The Conley index for fast-slow systems I: One-dimensional slow variable

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March 23, 1998

1 Introduction

Fast-slow systems of the form

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= \varepsilon g(x,y)\end{aligned} \tag{1.1}$$

arise frequently in applications and intuitively one expects that for small $\varepsilon > 0$ the dynamics of (1.1) can be described in terms of the fast dynamics $\dot{x} = f(x,y)$ and the slow dynamics $\dot{y} = g(x,y)$ restricted to $f(x,y) = 0$. In fact, a rather powerful technique, often referred to as geometric singular perturbation theory, has been developed by N. Fenichel, C. Jones, N. Kopell and others (see [1] for a survey and further references). These techniques are based on and provide extensions to the classical concepts of normal hyperbolicity and transversality and when applicable provide sharp results concerning the dynamics for the full system.

However, there are problems for which it is not always possible to satisfy the hyperbolicity assumptions or verify the transversality conditions. It was, at least in part, with this in mind that C. Conley promoted the use of isolating neighborhoods and what is now called the Conley index theory. These ideas have proven useful in the study of differential equations, and hence, it is natural to ask whether they can be applied in this context. The first difficulty is that typically for $\varepsilon = 0$ one loses isolation. Conley addressed this issue in [4] and introduced the notion of a singular isolating neighborhood. These ideas seem to have been ignored, in part one presumes, because of the second issue; the real interest in dynamical systems is not in the existence of isolating neighborhoods, but rather in the structure of the associated isolated invariant set. In typical applications, given an isolating neighborhood the Conley index theory is used to obtain information about the dynamics of the invariant set. In [11] K. Mischaikow, M. Mrozek, and J. Reineck showed that it was possible to compute the index for small $\varepsilon > 0$ from information in the $\varepsilon = 0$ system. The problem with the approach presented there is that it is extremely geometrical in nature, and thus, difficult to apply in high dimensional settings. The purpose of this paper is to show that if information concerning the fast dynamics has been obtained using the Conley index theory, then this information can be used to compute the index for $\varepsilon > 0$.

Consider the family of differential equations on $\mathbb{R}^n \times \mathbb{R}$ given by

$$\begin{aligned}\dot{x} &= f(x,\lambda)\\ \dot{\lambda} &= \varepsilon g(x,\lambda)\end{aligned} \tag{1.2}$$

where $f(x,\lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g(x,\lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are C^1 functions and $\varepsilon \geq 0$. This is clearly a special case of (1.1) since λ is taken to be a real number rather than a vector. The solutions to this equation generate a flow

$$\varphi^\varepsilon : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}.$$

In the special case $\varepsilon = 0$, (1.2) has a simpler form since λ is a constant. We can view λ as a parameter for the flows on $\mathbb{R}^n$, and for each λ we define a flow ψ_λ on $\mathbb{R}^n$ by

$$(\psi_\lambda(t,x),\lambda) = \varphi^0(t,x,\lambda). \tag{1.3}$$

If we fix a range of values of λ i.e., $\lambda \in \Lambda = [\lambda_0, \lambda_1]$, one can define a *parameterized flow*

$$\psi^\Lambda : \mathbb{R} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \times \Lambda$$

by $\psi^\Lambda(t, x, \lambda) := (\psi_\lambda(t, x), \lambda)$.

Because the final result is fairly abstract in nature and involves the introduction of a considerable amount of technical language, we shall in this introduction use a classical example, the Nagumo equation (see [14] for a derivation of the equations),

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \theta v - f(u) + \lambda \\ \dot{\lambda} &= \frac{\varepsilon}{\theta} u, \end{aligned} \tag{1.4}$$

to provide a framework for the development of these concepts. Recall that the nonlinearity f is a cubic-like function as indicated in Figure 1. When discussing these equations φ^ε is the flow on $\mathbb{R}^3$ generated by (1.4) for a fixed ε.

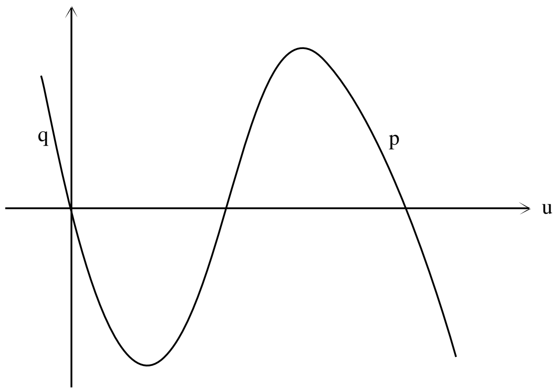

Figure 1: The nonlinearity of the Nagumo equation.

