibers. We denote the exterior of $L_1$ in $Y_2$ by $M_2$ and realize it concretely as $M_2 = Y_2 \setminus \text{int}(N)$. The boundary of $M_2$ consists of the tori $T_{2,1}, T_{2,2}$, and a third one $T_{2,3} = \partial N$. Now take $M$ to be the exterior of $L_1$ in $Y$ realized as $M = Y_1 \cup M_2$. The new fibration of $Y_2$ yields a decomposition $M_2 = \bar{R}_2^0 \times S^1$, where $\bar{R}_2^0 \subset \bar{R}_2$ is a once-punctured annulus. Hence $M$ is a graph manifold whose characteristic tori are $T_1$ and $T_2$.

Let $\bar{B}_2^0$ be the image of a section of $M_2 \to \bar{R}_2^0$, oriented arbitrarily, and endow $\partial\bar{B}_2^0$ with the induced orientation. Fix an orientation for the circle factor $\bar{\phi}$ of $M_2$.

Denote by $\bar{\alpha}_{2,j}$ the component of $\partial\bar{B}_2^0$ on $T_{2,j}$ $(j = 1, 2)$ and let $\bar{\alpha}_{2,3}$ be the remaining component. Finally, let $\bar{\phi}_{2,j}$ be a fixed new fiber with the given orientation in $T_{2,j}$ $(j = 1, 2, 3)$. Then $\{\bar{\alpha}_{2,j}, \bar{\phi}_{2,j}\}$ forms a basis for $H_1(T_{2,j})$ $(j = 1, 2, 3)$. The relation between the old basis $\{\alpha_{2,1}, \phi_{2,1}\}$ of $H_1(T_{2,1})$ and the new one $\{\bar{\alpha}_{2,1}, \bar{\phi}_{2,1}\}$ is given by

$$\bar{\alpha}_{2,1} = a\alpha_{2,1} + b\phi_{2,1} \qquad \bar{\phi}_{2,1} = c\alpha_{2,1} + d\phi_{2,1},$$

where $a, b, c, d$ are integers satisfying $ad - bc = \pm 1$. By reversing the orientation of the fibers $\phi$ if necessary, we may suppose that $ad - bc = 1$. Note that $c \neq 0$, since the slope of $L_1$ on $T$ is different from that of the original fiber. We can assume that $c > 0$ while maintaining the conditions above by simultaneously changing, if necessary, the orientation of $\bar{B}_2^0$ and the new fibers. In fact, $c = 2$ since $L_1$ double covers $L_1^*$, though we will not use this fact till later in the proof.

As homology classes in $Y_2$, we have $\bar{\alpha}_{2,1} = -\bar{\alpha}_{2,2}$ and $\bar{\phi}_{2,1} = \bar{\phi}_{2,2}$. Thus

$$\bar{\alpha}_{2,2} = a\alpha_{2,2} - b\phi_{2,2} \qquad \bar{\phi}_{2,2} = -c\alpha_{2,2} + d\phi_{2,2}.$$

Hence, with respect to the bases $\{\alpha_{1,j}, \phi_{1,j}\}$ of $H_1(T_{1,j})$ and $\{\bar{\alpha}_{2,j}, \bar{\phi}_{2,j}\}$ of $H_1(T_{2,j})$ $(j = 1, 2)$, the gluing maps $g_1 : T_{2,1} \to T_{1,1}$ and $g_2 : T_{2,2} \to T_{1,2}$ can be expressed as

$$(g_1)_*(\bar{\alpha}_{2,1}) = -a\alpha_{1,1} + b\phi_{1,1} \qquad (g_1)_*(\bar{\phi}_{2,1}) = -c\alpha_{1,1} + d\phi_{1,1}$$
$$(g_2)_*(\bar{\alpha}_{2,2}) = -a\alpha_{1,2} + (ae - b)\phi_{1,2} \qquad (g_2)_*(\bar{\phi}_{2,2}) = c\alpha_{1,2} + (d - ce)\phi_{1,2}.$$

The associated matrices are

$$G_1 = (g_1)_* = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix} \quad G_2 = (g_2)_* = \begin{pmatrix} -a & ae - b \\ c & d - ce \end{pmatrix},$$

with inverses

$$G_1^{-1} = \begin{pmatrix} -d & b \\ -c & a \end{pmatrix} \quad G_2^{-1} = \begin{pmatrix} ce - d & ae - b \\ c & a \end{pmatrix}.$$

We claim that the method of [23] can be used to show that there are essential horizontal surfaces $H_1$ and $H_2$ contained in $Y_1$ and $M_2$, which piece together to form an essential surface $H$ in the graph manifold $M$. Further, $H$ is nonorientable so that $M$ is the total space of a semibundle. We will provide some of the details, though refer the reader to [23] for more complete explanations.

First, we calculate the $2 \times 2$ matrix $Y = (y_{ij})$ and the $2 \times 2$ diagonal matrix $Z = \mathrm{diag}(z_1, z_2)$ defined on [23, p. 450]. The graph of the JSJ-decomposition of $M$ consists of two vertices $v_1$ and $v_2$, corresponding to the pieces $Y_1$ and $M_2$, and two edges corresponding to $T_1$ and $T_2$. Thus, referring to [23] shows that $y_{11} = y_{22} = 0$, $y_{12} = y_{21} = \frac{2}{c}$, while $z_1 = (d/c) + (ce - d/c) = e$, $z_2 = (a/c) + (-a/c) = 0$. The matrix equation (1.6) of [23] becomes

$$\begin{pmatrix} -e & \frac{2}{c} \\ \frac{2}{c} & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \star \end{pmatrix},$$

which we solve in integers: $\lambda = (2r/c)$, $\bar{\lambda} = re$ for some $r \in \mathbb{Z}$ such that $2r$ is divisible by $c$. This solution yields horizontal surfaces $H_1, H_2$ as above such that the projection of $H_1$ to $R_1$ has degree $|\lambda|$, and that of $H_2$ to the base of $M_2$ has degree $|\bar{\lambda}|$ [23]. The boundary slopes of $H_1, H_2$ can be determined as follows. Assume that $r \neq 0$ and recall that $e \neq 0$. Then

$$\frac{\lambda}{\bar{\lambda}} = \frac{2}{ce}, \quad \frac{\bar{\lambda}}{\lambda} = \frac{ce}{2}.$$

Suppose that the slope of $H_1$ on $T_j$ is given by $u_j\alpha_{1,j} + t_j\phi_{1,j}$ $(j = 1, 2)$ and that of $H_2$ on $T_j$ is $\bar{u}_j\bar{\alpha}_{2,j} + \bar{t}_j\bar{\phi}_{2,j}$ $(j = 1, 2, 3)$. Then there are $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, so that equation (1.2) of [23] takes on the form

$$\frac{t_1}{u_1} = \frac{\epsilon_1\bar\lambda}{-\lambda c} - \frac{d}{c} = \frac{-\epsilon_1 ce - 2d}{2c} \qquad \frac{\bar t_1}{\bar u_1} = \frac{\epsilon_1\lambda}{-\bar\lambda c} + \frac{a}{-c} = \frac{-2\epsilon_1 - ace}{ec^2}$$

$$\frac{t_2}{u_2} = \frac{\epsilon_2\bar\lambda}{\lambda c} - \frac{ce-d}{c} = \frac{2d-(2-\epsilon_2)ce}{2c} \qquad \frac{\bar t_2}{\bar u_2} = \frac{\epsilon_2\lambda}{\bar\lambda c} + \frac{a}{c} = \frac{2\epsilon_2 + ace}{ec^2}.$$

