IS $\square^\omega (\omega + 1)$ PARACOMPACT?

by

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If $\{X_n : n \in \omega\}$ is a family of spaces, $\prod_{n \in \omega} X_n$, called the box product of those spaces, denotes the Cartesian product of the sets with the topology generated by all sets of the form

$$\Pi G_n,$$

where $G_n$ need only be open in each factor space $X_n$. If $X_n = X \forall n \in \omega$, we denote $\prod_{n \in \omega} X_n$ by $\boxtimes X$.

Box products have generated considerable interest during the past ten years, as first as "counter-example producing machines," later, as mathematical objects in their own right. Yet, except for a few surprising counter-examples there have been no non-trivial absolute results. As corollaries to more general results, M. E. Rudin and K. Kunen have proved that if the Continuum Hypothesis (CH) is assumed, then $\omega(\omega + 1)$ is paracompact; however, in [6,8] they question what occurs when CH is false. Kunen [6] has proved that if Martin's Axiom (MA) is assumed, then $\prod_{n \in \omega} X_n$ is paracompact whenever each $X_n$ is compact first countable; however, as stated in [2], the really interesting case occurs when $\omega(\omega + 1)$ when both CH and MA + $\neg$CH fail, as they do in the "random real" models of Solovay [10]. We prove:

**Theorem 1:** If $\omega(\omega + 1)$ is paracompact $\forall \alpha < \omega_1$, then $\omega(\omega + 1)$ is paracompact.

**Theorem 2:** If there exists a $\lambda$-scale in $\omega$, then $\omega(\omega + 1)$ is paracompact.

Suppose that for each $n \in \omega X_n$ is a set, then for each

1"Box Products" is the title of Chapter X of [9] where all the results attributed by this author to others may be found, if not referenced here.
x \in \prod_{n \in \omega} X_n,
\overline{x} = \{ y \in \prod_{n \in \omega} X_n : \exists m \in \omega \exists n > m \Rightarrow y(n) = x(n) \}

defines an equivalence relation on \( X \) and the ensuing quotient set is denoted by \( \nabla X \) and called the reduced Frechet product \( \nabla X \)
[6]. If \( \nabla X \) is given the quotient topology from \( \prod_{n \in \omega} X_n \), then \( \nabla X \)
\( \mathcal{G}_\delta \)-sets are open; therefore, \( \nabla X \) is paracompact if, and only
if, every open cover has a pairwise-disjoint open refinement.
Kunen first observed [6] that when each \( X_n \) is compact, \( \prod_{n \in \omega} X_n \)
is paracompact if, and only if, \( \nabla X \) is paracompact.

**Proof of Theorem 1:**

We suppose \( \mathcal{F} \) is a basic open covering of \( \nu^\omega(\omega_1+1) \). For
each \( \alpha < \omega_1 \) and \( A \subseteq \omega \) define
\[
A(\alpha)(n) = \begin{cases} 
[a+1, \omega_1] & \text{if } n \in A \\
[0, a] & \text{if } n \notin A,
\end{cases}
\]

\( A(\alpha) = \prod_{n \in \omega} A(\alpha)(n) \), and \( \overline{A(\alpha)} = \{ \overline{x} : x \in A(\alpha) \} \). The sets \( \overline{A(\alpha)} \)
are clopen and form a partition of \( \nu^\omega(\omega_1+1) \) since \( \overline{A(\alpha)} \neq \overline{B(\alpha)} \) iff
\( (A-B) \cup (B-A) \) is infinite.

We construct for each \( \alpha < \omega_1 \) a collection \( \mathcal{F}(\alpha) \) satisfying
(1) \( \forall G \in \mathcal{F}(\alpha) \Rightarrow G \) is clopen and contained in a member of \( \mathcal{F} \),
(2) \( \exists \mathcal{F}(\alpha) \) is clopen and \( \mathcal{F}(\alpha) \) is a pairwise disjoint co-

lection,
(3) \( \beta < \alpha < \omega_1 \Rightarrow \mathcal{F}(\beta) \subseteq \mathcal{F}(\alpha) \),
(4) \( \exists \{ \mathcal{F}(\alpha) : \alpha < \omega_1 \} \) is a cover of \( \nu^\omega(\omega_1+1) \).