For $\varepsilon = 0$ we view the Nagumo equation as a parameterized family of differential equations in the plane with parameter λ. In particular, for each fixed λ we have a flow

$$\psi_\lambda : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

which as an ensemble captures the dynamics of the fast system.

With regard to the Nagumo equation, for an appropriately chosen value of θ there is an interval $[\lambda_0, \lambda_1]$ over which the interesting fast dynamics occurs. Over this interval the slow motion manifold given by $\{(u, v) \mid \theta v - f(u) + \lambda = 0,\ v = 0\}$ consists of three branches. For our purposes only the left and right branches, labeled $q(\lambda)$ and $p(\lambda)$ respectively, are of interest. The slow dynamics is particularly simple on these branches; along q, $\dot{\lambda} < 0$, while on p, $\dot{\lambda} > 0$. Observe that for each value of λ, $q(\lambda)$ and $p(\lambda)$ are fixed points for ψ_λ. It is, also, assumed that θ is chosen such that at the parameter values λ_* and λ^*, where $\lambda_0 < \lambda_* < \lambda^* < \lambda_1$, there are heteroclinic connections from $q(\lambda_*)$ to $p(\lambda_*)$ and $p(\lambda^*)$ to

$q(\lambda^*)$, respectively. Combining this information from the fast and slow dynamics leads to the schematic picture of Figure 2. As will be clear by the end of this introduction, it follows immediately from the results of this paper that for $\varepsilon > 0$, there is a periodic solution to the Nagumo equation which is close to the closed curve made up of the heteroclinic orbits and branches of the slow manifold. However, before the abstract results can be stated some notions from the Conley index theory must be recalled. General references are [1, 3, 13].

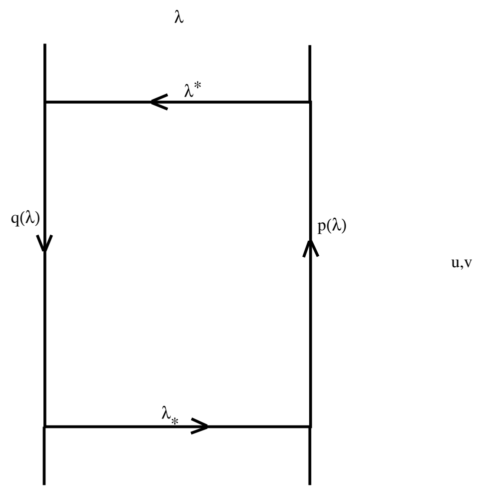

Figure 2: A schematic picture of the periodic orbit for the Nagumo equation.

Consider for the moment an arbitrary flow γ defined on a locally compact metric space X, a compact set $N \subset X$ is an *isolating neighborhood* if

$$\operatorname{Inv}(N, \gamma) := \{x \in X \mid \gamma(\mathbb{R}, x) \subset N\} \subset \operatorname{int} N.$$

If $S = \operatorname{Inv}(N, \gamma)$ for some isolating neighborhood N, then S is referred to as an *isolated invariant set.* The Conley index is an index of isolating neighborhoods with the property that if $\operatorname{Inv}(N, \gamma) = \operatorname{Inv}(N', \gamma)$ then the Conley index of N equals the Conley index of N'. In this way one may, also, view the Conley index as an index of isolated invariant sets. We shall make use of the cohomological Conley index which is denoted by $CH^*(S)$ and is an Alexander-Spanier cohomology group.

As was mentioned earlier, given an isolating neighborhood its Conley index can be used to describe the dynamics of the associated isolated invariant set. In our case we will present theorems which can be used to prove the existence of periodic and heteroclinic orbits.

The first step is to find the appropriate isolating neighborhoods. This is done by choosing compact neighborhoods of the connecting orbits and segments of the branches of equilibria. Observe, however, that this cannot produce an isolating neighborhood under the singular flow φ^0. For example, if one returns to the Nagumo equation, then one sees that for $\varepsilon = 0$ the branches of fixed points extend to infinity. Thus, no compact set which contains the equilibria can be isolating. On the other hand, our interest is in the dynamics for $\varepsilon > 0$. Therefore, it is only important that the constructed neighborhood isolate under φ^ε when $\varepsilon > 0$. That this is the case will be shown in Section 3; for now we concentrate on the construction.