We require that $(t_1/u_1) + (t_2/u_2) = 0$ [23], so $1 = \epsilon_2 = -\epsilon_1$. Hence

$$\frac{t_1}{u_1} = \frac{ce-2d}{2c} \qquad \frac{\bar t_1}{\bar u_1} = \frac{2-ace}{ec^2}$$

$$\frac{t_2}{u_2} = \frac{2d-ce}{2c} \qquad \frac{\bar t_2}{\bar u_2} = \frac{2+ace}{ec^2}.$$

Next we require $(\bar t_1/\bar u_1) + (\bar t_2/\bar u_2) + (\bar t_3/\bar u_3) = 0$, so

$$\frac{\bar t_3}{\bar u_3} = -\left[\frac{\bar t_1}{\bar u_1} + \frac{\bar t_2}{\bar u_2}\right] = -\left[\frac{2-ace}{ec^2} + \frac{2+ace}{ec^2}\right] = -\frac{4}{ec^2}.$$

Table 2 summarizes these calculations.

Next recall that $c = 2$, and therefore

$$\bar\lambda = e, \quad \lambda = 1.$$

Further, by replacing $B_0$ and $\bar B_2$ by appropriate new sections of $Y_0 \to R_0$ and $M_2 \to \bar R_2$, we can take $b = 0$, $a = d = 1$. Then $\alpha_{2,1} = \bar\alpha_{2,1}$ and without loss of generality, $B_2 = \bar B_2$. Since $e$ is odd, these values determine $t_i, u_i, \bar t_i$, and $\bar u_i$. See Table 3.

It follows from [23, p. 449], that $\partial H_1 \cap T_{1,i}$ has $|\lambda/u_i| = 1$ components $(i = 1, 2)$ and $\partial H_2 \cap T_{2,i}$ has $|\bar\lambda/\bar u_i| = 1$ components $(i = 1, 2, 3)$. Further, the fact that $\epsilon_1 = -\epsilon_2$ implies that any loop in $H$ consisting of an arc between $T_{1,1}$ and $T_{1,2}$ on $H_1$ and an arc between $T_{2,2}$ and $T_{2,1}$ on $H_2$ is orientation reversing on $H$. Hence $M$ is a surface semibundle. □

Next we give concrete constructions for $H_1$ and $H_2$ and establish some of their specific properties for later use. We begin with $H_2$.

Recall that $Y_2$ is identified with $T \times [-\epsilon, \epsilon]$ for small $\epsilon > 0$, where $T = T \times \{0\}$, $T_{2,1} = T \times \{-\epsilon\}$, and $T_{2,2} = T \times \{\epsilon\}$. Set $\bar\alpha_0 = (\bar B_2 \cap T)$ and recall that it is a geodesic in $T$. We may

TABLE 2. *The slopes of $\partial H_1$ and $\partial H_2$ when $K$ has one component.*

| $\partial H_1$ | $i=1$ | $i=2$ | $\partial H_2$ | $i=1$ | $i=2$ | $i=3$ |
|---|---|---|---|---|---|---|
| $\frac{t_i}{u_i}$ | $\frac{ce-2d}{2c}$ | $\frac{2d-ce}{2c}$ | $\frac{\bar t_i}{\bar u_i}$ | $\frac{2-ace}{ec^2}$ | $\frac{2+ace}{ec^2}$ | $-\frac{4}{ec^2}$ |

TABLE 3. *The coefficients of the slopes of $\partial H_1$ and $\partial H_2$ when $K$ has one component.*

| $\partial H_1$ | $j=1$ | $j=2$ | $\partial H_2$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|---|---|---|
| $t_j$ | $\frac{e-1}{2}$ | $\frac{1-e}{2}$ | $\bar t_j$ | $\frac{1-e}{2}$ | $\frac{1+e}{2}$ | $-1$ |
| $u_j$ | $1$ | $1$ | $\bar u_j$ | $e$ | $e$ | $e$ |

suppose that $\bar{B}_2 = \bar{\alpha}_0 \times [-\epsilon, \epsilon]$. Orient $\bar{\alpha}_0$ to be homologous with $\bar{\alpha}_{2,1}$ in $Y_2$ and fix a new fiber $\bar{\phi}_0$ (with its given orientation) in $T$. Then $T$ is identified with $\bar{\alpha}_0 \times \bar{\phi}_0$ and $Y_2$ with $\bar{\alpha}_0 \times \bar{\phi}_0 \times [-\epsilon, \epsilon]$. The first homology of each $\bar{\phi}$-vertical torus $T_* = \bar{\alpha}_0 \times \bar{\phi}_0 \times \{*\}$ has a canonical basis $\{\bar{\alpha}_*, \bar{\phi}_*\}$ where $\bar{\alpha}_* = \bar{B}_2 \cap T_*$ and $\bar{\phi}_* = (\bar{\phi}_0 \times [-\epsilon, \epsilon]) \cap T_*$ are oriented consistently with $\bar{\alpha}_0$ and $\bar{\phi}_0$. Note that the boundary slope of $H_2$ on $T_{-\epsilon} = T_{2,1}$ is $\frac{1}{2e} - \frac{1}{2}$ with respect to the ordered basis $\{\bar{\alpha}_{-\epsilon}, \bar{\phi}_{-\epsilon}\}$, while that on $T_\epsilon = T_{2,2}$ is $-\frac{1}{2e} - \frac{1}{2}$ with respect to $\{\bar{\alpha}_\epsilon, \bar{\phi}_\epsilon\}$ (cf. the proof of Proposition 6.1 and Table 2).