There is a first \( \lambda \in \omega_1 \) such that \( \overline{\omega(\lambda)} \) is contained in an ele-
ment of \( \mathcal{F} \), let \( \mathcal{F}(0) = \{ \overline{\omega(\lambda)} \} \) and suppose that for \( \alpha < \omega_1 \) we
have constructed \( \mathcal{F}(\beta) \forall \beta < \alpha \) to satisfy (1), (2), and (3).
If \( \alpha \) is a limit ordinal, then let
\[
\mathcal{F}(\alpha) = \bigcup \{ \mathcal{F}(\beta) : \beta < \alpha \}.
\]
If \( \alpha \) is a non-limit ordinal, suppose \( A \subseteq \omega \) and let
\[
\mathcal{T}(A) = \{ \overline{y} \in \overline{A(\alpha)} : y^{-1}(\omega_1) = A \}.
\]
Since $T(A)$ is homeomorphic to $\nu^\omega_{\omega+1}$, we may find a pairwise disjoint basic open covering $S(A)$ of $T(A)$ to satisfy

(i) $\bar{W} \in S(A)$, $n, m \in A \Rightarrow \inf W(n) = \inf W(m)$ is a successor ordinal $> \alpha + 1$.

(ii) $\bar{W} \in S(A) \Rightarrow \exists G \in \mathcal{F} \ni \bar{W} \subseteq G$.

By choosing only one representative $A$ for each equivalence class $A(a)$, we let

$$\mathcal{F}(a) = \mathcal{F}(a-1) \cup \{\bar{W} - \cup \mathcal{F}(a-1) : \bar{W} \in S(A), A \subseteq \omega\}.$$ 

In order to show $\mathcal{F}(a)$ satisfies (1), (2), and (3) we need only show $\cup S(A)$ is closed for each $A \subseteq \omega$. So we suppose

$$\bar{x} \in \bar{A}(\omega) - \cup S(A)$$

and $\bar{y} \in T(A)$ such that

$$y(n) = \begin{cases} x(n) & \text{if } n \not\in A \\ \omega_1 & \text{if } n \in A. \end{cases}$$

Now choose $\bar{W} \in S(A)$ such that $y \in W$ and define

$$V_x(n) = \begin{cases} W(n) & \text{if } x(n) \in W(n) \\ [\alpha+1, \inf W(n)) & \text{if } x(n) \notin W(n). \end{cases}$$

From (i) $\bar{x} \in \bar{V}_x \subseteq \bar{A}(\omega)$; moreover, if $\bar{U} \in S(A)$ and $\bar{U} \neq \bar{W}$, then we may assume

$$\left( \forall \bar{n} \in \mathcal{A}(\omega) \right) \cap \left( \forall \bar{n} \in W(n) \right) = \emptyset.$$ 

Thus, $\bar{U} \cap \bar{V}_x = \emptyset$. Clearly, $\bar{A}(\omega) - \cup S(A)$ is open and our induction is completed.

To see (4) we observe that $\bar{x} \in \nu^\omega(\omega+1) \Rightarrow$ either $\bar{x} = \omega_1$ or $\exists a$ a first $a \not\in A$

$$a > \sup \{x(n) : x(n) \neq \omega_1\}.$$ 

In the first case $\bar{x} \in \cup \mathcal{F}(0)$, and in the second case $\bar{x} \in \cup \mathcal{F}(a)$. Therefore, our proof is complete.

If $\lambda$ is an ordinal, a $\lambda$-scale in $\omega_\omega$ is an order-preserving injection $\Psi : \lambda \rightarrow \omega_\omega$; given any $x \in \omega_\omega \exists \alpha < \lambda$ with $x(n) < \Psi(\alpha)(n)$ for all but finitely many $n \in \omega$. It should be clear that there

$^2T(A)$ may actually be a singleton; however, this causes no disturbance.
can be no $\omega$-scales in $\omega_\omega$; however, it is a fact, probably due to Hausdorff, that

$$CH \rightarrow \exists \text{ an } \omega_1 \text{-scale in } \omega_\omega.$$  

However, in the random real models for $\neg CH$, with the ground model "satisfying" $CH$, there is an $\omega_1$-scale in $\omega_\omega$ [4]. Booth's theorem [9, pg. 40] says

$$MA \Rightarrow \exists a 2^\omega \text{-scale in } \omega_\omega.$$  

In Cohen's original model for $\neg CH$ there is no $\lambda$-scale in $\omega_\omega$. In [4] S. Hechler has shown that given cardinals $\lambda$ and $\kappa$ and a model $M$ of $\text{ZFC}$ in which

$$\omega < \text{cf}(\lambda) \leq \lambda \leq \min(2^\omega, \text{cf}(\kappa))$$  
then one can "extend" $M$ to a model $N$ in which $N = 2^\omega$ and $\omega_\omega$ has a $\lambda$-scale.

van Douwen [1] and Hechler [3] have examined a number of topological cardinal functions which are implied by or are equivalent to the existence of a $\lambda$-scale. Kunen [5] proved

(a) $\exists \lambda$-scale in $\omega_\omega \Rightarrow \lambda \times \square^\omega(\omega+1)$ is not normal,

(b) $\exists 2^\omega$-scale in $\omega_\omega \Rightarrow \lambda \times \square^\omega(\omega+1)$ is normal for any ordinal $\lambda$ such that $\text{cf}(\lambda) \geq \omega$.