The second step is to compute the Conley index of the isolating neighborhood for $\varepsilon > 0$. Again, the details of this will occupy Sections 3 through 5. However, as will be made clear in this introduction, the goal is to perform this computation in terms of index information concerning the connecting orbits and segments of the branches of equilibria. Therefore, along with the construction of the isolating neighborhood we will need to make assumptions concerning the index.

The segments around the branches of equilibria are the simplest to define. Let ψ_λ be as in (1.3).

Definition 1.1 $\mathcal{T} \subset \mathbb{R}^n \times \mathbb{R}$ is a *tube* if:

(1) There exists an interval $[a, b]$ such that $\mathcal{T} \subset \mathbb{R}^n \times [a, b]$ and $\mathcal{T}$ is an isolating neighborhood for

$$\begin{array}{rcl} \psi^{\mathcal{T}} : \mathbb{R} \times \mathbb{R}^n \times [a,b] & \to & \mathbb{R}^n \times [a,b], \\ (t, x, \lambda) & \mapsto & (\psi_\lambda(t,x), \lambda). \end{array}$$

(2) There exists $\delta(\mathcal{T}) \in \{\pm 1\}$ such that for all $(x, \lambda) \in \mathcal{T}$ we have

$$\delta(\mathcal{T}) g(x, \lambda) > 0.$$

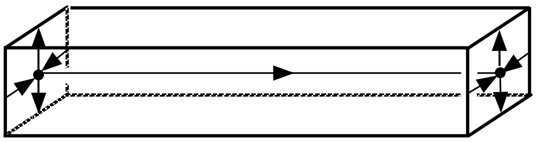

Figure 3: A tube.

See Figure 3. In the setting of the Nagumo equation we can choose $\mathcal{T}(i)$, $i = 1, 2$ to be tubular neighborhoods of $p(\lambda)$ and $q(\lambda)$ over the interval $[\lambda_*, \lambda^*]$, respectively. Since, the set of equilibria over the interval $[\lambda_0, \lambda_1]$ are normally hyperbolic, a tubular neighborhood is an isolating neighborhood. Furthermore, $\delta(\mathcal{T}(1)) > 0$ and $\delta(\mathcal{T}(2)) < 0$.

We now turn to the neighborhoods of the connecting orbits and the non-trivial problem of how to relate the index information between the various tubes. The Conley index theory provides a variety of techniques for proving the existence of heteroclinic connections. We shall use the following. Recall that a *Morse decomposition*

$$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$$

of an isolated invariant set S is a finite collection of disjoint compact invariant subsets $M(p)$, called *Morse sets*, indexed by a partially ordered set $(\mathcal{P}, >)$, with the property that; if $x \in S \setminus \bigcup_{p \in P} M(p)$, then there exist $q > p$ such that the alpha limit set of x is contained in $M(q)$ and the omega limit set of x is contained in $M(p)$.

In the context of a parametrized flow $\psi^\Lambda : \mathbb{R} \times X \times \Lambda \to X \times \Lambda$, a Morse decomposition is said to *continue over* Λ if there is an isolated invariant set $S = \mathrm{Inv}(N, \psi^\Lambda)$ with a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$. Observe that if one defines

$$S_\lambda := S \cap (\mathbb{R}^n \times \{\lambda\}),$$

then S_λ is an isolated invariant set for ψ_λ. Similarly, $\{M_\lambda(p) \mid p \in (\mathcal{P}, >)\}$ is a Morse decomposition for S_λ. Since Morse sets are isolated invariant sets, $CH^*(M_\lambda(p))$ is defined. Furthermore, the index of each Morse set remains constant over Λ. Let $\lambda_0, \lambda_1 \in \Lambda$ and assume that

$$S_{\lambda_i} = \bigcup_{p \in \mathcal{P}} M_{\lambda_i}(p) \quad i = 0, 1.$$

Then, there exists a lower triangular (with respect to the order $>$) degree 0 isomorphism

$$T^{\lambda_1,\lambda_0} : \bigoplus_{p \in \mathcal{P}} CH^*(M_{\lambda_1}(p)) \to \bigoplus_{p \in \mathcal{P}} CH^*(M_{\lambda_0}(p))$$

called a *topological transition matrix* (see [8, 9]). Roughly, if the p, q off diagonal entry of T^{λ_1,λ_0} is non-zero, then for some parameter value $\lambda \in (\lambda_0, \lambda_1)$ there exists a connecting orbit between $M_\lambda(p)$ and $M_\lambda(q)$. As will become clear later, these off diagonal entries play a crucial role in the desired computation of the Conley index.