Decompose $\bar{\alpha}_0$ as the union of two intervals $I, J$ which intersect along their boundaries. Let $0 < \delta < \epsilon$ and think of $M_2$ as

$$M_2 = (\bar{\alpha}_0 \times \bar{\phi}_0 \times [-\epsilon, \epsilon]) \setminus (\mathrm{int}(I) \times \bar{\phi}_0 \times (-\delta, \delta)).$$

Fix a straight connected simple closed curve $\gamma_1$ in $T_{-\epsilon}$ of slope $\frac{1}{2e} - \frac{1}{2}$ with respect to the basis $(\bar{\alpha}_{-\epsilon}, \bar{\phi}_{-\epsilon})$. (Recall that an arc $\sigma \times \{*\} \subset T_*$ is *straight* if $\sigma$ is geodesic in $T$ and note that each new fiber of $Y_2$ is straight.) Set $\Theta_- = \gamma_1 \times [-\epsilon, -\delta]$. Then $\Theta_- \cap (J \times \bar{\phi}_0 \times \{-\delta\})$ consists of $|e|$ straight arcs of slope $\frac{1}{2e} - \frac{1}{2}$ in $T_{-\delta}$ evenly spaced around $\bar{\phi}$. Rotate these arcs about their centers as we pass from $-\delta$ to $\delta$ in $J \times \bar{\phi}_0 \times [-\delta, \delta]$, so that when we reach $J \times \bar{\phi}_0 \times \{\delta\}$, they have slope $-\frac{1}{2e} - \frac{1}{2}$. The trace of this rotation is a surface $\Theta_0$ in $J \times \bar{\phi}_0 \times [-\delta, \delta]$ such that for each $t \in (-\delta, \delta)$, $\Theta_0 \cap T_t$ is a union of $|e|$ evenly spaced straight arcs of slope $-\frac{1}{2} - \frac{t}{2\delta e}$. When $t = \delta$, the arcs are contained in a straight loop of $T_\delta$ of slope $-\frac{1}{2e} - \frac{1}{2}$, which we denote by $\gamma_2$. Set $\Theta_+ = \gamma_2 \times [\delta, \epsilon]$. Finally define

$$H_2 = \Theta_- \cup \Theta_0 \cup \Theta_+.$$

The reader will verify that $H_2 \cap T_{2,3}$ has a single-boundary component representing the slope $e\bar{\alpha}_{2,3} - \bar{\phi}_{2,3}$ and that $H_2$ is everywhere transverse to the new fibers. Thus we can form a surface fibration $\mathcal{F}_2$ in $M_2$ obtained by isotoping $H_2$ around the new fibers. Note also that $H_2$ is transverse to the original fibers except when $t = 0$. In this case, $H_2 \cap (J \times \bar{\phi}_0 \times \{0\})$ consists of $|e|$ subarcs of the original fibers. Figure 6 illustrate one fiber of $\mathcal{F}_2$ in $M_2$ when $e = 5$.

Next we construct $H_1$. Fix a properly embedded arc $\sigma$ in $B_1$ running from $T_{1,1}$ to $T_{1,2}$, and let $\sigma \times [-1, 1]$ be a regular neighborhood of $\sigma$ in $B_1$ such that $(\sigma \cap T_{1,1}) \times [-1, 1]$ is a subarc of $\alpha_{1,1}$ which follows the orientation of $\alpha_{1,1}$ as we pass from $-1$ to $1$ (and therefore $(\sigma \cap T_{1,2}) \times [-1, 1]$ is a subarc of $\alpha_{1,2}$ which follows the opposite orientation of $\alpha_{1,2}$ as we pass from $-1$ to $1$). The surface $H_1$ is obtained by replacing $\sigma \times [-1, 1]$ in $B_1$ by a reimbedding of it which wraps around the $\phi$-direction $((1 - e)/2)$ times as we pass from $-1$ to $1$ in $\sigma \times [-1, 1] \times \phi$. By construction, $H_1$ is horizontal in $Y_1$ and one can easily verify that the boundary slopes of $H_1$ in $T_{1,1}$ and in $T_{1,2}$ are those given in Table 2. Let $\mathcal{F}_1$ be the corresponding surface bundle in $Y_1$.

Since $H_2$ is transverse to the original fibers in $T_{2,1} \cup T_{2,2}$, we may suppose that $\partial H_1 = \partial H_2$ in $T_1 \cup T_2$ and therefore produce a horizontal surface $H$ in the graph manifold $M$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ match up forming the corresponding global semisurface bundle of $M$, which we denote by $\mathcal{F}$.

Recall that the torus $T = \hat{f}^{-1}(L_1^*)$ is vertical in the Seifert structure $\hat{f} : Y \to F$, and that $L_1 \subset T$ is geodesic in $Y$ and transverse to the original fibers. Similarly, each knot $L_i$ ($i = 2, \ldots, p$) is geodesic in $Y$, contained in the $\phi$-vertical torus $U_i = \hat{f}^{-1}(L_i^*)$, and transverse to the original circle fibers.

For each $i$ and $j \neq i$, we have $U_i \cap T = U_i \cap U_j = \phi_1 \cup \ldots \cup \phi_n$ where $\phi_k$ is the original fiber $\hat{f}^{-1}(\hat{c}_k)$. The knot $L_1$ intersects each $\phi_k$ twice. Let $U_i^0$ be the $2n$-punctured torus $M \cap U_i$ ($i = 2, \ldots, p$). It decomposes into $2n$ successive components $U_{i,m}^0$ ($m = 1, \ldots, 2n$), where

$$\begin{cases} U_{i,2k-1}^0 \text{ is contained in } Y_1 \\ U_{i,2k}^0 \text{ is contained in } M_2, \end{cases}$$

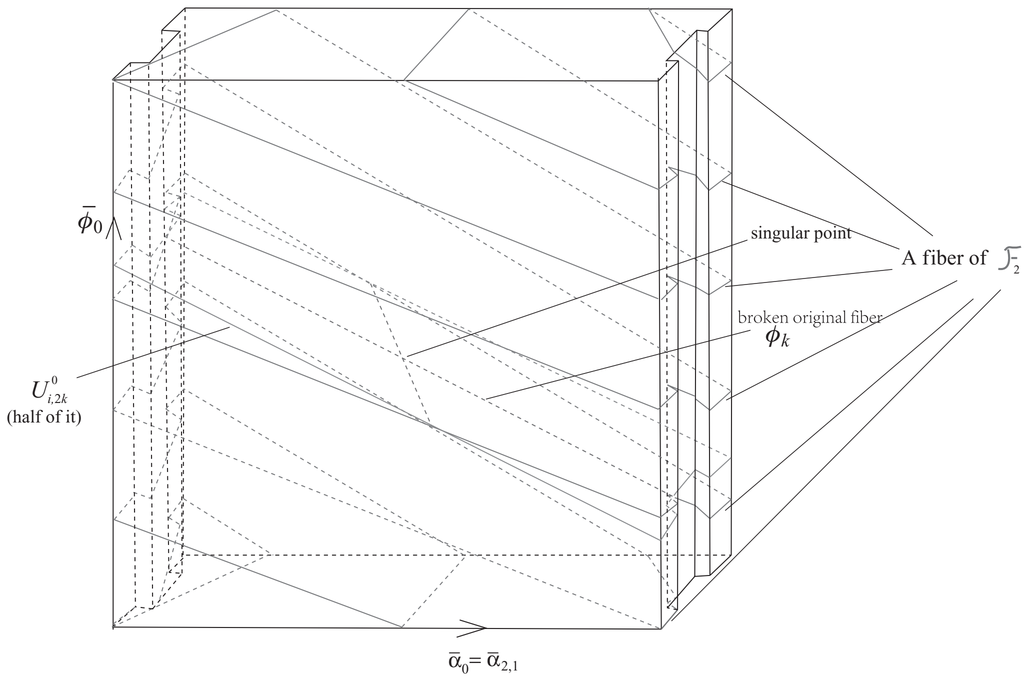



FIGURE 6. *A fiber of* $\mathcal{F}_2$.