Recall [7] that a space $Y$ is $\lambda$-metrizable for an ordinal $\lambda$, $\text{cf}(\lambda) > \omega$, whenever each $y \in Y$ has a local base $\{ B(y, \alpha) : \alpha < \lambda \}$ satisfying

(i) $\beta < \alpha \Rightarrow B(y, \alpha) \subseteq B(y, \beta)$

(ii) $y \in B(z, \alpha) \Rightarrow z \in B(y, \alpha)$

(iii) $y \in B(z, \alpha) \Rightarrow B(y, \alpha) \subseteq B(z, \alpha)$.

It is well known that $\lambda$-metrizable spaces are paracompact.

Our original proof of Theorem 2, presented during this conference, was similar to the proof of Theorem 1 and made use of:

If there is a $\lambda$-scale in $\omega_\omega$, then the intersection of less than $\text{cf}(\lambda)$ open sets of $\forall^\omega(\omega+1)$ is open.
We give thanks to Brian Scott who has provided us with the "if" part of the Lemma from which our theorem 2 is immediate.

**Proof of Theorem 2:**

**Lemma:** Let $\lambda$ be a regular cardinal. Then $\mathcal{V}^{\omega}(\omega+1)$ is $\lambda$-metrizable if, and only if, there is a $\lambda$-scale in $\omega$.

**Proof:** Suppose $\{B_\alpha : \alpha < \lambda\}$ is a well-ordered decreasing local base at $\omega$. It is easy to find $\{G_\alpha : \alpha < \lambda\} \subseteq \{B_\alpha : \alpha < \lambda\}$ and $\{x_\alpha : \alpha < \lambda\} \subseteq \omega$ such that whenever $\alpha < \beta < \lambda$,

$$G_\beta \subseteq \prod_{n \in \omega} [x_\beta(n), \overline{\omega}] \subseteq G_\alpha,$$

and $\{G_\alpha : \alpha < \lambda\}$ is a local base at $\omega$.

If $\forall(\alpha) = x_\alpha$, then $\forall : \lambda \to \omega$ is a $\lambda$-scale in $\omega$.

Conversely, suppose $\forall : \lambda \to \omega$ is a $\lambda$-scale in $\omega$. For each $x \in \mathcal{V}^{\omega}(\omega+1)$, let $d(x, x) = \lambda$, and if $y \neq x$, let

$$d(x, y) = \inf\{\alpha < \lambda : |\{ n \in \omega : \inf(x(n), y(n)) \leq \forall(\alpha)(n) \} | = \omega\}$$

We see that $d : \mathcal{V}^{\omega}(\omega+1) \times \mathcal{V}^{\omega}(\omega+1) \to \lambda + 1$ satisfies the criterion of [7, Theorem 4.8(B)], and hence $\mathcal{V}^{\omega}(\omega+1)$ is $\lambda$-metrizable.

The previous lemma establishes that the $\lambda$-metrizability of $\mathcal{V}^{\omega}(\omega+1)$ is independent of the axioms of ZFC whenever $\text{cf}(\lambda) > \omega$.

In answer to one of the questions we presented at this conference, Eric van Douwen has recently shown that $\mathcal{V}^{\omega}(\omega+1)$ in the previous lemma may be replaced by $\mathcal{V} X_{n}$ whenever each $X_{n}$ is a compact metrizable space. In answer to another of our questions, Judith Roitman has proved:

In a model of set theory which is an iterated CCC extension of length $\lambda$, $\text{cf}(\lambda) > \omega \Rightarrow \mathcal{V} X_{n}$ is paracompact if each $X_{n}$ is regular and separable. Furthermore, if $\lambda$ is regular and $\lambda > 2^{\omega}$ in the ground model, then $\mathcal{V} X_{n}$ is paracompact whenever each $X_{n}$

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is compact first countable.

The following questions are outstanding:

1. Is $\square^\omega(\omega+1)$ always paracompact or normal?
2. Is $\square^\omega(\omega+1)$ normal in any model of ZFC?
3. Can there be a normal non-paracompact box product of compact spaces?
4. Is the box product of countably many compact linearly ordered topological spaces paracompact?

References

1. E. K. van Douwen, *Functions from $\omega$ to $\omega$*, this conference.

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