In order to insure the existence of topological transition matrices in the abstract setting of the fast-slow systems we introduce the following neighborhoods of the connecting orbits.

Definition 1.2 A set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}$ is a *box* if:

(1) There exists an interval $[c, d]$ such that $\mathcal{B} \subset \mathbb{R}^n \times [c, d]$ and $\mathcal{B}$ is an isolating neighborhood for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{aligned} \psi^{\mathcal{B}} : \mathbb{R} \times \mathbb{R}^n \times [c, d] &\to \mathbb{R}^n \times [c, d], \\ (t, x, \lambda) &\mapsto (\psi_\lambda(t, x), \lambda). \end{aligned}$$

(2) Let $S(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}, \psi^{\mathcal{B}})$. There exists a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) := \{M(p, \mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as the admissible ordering. Let $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathbb{R} \times \{\lambda\})$, $S_\lambda(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}_\lambda, \psi_\lambda)$ and let $\{M_\lambda(p, \mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\}$ be the corresponding Morse decomposition of $S_\lambda(\mathcal{B})$. Then

$$S_c(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_c(p, \mathcal{B}) \quad \text{and} \quad S_d(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_d(p, \mathcal{B}).$$

(3) There are isolating neighborhoods $V(p, \mathcal{B})$ for $M(p, \mathcal{B})$ such that

$$V(p, \mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad V(p, \mathcal{B}) \cap V(q, \mathcal{B}) = \emptyset$$

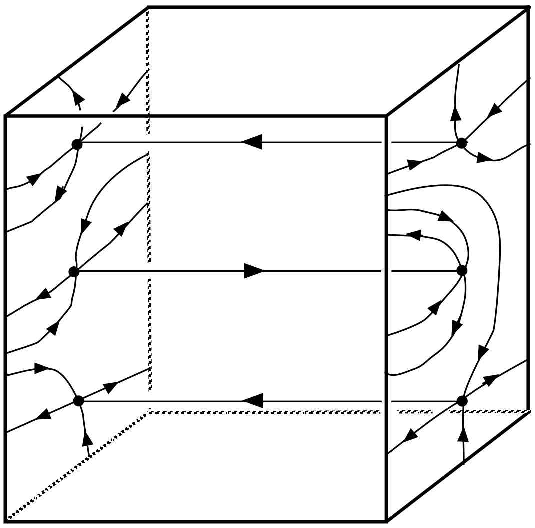

Figure 4: A box with three Morse components.

for $p \neq q$ and for every $\lambda \in [c, d]$

$$V_\lambda(p, \mathcal{B}) \subset \operatorname{int}(\mathcal{B}_\lambda)$$

Furthermore, there are $\delta(p, \mathcal{B}) \in \{\pm 1\}$, $p = 1, \ldots, P_{\mathcal{B}}$, such that

$$\delta(p, \mathcal{B}) g(x, \lambda) > 0 \qquad \text{for all } (x, \lambda) \in V(p, \mathcal{B})$$

See Figure 4. Notice that Definition 1.2(2) implies that there are no connecting orbits between the Morse sets at the parameter values c and d, and by the construction, the sets $S_c(\mathcal{B})$ and $S_d(\mathcal{B})$ are related by continuation.

From the last property, one can define $M_{in}(p, \mathcal{B})$ and $M_{out}(p, \mathcal{B})$ as follows:

$$M_{in}(p, \mathcal{B}) = \begin{cases} M_c(p, \mathcal{B}) & \text{if } \delta(p, \mathcal{B}) > 0, \\ M_d(p, \mathcal{B}) & \text{if } \delta(p, \mathcal{B}) < 0; \end{cases} \tag{1.5}$$

similarly,

$$M_{out}(p, \mathcal{B}) = \begin{cases} M_d(p, \mathcal{B}) & \text{if } \delta(p, \mathcal{B}) > 0, \\ M_c(p, \mathcal{B}) & \text{if } \delta(p, \mathcal{B}) < 0. \end{cases} \tag{1.6}$$