$(k = 1, \ldots, n)$ and $U^0_{i,2k} \cap \phi_k \neq \emptyset$ $(2 \leqslant i \leqslant p, 1 \leqslant k \leqslant n)$. Similarly, the knot $L_i$ decomposes into $4n$ successive segments $L_{i,m}$ $(2 \leqslant i \leqslant p, 1 \leqslant m \leqslant 4n)$ such that

$$\begin{cases} L_{i,2k-1} \cup L_{i,2k-1+2n} \text{ is contained in } U^0_{i,2k-1}, \\ L_{i,2k} \cup L_{i,2k+2n} \text{ is contained in } U^0_{i,2k}, \end{cases}$$

for $k = 1, \ldots, n$. Note that

$$L^0_{i,2k} \cap \phi_k \neq \emptyset, \qquad L^0_{i,2k+2n} \cap \phi_k \neq \emptyset$$

$(2 \leqslant i \leqslant p, 1 \leqslant k \leqslant n)$. *Reorient* $L_i$ so that when $L_{i,m}$ is given the induced orientation,

$$\text{the tail of } \begin{cases} L_{i,2k-1} \text{ lies in } T_2, \\ L_{i,2k} \text{ lies in } T_1, \end{cases}$$

for each $i = 2, \ldots, p$ and $k = 1, \ldots, 2n$. Finally, let $L^*_{i,m}$ be the image of $L_{i,m}$ in $F$ $(2 \leqslant i \leqslant p, 1 \leqslant m \leqslant 2n)$.

Figure 6 illustrates how $U^0_{i,2k}$ intersects one leaf of $\mathcal{F}_2$ (where we assumed that the broken original fiber $\phi_k$ is contained in the leaf). From this it is not hard to see that $U^0_{i,2k} \cap \mathcal{F}_2$ is as shown in Figure 7, up to a small isotopy in $U_{i,2k}$. Thus, the induced singular foliation on $U^0_{i,2k}$ by $\mathcal{F}_2$ has only two singular points and they are contained in $\phi_k$. Further, this singular foliation varies continuously under the isotopy of $U^0_{i,2k}$ to $U^0_{j,2k}$ which lies over a rotation of $L^*_{i,2k}$ to $L^*_{j,2k}$ centered at $\hat{c}_k$ which does not pass through $L^*_1$.

The following proposition will be needed later.

PROPOSITION 6.2. *Fix a transverse orientation of* $\mathcal{F}_2$. *There is a smooth* $\phi$*-vertical isotopy of the link* $L_2 \cup \ldots \cup L_p$ *in* $\cup^p_{i=2} U^0_i$, *fixed outside a small neighborhood of* $(\cup^p_{i=2} U^0_i) \cap M_2$, *which repositions* $L_{i,2k}$ *to be transverse to* $\mathcal{F}_2$ *and to pass from the negative to the positive side of the leaves of* $\mathcal{F}_2$ *leaves while traveling from* $T_1$ *to* $T_2$ *in* $M_2$ $(i = 2, \ldots, p, k = 1, \ldots, 2n)$ *(cf. Figure 7 for the position of* $L_{i,2k} \cup L_{i,2k+2n}$ *in* $U^0_{i,2k}$ *after the isotopy).*

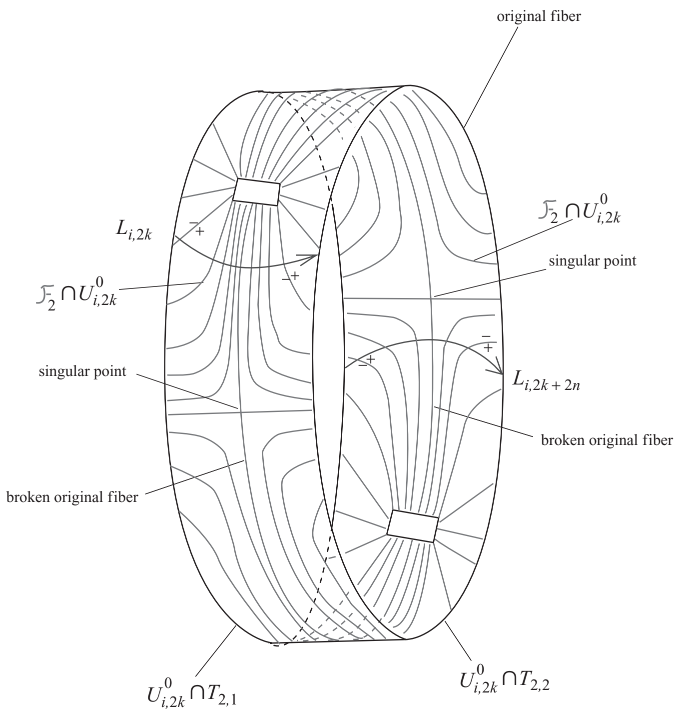



FIGURE 7. $U^0_{i,2k}$.

*Proof.* Fix $i \in \{2,3,\ldots,p\}$ and $k \in \{1,2,\ldots,n\}$. As the geodesic arcs $L_{i,2k}, L_{i,2k+2n} \subset U^0_{i,2k}$ are perpendicular to the original fibers in $U_{i,2k}$, there is a smooth $\phi$-vertical isotopy supported in $U^0_{i,2k}$ which places them in the positions depicted in Figure 7 near $U^0_{i,2k} \cap (T \times [-\delta,\delta])$. A further $\phi$-vertical isotopy, supported in a small neighborhood of $U^0_{i,2k}$ in $U^0_i$, places them in the desired position throughout $U^0_{i,2k}$.

We would like to perform the isotopies of the previous paragraph simultaneously, but this is potentially obstructed, since $U^0_{i,2k} \cap U^0_{j,2l} = \phi_k \cap M_2$ if $k = l$ (and is empty otherwise). In fact, there is no obstruction. To see this, observe that the broken fiber $U_{i,2k} \cap \phi_k$ consists of two arcs, each of which is split into two halves by a singular point of $U^0_{i,2k} \cap \mathcal{F}_2$. These halves inherit a label '+' and '−' from the given orientation of $\phi_k$. The reader will verify that for each $k$, the isotopy of the previous paragraph moves the finite set $(L_2 \cup \ldots \cup L_p) \cap \phi_k$ into either the '+'-sides or the '−'-sides of the arcs. It follows that the isotopies can be performed so as not to interfere with each other. This completes the proof. □

