MY FAVORITE FUNCTIONS
OR
Continuous from *what* to WHAT?!?
Section 1. Accordions

\[ y = \sin x \]
$y = \sin x$

$y = \sin \frac{1}{x}$

not continuous at 0
\[ f(x) = x \sin \frac{1}{x} \]

\[ y = \sin \frac{1}{x} \]

\[ f \text{ is continuous at } 0 \text{ even though there are nearly vertical slopes as you approach } 0. \]
\[ g(x) = x^2 \sin \frac{1}{x} \]

Here \( \lim_{h \to 0} \frac{g(h)}{h} = 0 \)
\[ g(x) = x^2 \sin \frac{1}{x} \]

So \( g \) has a derivative at 0,

\[ g'(0) = \lim_{h \to 0} \frac{g(h)}{h} = 0 \]
\[ g(x) = x^2 \sin \frac{1}{x} \]

\[ g'(0) = 0. \]

Still we have nearly vertical tangents.
\[ g(x) = x^2 \sin \frac{1}{x} \]

\[ g'(0) = 0. \]

We have nearly vertical tangents.

Further, there are sequences \(<a_n>\) and \(<b_n>\) such that

\[ \lim_{n \to \infty} b_n - a_n = 0, \text{ but } \lim_{n \to \infty} \frac{g(b_n) - g(a_n)}{b_n - a_n} = \infty. \]
Section 2. **Le Blancmange function**

Fix a non-negative integer \( n \). Given a real number \( x \), let \( k \) be the greatest non-negative integer such that

\[
a_{(x,n)} = 2^{-n}k \leq x \text{ and let } b_{(x,n)} = 2^{-n}(k+1).
\]

So \( x < b_{(x,n)} \).

Define \( f_n : \mathbb{R} \to [0,1] \) by \( f_n(x) = \min\{x-a_{(x,n)}, b_{(x,n)}-x\} \).
Fix a non-negative integer $n$. Given a real number $x$, let $k$ be the greatest non-negative integer such that

$$a_{(x,n)} = 2^{-n}k \leq x$$

and let $b_{(x,n)} = 2^{-n}(k+1)$. So $x < b_{(x,n)}$.

Define $f_n : \mathbb{R} \rightarrow [0,1]$ by

$$f_n(x) = \min\{x - a_{(x,n)}, b_{(x,n)} - x\}.$$
\[ f_0 + f_1 \]

\[ f_0 + f_1 + f_2 \]

\[ f_0 + f_1 + f_2 + f_3 \]

\[ f_0 + f_1 + f_2 + f_3 + f_4 \]
THEOREM 2. There is a function continuous at each real $x$ but differentiable at no real $x$.

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Example:

$$f_0\left(\frac{7}{16}\right) = \frac{7}{16}; \quad f_1\left(\frac{7}{16}\right) = \frac{7}{16}; \quad f_2\left(\frac{7}{16}\right) = \frac{1}{16}; \quad f\left(\frac{7}{16}\right) = \frac{1}{16}.$$
\[ f(x) = \sum_{n=1}^{\infty} f_n(x) \leq \sum_{n=1}^{\infty} 2^{-n} \]
Lemma 2: Suppose a function $h : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $x$. If $a_n$ and $b_n$ are such that $\forall n, \ a_n \leq x \leq b_n$, then

$$h'(x) = \lim_{n \to \infty} \frac{h(b_n) - h(a_n)}{b_n - a_n}$$
Section 3. Stretching zero to one.

Cantor's Middle Third Set $C$ is a subset of $[0,1]$ formed inductively by deleting middle third open intervals. Say $(1/3, 2/3)$ in step one.
In step two, remove the middle-thirds of the remaining two intervals of step one, they are \( (\frac{1}{9}, \frac{2}{9}) \) and \( (\frac{7}{9}, \frac{8}{9}) \).

In step three, remove the middle thirds of the remaining four intervals.

and so on for infinitely many steps.
What we get is **C, Cantor’s Middle Thirds Set.**

**C** is very “thin” and a “spread out” set whose measure is 0 (since the sum of the lengths of intervals removed from [0,1] is 1.

![Cantor's Middle Thirds set](image-url)
As $[0,1]$ is thick and as $C$ is a thin subset of $[0,1]$, the following is surprising:

**THEOREM 3.**

There is a continuous function from $C$ onto $[0,1]$. 

Cantor's Middle Thirds set
THEOREM 3. There is a continuous function from $C$ onto $[0,1]$.

The points of $C$ are the points equal to the sums of infinite series of form

$$\sum_{n=1}^{\infty} 2s(n)3^{-n}$$

where $s(n) \in \{0,1\}$.
\[
F(\sum_{n=1}^{\infty} 2s(n)3^{-n}) = \sum_{n=1}^{\infty} s(n)2^{-n}
\]
defines a continuous surjective function whose domain is \(\mathbb{C}\) and whose range is \([0,1]\).