It follows from this that the *directional transition matrix*

$$D^{\mathcal{B}} : \bigoplus_{p=1}^{\mathcal{P}_{\mathcal{B}}} CH^*(M_{out}(p, \mathcal{B})) \to \bigoplus_{p=1}^{\mathcal{P}_{\mathcal{B}}} CH^*(M_{in}(p, \mathcal{B}))$$

can be defined by using the topological transition matrix. See the next section and [7] for more details. Let

$$D^{\mathcal{B}}(P, 1) : CH^*(M_{out}(1, \mathcal{B})) \to CH^*(M_{in}(P, \mathcal{B}))$$

denote the $(P,1)$-entry of the matrix $D^{\mathcal{B}}$.

Let us review this definition in the setting of the Nagumo equations. There are two boxes $\mathcal{B}(i)$, $i=1,2$, which can be obtained by taking the isolating neighborhoods of Figure 5 over the intervals $[\lambda_* - \mu, \lambda_* + \mu]$ and $[\lambda^* - \mu, \lambda^* + \mu]$, respectively. The associated Morse decompositions are

$$\begin{aligned}\mathcal{M}(S(\mathcal{B}(1))) &:= \{M_\lambda(1,\mathcal{B}(1)) = p(\lambda), M_\lambda(2,\mathcal{B}(1)) = q(\lambda)\},\\ \mathcal{M}(S(\mathcal{B}(2))) &:= \{M_\lambda(1,\mathcal{B}(2)) = q(\lambda), M_\lambda(2,\mathcal{B}(2)) = p(\lambda)\}.\end{aligned}$$

Finally, $\delta(2,\mathcal{B}(1)) < 0$, $\delta(1,\mathcal{B}(1)) > 0$, $\delta(2,\mathcal{B}(2)) > 0$, and $\delta(1,\mathcal{B}(2)) < 0$.

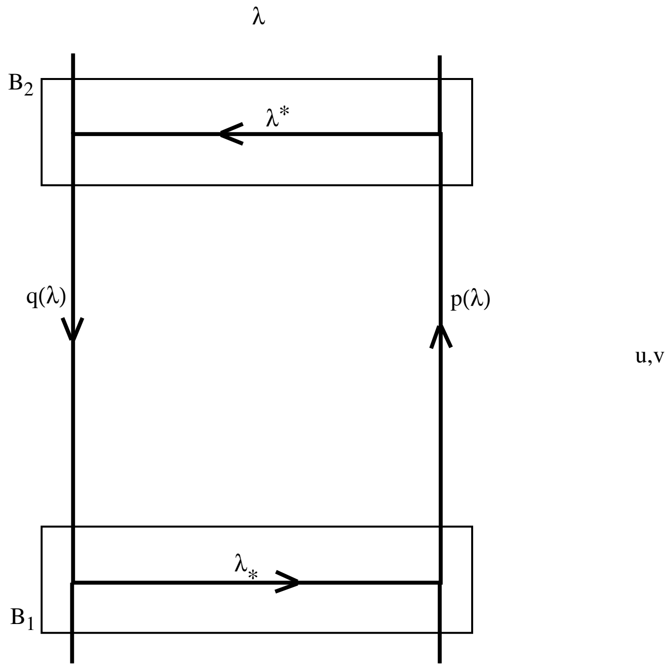

Figure 5: The boxes for the Nagumo equation.

If one is attempting to prove the existence of heteroclinic orbits, an additional type of neighborhood which surrounds the critical points for the perturbed system is necessary.

Definition 1.3 A set $\mathcal{C}(R)$ $(\mathcal{C}(A))$ is a *repelling (attracting) cap* if:

(1) There exists an interval $[e,f]$ such that $\mathcal{C} \subset \mathbb{R}^n \times [e,f]$ and $\mathcal{C}$ is an isolating neighborhood for

$$\begin{aligned}\psi^{\mathcal{C}} : \mathbb{R} \times \mathbb{R}^n \times [e,f] &\to \mathbb{R}^n \times [e,f]\\ (t,x,\lambda) &\mapsto (\psi_\lambda(t,x),\lambda)\end{aligned}$$

(2)

$$x \in \mathcal{C}_e(R) \quad \Rightarrow \quad g(x,e) < 0$$

$$
\begin{array}{rcl}
x \in \mathcal{C}_f(R) & \Rightarrow & g(x,f) > 0 \\
x \in \mathcal{C}_e(A) & \Rightarrow & g(x,e) > 0 \\
x \in \mathcal{C}_f(A) & \Rightarrow & g(x,f) < 0,
\end{array}
$$

where $\mathcal{C}_\lambda(R) := \mathcal{C}(R) \cap \{\lambda\}$ and $\mathcal{C}_\lambda(A) := \mathcal{C}(A) \cap \{\lambda\}$.