Let $p_2 : \breve{M} \to M$ be the double covering determined by the homomorphism

$$\pi_1(M) \longrightarrow \mathbb{Z}/2, \delta \longmapsto \overline{[\delta]\cdot[T]},$$

where $[\delta] \in H_1(M)$ is the class of $\delta$ and $[T] \in H_2(M)$ corresponds to a fundamental class of $T$. The reader will verify that $\breve{H} = p_2^{-1}(H)$ is a connected, orientable surface, and the semifiber structure $\mathcal{F}$ on $M$ lifts to a fiber structure $\breve{\mathcal{F}}$ on $\breve{M}$ with fiber $\breve{H}$. Note that $p_2^{-1}(Y_1), p_2^{-1}(M_2)$, and $p_2^{-1}(T_i)$ $(i = 1,2)$ have two components each, say

$$p_2^{-1}(Y_1) = \breve{Y}_{1,1} \cup \breve{Y}_{1,2},$$
$$p_2^{-1}(M_2) = \breve{M}_{2,1} \cup \breve{M}_{2,2},$$
$$p_2^{-1}(T_i) = \breve{T}_{i,1} \cup \breve{T}_{i,2}\ (i = 1,2).$$

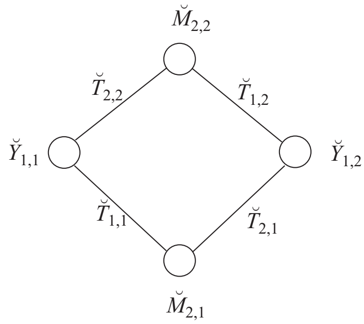



FIGURE 8. *The graph decomposition of* $\breve{M}$.

Figure 8 illustrates the graph of the JSJ-decomposition of $\breve{M}$. The cover $p_2 : \breve{M} \to M$ extends to a (free) double cover $p_2 : \breve{Y} \to Y$ where $\breve{Y}$ inherits the structure of the total space of a locally trivial oriented circle bundle with fiber $\breve{\phi}$. Note as well that $\breve{Y}$ decomposes as

$$\breve{Y} = \breve{Y}_{2,1} \cup \breve{Y}_{1,1} \cup \breve{Y}_{2,2} \cup \breve{Y}_{1,2},$$

where $\breve{Y}_{2,j}$ $(j = 1, 2)$ are the two components of $p_2^{-1}(Y_2)$ with $\breve{M}_{2,j} \subset \breve{Y}_{2,j}$. Since $L_i$ double covers $L_i^*$ $(i = 1, \dots, p)$, $p_2^{-1}(L_i)$ has two components which we denote by $\breve{L}_{i,d}$ $(d = 1, 2)$. Then $\breve{M}$ is the exterior of $\breve{L}_{1,1} \cup \breve{L}_{1,2}$ in $\breve{Y}$. Lemma 2.1 implies that wherever $\breve{L}_{i,d}$ meets a circle fiber of $\breve{Y}$, it meets it transversely.

Recall the surface bundles $\mathcal{F}_1, \mathcal{F}_2$ on $Y_1, M_2$, respectively, which match up to form the semibundle $\mathcal{F}$ in $M$. We denote by $\breve{\mathcal{F}}_{1,j}$ and $\breve{\mathcal{F}}_{2,j}$ the corresponding surface bundles in $\breve{Y}_{1,j}$ and $\breve{M}_{2,j}$ $(j = 1, 2)$, respectively. They match up in $\breve{M}$ to form the surface bundle $\breve{\mathcal{F}}$ in $\breve{M}$ described above. Fix a transverse orientation for $\breve{\mathcal{F}}$ and let $\breve{\mathcal{F}}_{i,j}$ $(i, j = 1, 2)$ have the induced transverse orientation. Thus, each leaf of these foliations has a '$-$'-side and a '$+$'-side in such a way that the transverse orientation points from the former to the latter.

Set $\breve{U}_i = p_2^{-1}(U_i)$ (a $\breve{\phi}$-vertical torus) and $\breve{U}_i^0 = \breve{U}_i \cap \breve{M}$. Define $\breve{U}_{i,2k,j}^0 \subset \breve{M}_{2,j}$ to be the lift of $U_{i,2k} \subset M_2$, and $\breve{U}_{i,2k-1,j}^0 \subset \tilde{Y}_{1,j}$ that of $U_{i,2k-1}^0 \subset Y_1$. The intersection $\breve{U}_{i,2k,j} \cap \breve{\mathcal{F}}_{2,j}$ is analogous to what is shown in Figure 7. Similarly, define $\breve{L}_{i,2k,j} \cup \breve{L}_{i,2k+2n,j} \subset \breve{U}_{i,2k,j}^0$ to be the lift of $L_{i,2k} \cup L_{i,2k+2n} \subset U_{i,2k}^0$, and $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j} \subset \breve{U}_{i,2k-1,j}^0$ to be the lift of $L_{i,2k-1} \cup L_{i,2k+2n-1} \subset U_{i,2k-1}^0$. Endow $\breve{L}_{i,m,j}$ with the induced orientation.

By the nature of our constructions, the obvious analog of Proposition 6.2 holds in each $\breve{M}_{2,j}$. In particular, there is a smooth $\breve{\phi}$-vertical isotopy of the link $\breve{L}_{2,1} \cup \breve{L}_{2,2} \cup \ldots \cup \breve{L}_{p,1} \cup \breve{L}_{p,2}$ in $\cup_{i=2}^p \breve{U}_i^0$, fixed outside a small neighborhood of $(\cup_{i=2}^p \breve{U}_i^0) \cap (\breve{M}_{2,1} \cup \breve{M}_{2,2})$, which repositions $\breve{L}_{i,2k,j}$ to be transverse to $\breve{\mathcal{F}}_{2,j}$ and to pass from the negative to the positive side of the leaves of $\breve{\mathcal{F}}_{2,j}$ while traveling from $\breve{T}_{1,j}$ to $\breve{T}_{2,j}$ in $\breve{M}_{2,j}$ $(i = 2, \dots, p; k = 1, \dots, 2n; j = 1, 2)$.

For $j = 1, 2$, the surface bundle $\breve{\mathcal{F}}_{1,j}$ is transverse to the $\breve{\phi}$ Seifert structure on $\breve{Y}_{1,j}$. Hence it induces a fibering of $\breve{U}_{i,2k-1,j}^0$ by intervals transverse to the original circle foliation of $\breve{U}_{i,2k-1,j}^0$ $(i = 2, \dots, p; k = 1, \dots, n)$. By construction, we may assume that near the boundary of $\breve{U}_{i,2k-1,j}^0$, the oriented arcs $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j} \subset \breve{U}_{i,2k-1,j}^0$ are transverse to the interval foliations and point from its negative to positive side (cf. Figure 9(a)). A typical interval fiber of $\breve{\mathcal{F}}_{1,j} \cap \breve{U}_{i,2k-1,j}^0$ is depicted in Figure 9(a).