Example. The two geometric series show \(F(1/3) = F(2/3) = 1/2\).
Picturing the proof.

Stretch the two halves of step 1 until they join at 1/2.
Now stretch the two halves of each pair of step 2
Until they join at 1/4 and 3/4…
Each point is moved to the sum of an infinite series.
Section 4. Advancing Dimension

\[ \mathbb{N} \leftrightarrow \mathbb{N}^2 \]

\[ 2^{n-1}(2m-1) \leftrightarrow \langle m,n \rangle \]

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Theorem 4. There is a continuous function from \([0,1]\) onto the square.

We’ll cheat and do it with the triangle.
An java animated version of a different Space Filling Curve can be found at http://www.geom.uiuc.edu/~dpvc/CVM/1998/01/vsfcf/article/sect2/brief_history.html
Section 5. An addition for the irrationals

By an addition for those objects $X \in [0, \infty)$ we mean a continuous function $s : X \times X \rightarrow X$ (write $x+y$ instead of $s(<x,y>)$) such that for $x+y$ the following three rules hold:

1. $x+y = y+x$ (the commutative law) and
2. $(x+y)+z = x+(y+z)$ (the associative law).

With sets like $\mathbb{Q}$, the set of positive rationals, the addition inherited from the reals $\mathbb{R}$ works, but with the set $\mathbb{P}$ of positive irrationals it does not work: $(3+\sqrt{2})+(3-\sqrt{2}) = 6$. 
THEOREM 5.

The set $P$ of positive irrationals has an addition.

Our aim is to consider another object which has an addition And also "looks like" $P$.

Continued fraction
Given an irrational \( x \), the sequence \( \langle a_n \rangle \) is computed as follows:

Let \( G(x) \) denote the greatest integer \( \leq x \). Let \( a_0 = G(x) \).

If \( a_0, \ldots, a_n \) have been found as below, let \( a_{n+1} = G(1/r) \).

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + r}}}}
\]
Continuing in this fashion we get a sequence which converges to \(x\). Often the result is denoted by

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.
\]

However, here we denote it by \(\text{CF}(x) = <a_0, a_1, a_2, a_3, \ldots>\).
We let \( \langle 2 \rangle \) denote the constant \( \langle 2, 2, 2, 2, \ldots \rangle \). Note \( \langle 2 \rangle = \text{CF}(1+\sqrt{2}) \) since

\[
2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}} = 1 + \sqrt{2}
\]

Hint: A quick way to prove the above is to solve for \( x \) in \( x = 2 + \frac{1}{x} \) or \( x^2 - 2x - 1 = 0 \).
Prove $<1> = \text{CF}(\frac{1+\sqrt{5}}{2})$ and $<1,2> = \text{CF}(\frac{2+\sqrt{3}}{2})$

We add two continued fractions “pointwise,” so $<1> + <1,2> = <2,3>$ or $<2,3,2,3,2,3,2,3,\ldots>$.  

Here are the first few terms for $\pi$, $<3,7,15,1,\ldots>$. No wonder your grade school teacher told you $\pi = 3 + \frac{1}{7}$. The first four terms of $\text{CF}(\pi)$, $<3,7,15,1>$ approximate $\pi$ to 5 decimals.

Here are Euler’s first few terms for $e$, $<2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,\ldots>$.  


Lemma. Two irrationals \( x \) and \( y \) are "close" as real numbers iff the "first few" partial continued fractions of \( \text{CF}(x) \) and \( \text{CF}(y) \) are identical.

For example \(<2,2,2,2,2,1,1,1,...>\) and \(<2>\) are close, but \(<2,2,2,2,2,2,2,2,2,2,91,5,5,...>\) and \(<2>\) are closer.

Here is the “addition:” We define \( x \oplus y = z \) if \( \text{CF}(z) = \text{CF}(x) + \text{CF}(y) \).
Then the lemma shows \( \oplus \) is continuous.
However, strange things happen:

\[
\frac{1 + \sqrt{5}}{2} \oplus \frac{1 + \sqrt{5}}{2} = 1 + \sqrt{2}
\]
Problems

1. How many derivatives has

\[ g(x) = x^2 \sin \frac{1}{x} \]

2. Prove that each number in [0,2] is the sum of two members of the Cantor set.
Problems

3. Prove there is no distance non-increasing function whose domain is a closed interval in \( \mathbb{N} \) and whose range is the unit square \([0,1] \times [0,1]\) .

4. Determine \(\sqrt{2} + \frac{\sqrt{2} + \sqrt{3}}{2}\) .
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