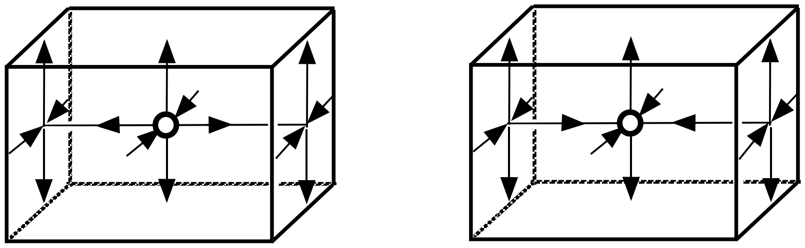

Figure 6: Attracting and repelling caps.

See Figure 6. Finally, in order to construct a global isolating neighborhood, these boxes, tubes, and possibly caps must to be related in a consistent manner. The primary requirement is that the tubes and boxes overlap at the appropriate Morse sets. To simplify the notation we let $P_i = P_{\mathcal{B}_i}$ and $M(p,i) := M(p, \mathcal{B}(i))$.

Definition 1.4 A set of tubes $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$ and boxes $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ forms a *tubes and boxes collection* (TB collection) if the following compatibility conditions are satisfied:

(1) for $i = 1, \ldots, I$

  (a) $\mathcal{T}(i) \cap (\mathbb{R} \times [c_i, d_i]) \subset V(1, \mathcal{B}(i))$ and $\mathcal{T}(i) \cap \mathcal{B}(i)$ isolates $M(1,i)$.

  (b) $\mathcal{T}(i+1) \cap (\mathbb{R} \times [c_i, d_i]) \subset V(P_i, \mathcal{B}(i))$ and $\mathcal{T}(i+1) \cap \mathcal{B}(i)$ isolates $M(P_i, i)$.

(2) for $i = 1, \ldots, I$ either

$$\delta(\mathcal{T}(i+1)) > 0 \text{ and } \delta(P_i, \mathcal{B}(i)) > 0 \text{ in which case } b_{i+1} = d_i$$

or

$$\delta(\mathcal{T}(i+1)) < 0 \text{ and } \delta(P_i, \mathcal{B}(i)) < 0 \text{ in which case } a_{i+1} = c_i$$

where a, b, c, and d are as in Definitions 1.1 and 1.2.

(3) for $i = 1, \ldots, I$ either

$$\delta(\mathcal{T}(i)) > 0 \text{ and } \delta(1, \mathcal{B}(i)) > 0 \text{ in which case } a_i = c_i$$

or

$$\delta(\mathcal{T}(i)) < 0 \text{ and } \delta(1, \mathcal{B}(i)) < 0 \text{ in which case } b_i = d_i$$

where a, b, c, and d are as in Definitions 1.1 and 1.2.