Note that both $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ are properly embedded arcs in $\breve{Y}_{1,j}$ joining the two boundary components of $\breve{Y}_{1,j}$. Hence if we take a $\breve{\phi}$-vertical torus $\breve{V}_j$ in $\mathrm{int}(\breve{Y}_{1,j})$ which is parallel to one of the boundary components of $\breve{Y}_{1,j}$, then we may assume that $\breve{V}_j$ intersects each of $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ transversely in exactly one point. Our next adjustment is to perform a Dehn twist operation on $\breve{\mathcal{F}}_{1,j}$ in a neighborhood of $\breve{V}_j$ a sufficiently large number

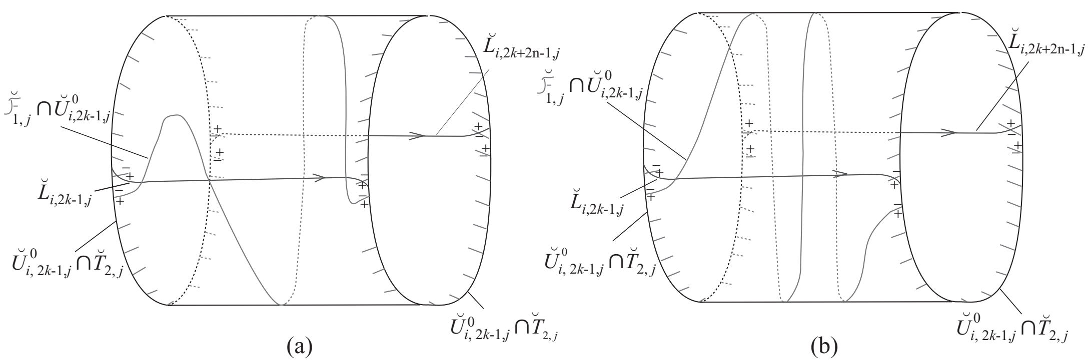



Figure 9. $\breve{U}^0_{i,2k-1,j}$.

of times so that after a further isotopy, the resulting surface bundle in $\breve{Y}_{1,j}$ is transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$ everywhere $(i = 2, \ldots, p;\ k = 1, \ldots, n)$. The details of this operation are as follows.

Let $N(\breve{V}_j) = \breve{V}_j \times I$ be a small neighborhood of $\breve{V}_j$ in $\operatorname{int}(\breve{Y}_{1,j})$ consisting of $\breve{\phi}$-circle fibers. The boundary of $N(\breve{V}_j)$ consists of two $\breve{\phi}$-vertical tori, which we denote by $\breve{\partial}_1$ and $\breve{\partial}_2$. Note that $N(\breve{V}_j)$ intersects each of $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ in a single arc. We may assume that the tail of the arc (with the induced orientation) is contained in $\breve{\partial}_1$ and its head in $\breve{\partial}_2$ $(i = 2, \ldots, p; k = 1, \ldots, n)$. Since $\breve{\mathcal{F}}_{1,j}$ is transverse to the old fibers, $\breve{\mathcal{F}}_{1,j} \cap N(\breve{V}_j)$ is a foliation of $N(\breve{V}_j)$ by annuli. We now perform a Dehn twist operation which wraps these annuli many times around the $\breve{\phi}$-fibers in the direction opposite to the transverse orientation of $\mathcal{F}_{1,j}$ as we pass from $\breve{\partial}_1$ to $\breve{\partial}_2$. This is illustrated in Figure 10. Let $\breve{\mathcal{F}}'_{1,j}$ denote the resulting surface bundle in $\breve{Y}_{1,j}$.

Note that the foliation by intervals of $\breve{U}^0_{i,2k-1,j}$ determined by $\breve{U}^0_{i,2k-1,j} \cap \breve{\mathcal{F}}'_{1,j}$ is obtained from the foliation $\breve{U}^0_{i,2k-1,j} \cap \breve{\mathcal{F}}_{1,j}$ by the corresponding Dehn twists in a neighborhood of some $\breve{\phi}$-circle fiber in $\breve{U}^0_{i,2k-1,j}$ in the direction opposite to the transverse orientation of $\breve{\mathcal{F}}_{1,j}$ as we pass from $\breve{\partial}_1 \cap \breve{U}^0_{i,2k-1,j}$ to $\breve{\partial}_2 \cap \breve{U}^0_{i,2k-1,j}$. Finally, we adjust $\breve{\mathcal{F}}'_{1,j}$ by an isotopy which is the identity in a small regular neighborhood of $\partial \breve{Y}_{1,j}$ and outside a small regular neighborhood of $\cup_{i \geqslant 2,k,j} \breve{U}^0_{i,2k-1,j}$, so that the interval foliation in each $\breve{U}^0_{i,2k-1,j}$ becomes transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$. After the isotopy, a typical interval fiber in $\breve{U}^0_{i,2k-1,j}$ is depicted in Figure 9(b). The resulting surface bundle, denoted $\breve{\mathcal{F}}''_{1,j}$, is everywhere transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$ $(i = 2, \ldots, p; k = 1, \ldots, n)$. As this final operation is the identity in a neighborhood of a small regular neighborhood of $\partial \breve{Y}_{1,j}$, the resulting surface bundle $\breve{\mathcal{F}}''$ on $\breve{M}$ is transverse to the link $\cup_{i=2}^{p} \cup_{d=1}^{2} \breve{L}_{i,d}$ everywhere. The proof of Theorem 1.9 when $K$ is a classic Montesinos knot is now complete.

## 6.2. *K is a link of two components*

The proof in this case is the same in most essentials and we continue to use the notation and conventions developed above except for the following changes.

In our present case, $\tilde{K} \to K^*$ is a trivial 2-fold cover and $\Psi^{-1}(\tilde{K})$ is a geodesic link $L$ of $2p$ components $L_{1,1} \cup L_{1,2} \cup \ldots \cup L_{p,1} \cup L_{p,2}$, where $L_{i,1} \cup L_{i,2} = \hat{f}^{-1}(L_i^*)$. We take $M$ to be the exterior of $L_{1,1}$ in $Y$. As before, $M$ admits a graph manifold decomposition $M = Y_1 \cup M_2$. Note that $L_{1,2}$ is a fiber of the Seifert structure on $M_2$.

Since $K$ has two components and $p$ is odd, $q_1 + \ldots + q_n$ is even (see § 2). Note, moreover, that $0 \neq e(W_K) = -(q_1 + \ldots + q_n)/p$ so that $e = pe(W_K) = -(q_1 + \ldots + q_n)$ is even and nonzero.