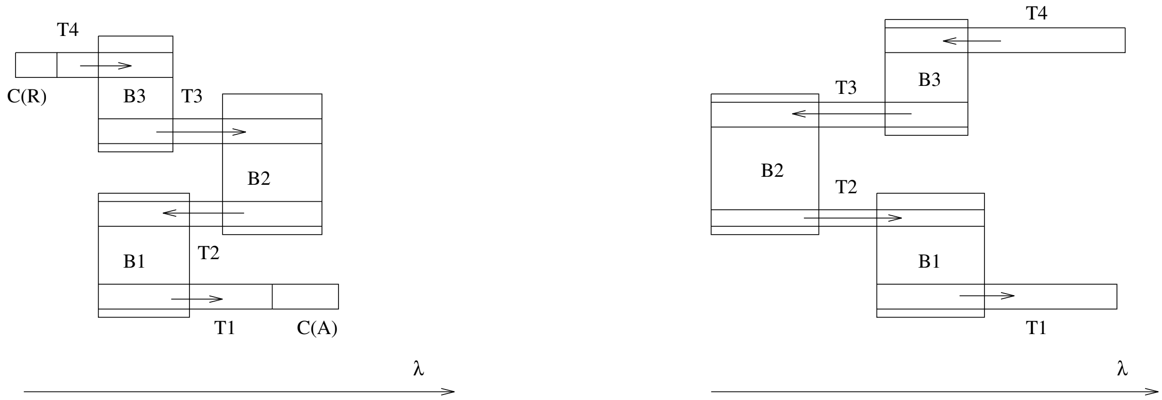

Figure 7: Schematic picture of a TBC collection on the left and a TB collection on the right. The horizontal direction is λ and arrows indicate the sign of $\dot{\lambda} = \varepsilon g(x, \lambda)$ in a tube φ^ε.

(4) If $i \neq j$, then $\mathcal{B}(i) \cap \mathcal{B}(j) = \emptyset$.

See Figure 7.

Returning yet again to the Nagumo equations observe that having fixed the boxes $\mathcal{B}(1)$ and $\mathcal{B}(2)$, if we choose our tubular neighborhoods sufficiently small in the (u, v) direction then $\mathcal{B}(1), \mathcal{B}(2)$ and $\mathcal{T}(1), \mathcal{T}(2), \mathcal{T}(3)$ with $\mathcal{T}(3) = \mathcal{T}(1)$ form a TB collection.

In the case in which one is looking for heteroclinic orbits the collection must, also, include caps.

Definition 1.5 A *tubes, boxes and caps collection* (TBC collection) is a collection of tubes $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$, boxes $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$, and caps $\mathcal{C}(R)$ and $\mathcal{C}(A)$ such that:

(1) the tubes $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$ and boxes $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ form a TB collection;

(2) $\mathcal{C}(R) \cap \mathcal{T}(I+1) \neq \emptyset$ and $\mathcal{C}(A) \cap \mathcal{T}(1) \neq \emptyset$. Furthermore,

$$\begin{aligned} \mathcal{C}(R) \cap \mathcal{T}(I+1) \cap (\mathbb{R}^n \times \{\lambda\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_\lambda(R) = \mathcal{T}_\lambda(I+1), \\ \mathcal{C}(A) \cap \mathcal{T}(1) \cap (\mathbb{R}^n \times \{\lambda\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_\lambda(A) = \mathcal{T}_\lambda(1). \end{aligned}$$

We introduce one final bit of notation before stating some of the results of this paper. Given a TB or TBC collection, let

$$D^i : \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{out}(p, i)) \to \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{in}(p, i))$$

denote the directional transition matrix associated with the box $\mathcal{B}(i)$ and let

$$D^i(P_i, 1) : CH^*(M_{out}(1, i)) \to CH^*(M_{in}(P_i, i)) \tag{1.7}$$

denote the corresponding entry. Again, having fixed the TB or TBC collection, we define a map

$$\Theta := \Theta(I) = D^I(P_I, 1) \circ D^{I-1}(P_{I-1}, 1) \circ \ldots \circ D^2(P_2, 1) \circ D^1(P_1, 1). \tag{1.8}$$

As stated, this definition obviously makes no sense since $CH^*(M_{in}(P_i, i)) \neq CH^*(M_{out}(1, i+1))$. However, as will be made clear in the next section (see Remark 2.12), the continuation of the Conley index allows for a natural identification between these spaces. Θ is introduced at this point to simplify the statements of the following theorems.

Theorem 1.6 *Let $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$ and $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ be a TB collection where $\mathcal{T}(1) = \mathcal{T}(I+1)$. Let*

$$\mathcal{N} := \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I} \mathcal{T}(i).$$

Then, for $\varepsilon > 0$ sufficiently small:

(1) $\mathcal{N}$ is an isolating neighborhood for φ^ε;

(2) Further, assume that for each $i = 1, \ldots, I$,

$$CH^j(M_{c_i}(1, i); \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$CH^j(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) = 0 \text{ if } j \neq k, k+1.$$

If Θ is an isomorphism, then

$$CH^k(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon); \mathbb{Z}_2) \cong CH^{k+1}(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon); \mathbb{Z}_2) \cong \mathbb{Z}_2;$$

otherwise

$$CH^k(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) \cong CH^{k+1}(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) = 0.$$

The importance of this result is that it can be used to prove the existence of periodic orbits. To do this, however, an additional piece of information is required.