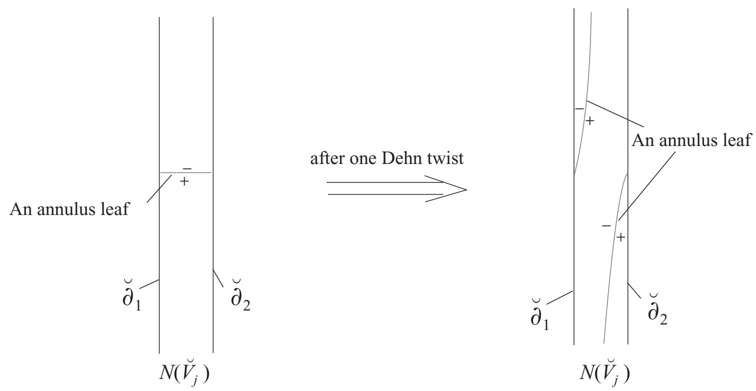



FIGURE 10. *Dehn twist in $N(\breve{V}_j)$ in the direction opposite to the transverse orientation of $\breve{\mathcal{F}}_{1,j}$.*

The numbers $a, b, c, d$ from the proof of Proposition 6.1 can be taken to be $a = c = d = 1, b = 0$. In other words, we have

$$\bar{\alpha} = \alpha,$$
$$\bar{\phi} = \alpha + \phi.$$

Our previous analysis can be easily modified to see that there are orientable horizontal surfaces $H_1, H_2$ in $Y_1, M_2$ which piece together to form the semifiber $H$ where the projection of $H_1$ to $R_1$ has degree $|\lambda| = 1$ and that of $H_2$ to the base of $M_2$ has degree $|\bar{\lambda}| = \frac{e}{2}$. Further, if the slope of $H_1$ on $T_j$ is given by $u_j\alpha_{1,j} + t_j\phi_{1,j}$ $(j = 1, 2)$ and that of $H_2$ on $T_j$ is $\bar{u}_j\bar{\alpha}_{2,j} + \bar{t}_j\bar{\phi}_{2,j}$ $(j = 1, 2, 3)$, then reference to Table 2 shows that these slopes are given by Table 4 below.

Since $e$ is even, the values of the coefficients $u_j, t_j, \bar{u}_j, \bar{t}_j$ are given in Table 5.

As before, we can think of $H_2$ as a surface which interpolates between the slope $-1 + \frac{2}{e}$ on $T_1$ and $-1 - \frac{2}{e}$ on $T_2$. The associated fibering $\mathcal{F}_2$ is transverse to all the new fibers, and in particular to $L_{1,2}$. The intersection of $U^0_{i,2k}$ with $\mathcal{F}_2$ is depicted in Figure 11. Our

TABLE 4. *The slopes of $\partial H_1$ and $\partial H_2$ when $K$ has two components.*

| $\partial H_1$ | $i = 1$ | $i = 2$ | $\partial H_2$ | $i = 1$ | $i = 2$ | $i = 3$ |
|---|---|---|---|---|---|---|
| $\frac{t_i}{u_i}$ | $\frac{e-2}{2}$ | $\frac{2-e}{2}$ | $\frac{\bar{t}_i}{\bar{u}_i}$ | $\frac{2-e}{e}$ | $\frac{2+e}{e}$ | $-\frac{4}{e}$ |

TABLE 5. *The coefficients of the slopes of $\partial H_1$ and $\partial H_2$ when $K$ has two components.*

| $\partial H_1$ | $j = 1$ | $j = 2$ | $\partial H_2$ | $j = 1$ | $j = 2$ | $j = 3$, $e = 4k + 2$ | $j = 3$, $e = 4k$ |
|---|---|---|---|---|---|---|---|
| $t_j$ | $\frac{e}{2} - 1$ | $1 - \frac{e}{2}$ | $\bar{t}_j$ | $1 - \frac{e}{2}$ | $1 + \frac{e}{2}$ | $-2$ | $-1$ |
| $u_j$ | $1$ | $1$ | $\bar{u}_j$ | $\frac{e}{2}$ | $\frac{e}{2}$ | $\frac{e}{2}$ | $\frac{e}{4}$ |

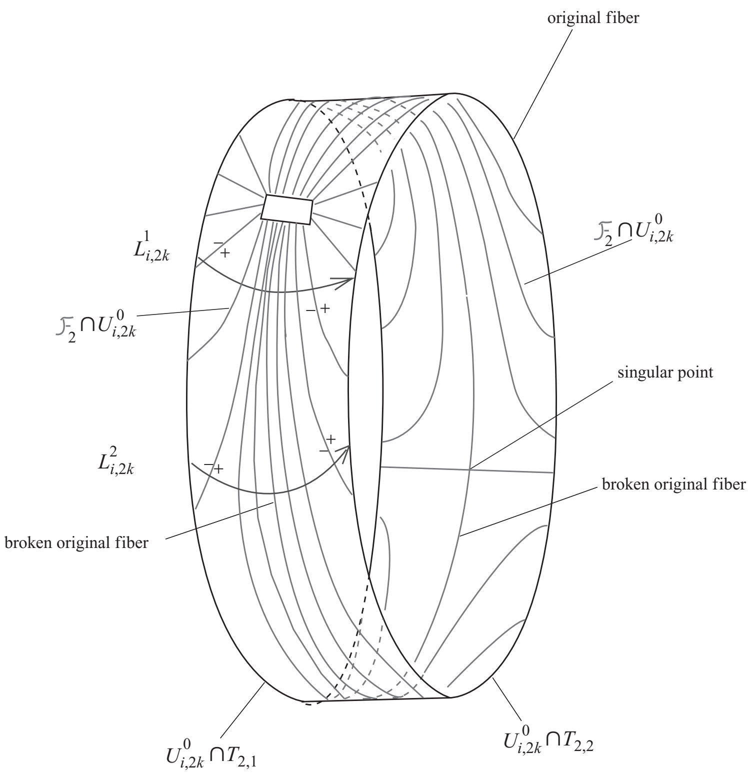



FIGURE 11. *Dehn twist in $N(\breve{V}_j)$ in the direction opposite to the transverse orientation of $\breve{\mathcal{F}}_{1,j}$.*

previous argument now proceeds without significant change to produce the desired surface bundle structure on $\breve{M}_2$ which is transverse to $L_{1,2} \cup \ldots \cup L_{p,1} \cup L_{p,2}$. One point to note is that the $\phi$-vertical isotopies of the geodesic arcs in $\breve{M}_{2,j}$ can be made while isotoping $L_{1,2}$ through new fibers in $T$ to a new position.

## 7. *Virtually biorderable fundamental groups*

In this section we prove Theorem 1.10. The argument is based on the following result of Dale Rolfsen and Bernard Perron [**17**].

THEOREM 7.1 (Perron–Rolfsen). *If the monodromy of a surface bundle $M^3 \to S^1$ has positive eigenvalues on the level of first homology, then $\pi_1(M)$ is biorderable.*

It suffices then to show that for each of the fiber bundle structures associated to the generalized Montesinos links considered in this paper, some power of the monodromy has positive eigenvalues. We begin with two elementary lemmata.

LEMMA 7.2. *Let $h_0$ be a homeomorphism of a compact, connected, orientable surface $F_0$, and $h$ the extension of $h_0$ to the surface $F$ obtained by attaching 2-disks to $\partial F_0$. Then the set of eigenvalues of $(h_0)_* : H_1(F_0) \to H_1(F_0)$ consists of that of $h_* : H_1(F) \to H_1(F)$, together with some roots of unity.*

*Proof.* Since $h_0$ permutes the boundary components of $F_0$, the lemma's conclusion is a simple consequence of the exactness of the following commutative diagram's rows.

$$\begin{array}{ccccc} H_1(\partial F_0) & \xrightarrow{i_*} & H_1(F_0) & \xrightarrow{j_*} & H_1(F) \\ \downarrow{\scriptstyle (h_0|\partial F_0)_*} & & \downarrow{\scriptstyle (h_0)_*} & & \downarrow{\scriptstyle h_*} \\ H_1(\partial F_0) & \xrightarrow{i_*} & H_1(F_0) & \xrightarrow{j_*} & H_1(F) \end{array}$$

□

LEMMA 7.3. *If $F$ is a compact, connected, orientable surface of genus 0 or 1, and $h : F \to F$ is an orientation-preserving homeomorphism, then some power of $h_* : H_1(F) \to H_1(F)$ has positive eigenvalues. Hence any locally trivial $F$-bundle has a virtually biorderable fundamental group.*

*Proof.* The previous lemma yields the genus zero 0 immediately, while it reduces the genus 1 case to the situation where $F$ is a torus and $h_* \in SL_2(\mathbb{Z})$. It is elementary to see that if the latter holds, the eigenvalues of $h_*$ are real if $|\text{trace}(h_*)| \geqslant 2$ and are roots of unity otherwise. Thus the lemma holds. □

We observed in the introductory section that generalized Montesinos links of spherical type virtually fiber with fiber a surface of genus 0, while the argument of § 4.1 shows that the same is true for a generalized Montesinos link of type $S^2 \times \mathbb{R}$ unless it is a trivial link of two components. In the latter case, the link group is free on two generators, which is well known to be biorderable [**15**]. Thus Theorem 1.10 holds in these cases. We noted in Remark 4.2 that generalized Montesinos links of type $\mathbb{E}^3$ or Nil virtually fiber with fiber a surface of genus 1, so the theorem follows in a similar fashion in these cases as well.

Suppose next that $K$ has type $\mathbb{H}^2 \times \mathbb{R}$. Recall from § 4.2 that there is a closed, connected, orientable, hyperbolic surface $F$ and cover $\Psi : (Y, L) = (F \times S^1, \Psi^{-1}(\tilde{K})) \to (W_K, \tilde{K})$, where $L_v$ is a union of circle fibers $\{x\} \times S^1$ and $L_h$ is contained in a finite number of sections $F \times \{u\}$. Lemma 3.2 shows that there is a finite-degree cover $\psi : \tilde{F} \to F$ such that each component of $\tilde{L}_h = (\psi \times 1_{S^1})^{-1}(L_h) \subset \tilde{Y} = \tilde{F} \times S^1$ is homologically nontrivial in $\tilde{F}$. Take $\tilde{F}_0$ to be the exterior of $X = \tilde{L}_v \cap (\tilde{F} \times \{1\})$ in $\tilde{F} \times \{1\}$. The exterior of $\tilde{L}_v$ in $\tilde{Y}$ is the product $\tilde{Y}_0 = \tilde{F}_0 \times S^1$, and Theorem 1.7 implies that there is a surface bundle $p : \tilde{Y}_0 \to S^1$ whose fibers are everywhere transverse to $\tilde{L}_h$. Let $S_0$ denote one such fiber and let $h : S_0 \to S_0$ be the monodromy of $p$. It is not hard to see that $\chi(S_0) < 0$ and so as $S_0$ is essential in $\tilde{Y}_0 = \tilde{F}_0 \times S^1$, it can be isotoped to be horizontal with respect to the product Seifert structure. Hence $h$ has finite order up to isotopy, so the eigenvalues of $h_* : H_1(S_0) \to H_1(S_0)$ are roots of unity. The exterior of $E(\tilde{L}_h)$ inherits a surface bundle structure whose monodromy $g$ is determined as follows.

Identify $\tilde{F}_0 \times S^1$ cut open along $\tilde{F}_0 \times \{1\}$ with $\tilde{F}_0 \times I$ and think of $\tilde{L}_h$ as an isotopy of the subset $\tilde{L}_h \cap (\tilde{F}_0 \times \{0\})$ of $\tilde{F}_0$. This isotopy extends to an ambient isotopy $f_t : \tilde{F}_0 \to \tilde{F}_0$. Then $f_1(\tilde{F}_0) = \tilde{F}_0$ and $g = (h \circ f_1)|\tilde{F}_0$. Since $(f_1)_* = 1_{H_1(\tilde{F}_0)}$, Lemma 7.2 implies that the eigenvalues of $g$ are roots of unity. Hence the Perron–Rolfsen theorem yields the desired conclusion.

Inspection of the construction in § 4.4 shows that the argument of the previous paragraph can be applied to see that the fundamental groups of the exterior of a nonclassic generalized Montesinos link of type $\widetilde{SL_2}$ are virtually biordered.

Finally, consider the case where $K$ is one of the links $K(-g; e_0; (q_1/p), \dots, (q_n/p))$ considered in Theorem 1.9. From above we can suppose that $K$ is a classic Montesinos link of type $\widetilde{SL_2}$.

We will describe the proof in the case where $K$ is a knot. The case when it is a link is dealt with similarly.

First we suppose that $n = 0$. Then $e_0 = -e(W_K) \neq 0$, and so the exterior of $K$ is Seifert fibered. Hence its fundamental group is virtually biordered by [**3**]. Suppose then that $n > 0$. We constructed an $\breve{H}$-bundle structure on the exterior $\breve{M}$ of $\breve{L}$ in $\breve{Y}$, where $\breve{H} = \breve{H}_{1,1} \cup \breve{H}_{2,1} \cup \breve{H}_{1,2} \cup \breve{H}_{2,2}$ where $H_{i,1} \cap H_{i,2} = \emptyset$ for $i = 1, 2$ and $H_{1,1} \cap H_{2,1} \cong H_{2,1} \cap H_{1,2} \cong H_{1,2} \cap H_{2,2} \cong H_{2,2} \cap H_{1,1} \cong S^1$. It follows that

$$H_1(\breve{H}) \cong \mathbb{Z} \oplus H_1(\breve{H}_{1,1}) \oplus H_1(\breve{H}_{2,1}) \oplus H_1(\breve{H}_{1,2}) \oplus H_1(\breve{H}_{2,2}).$$

By construction, the restriction of the monodromy $h$ of the $\breve{H}$-bundle exterior of $\breve{L}$ to each $H_{i,j}$ is periodic. Thus there is some $m > 0$ such that $h_*^m | H_1(\breve{H}_{i,j})$ is the identity and therefore has eigenvalues 1. Since $\det(h_*) = 1$, it follows that all the eigenvalues of $h_*^m$ are 1. Since the virtual fibering of the exterior of $K$ is obtained by removing a link (which is transverse to the fibers of the $\breve{H}$-bundle structure) from the interior of $\breve{M}$, Lemma 7.2 implies that the eigenvalues of the action of the monodromy on the first homology of the resulting surface fiber are roots of unity. Thus some power of it has eigenvalues 1. This observation is sufficient to complete the proof when $K$ is a knot.

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