Let $\Xi \subset \mathbb{R}^n$ and $\delta > 0$. Define a map $\phi_\delta : \Xi \times (-\delta, \delta) \to X$ by setting $\phi_\delta(x, t) := \varphi(t, x), x \in \Xi, t \in (-\delta, \delta)$. We call Ξ a *local section* if there is a $\delta > 0$ such that ϕ_δ is a homeomorphism with an open range.

Let N be an isolating neighborhood under the flow φ. Ξ is a *Poincaré section* for N if Ξ is a local section, $\Xi \cap N$ is closed, and for every $x \in N$, there exists $t_x > 0$ such that $\varphi(t_x, x) \in \Xi$.

Theorem 1.7 *Let $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$ and $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ be a TB collection with $\mathcal{T}(1) = \mathcal{T}(I+1)$. Let*

$$\mathcal{N} := \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I} \mathcal{T}(i).$$

Then, for $\varepsilon > 0$ sufficiently small there is an isolating neighborhood $\mathcal{N}' \subset \mathcal{N}$ with $\mathrm{Inv}(\mathcal{N}', \varphi^\varepsilon) = \mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)$ *such that $\mathcal{N}'$ admits a Poincaré section.*

The proof of this theorem is presented in Section 6.

Corollary 1.8 *Under the assumptions of Theorem 1.6, if Θ is an isomorphism, then for all sufficiently small $\varepsilon > 0$, $\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)$ contains a periodic orbit.*

Proof. This follows immediately from Theorem 1.6, Theorem 1.7 and [10, Theorem 1.3]. □

Corollary 1.9 *The Nagumo equations contain a periodic orbit for sufficiently small $\varepsilon > 0$.*

The following result can be used to find heteroclinic orbits. We begin with a concept concerning the dynamics within the isolating neighborhood.

The simplest non-trivial Morse decomposition of an isolated invariant set S consists of two Morse sets $M(1)$ and $M(0)$ with an admissible ordering $1 > 0$. In this case, $M(0)$ is called an *attractor* in S and $M(1)$ a *repeller*. Together, the pair $(M(0), M(1))$ is referred to as an *attractor repeller pair decomposition* of S.

Theorem 1.10 *Let $\{\mathcal{T}(i) \mid i = 1, \ldots, I+1\}$, $\{\mathcal{B}(i) \mid i = 1, \ldots, I\}$ and $\mathcal{C}(R), \mathcal{C}(A)$ be a TBC collection. Let*

$$\mathcal{N} := \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I+1} \mathcal{T}(i) \cup \mathcal{C}(R) \cup \mathcal{C}(A).$$

Then, for $\varepsilon > 0$ sufficiently small,

(1) $\mathcal{N}$ is an isolating neighborhood for φ^ε;

(2) $(\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon), \mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon))$ is an attractor-repeller pair for $\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)$;

(3) If $\Theta \neq 0$, then

$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) \ncong CH^*(\mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon)) \oplus CH^*(\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon)).$$

Corollary 1.11 *Under the assumptions of Theorem 1.10, if $\Theta \neq 0$, then for all sufficiently small $\varepsilon > 0$ there is a connecting orbit from $\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon)$ to $\mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon)$ in $\mathcal{N}$ under the flow φ^ε.*

Proof. By Theorem 1.10(2) $(\mathrm{Inv}(\mathcal{C}(R)), \mathrm{Inv}(\mathcal{C}(A)))$ is an attractor repeller pair. By Theorem 1.10(3),

$$CH^*(\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon)) \oplus CH^*(\mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon)) \ncong CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)).$$

Therefore, by [1, Theorem 3.3.1], there exists a connecting orbit. □

2 Preliminaries

This section contains a brief review of relevant portions of the Conley index theory. For the general theory the reader is referred to [1, 3, 13] and references therein. Throughout this section we shall let $\varphi : \mathbb{R} \times X \to X$ denote a flow on a locally compact space X.

2.1 Isolating blocks

The Conley index of an isolated invariant set is central to our discussion. As was indicated in the introduction, it is defined in terms of an index pair.

Definition 2.1 Let S be an isolated invariant set. A pair of compact sets (N, L) with $L \subset N$ is an *index pair* for S if:

(1) $S = \operatorname{Inv}(cl(N \setminus L))$ and $N \setminus L$ is a neighborhood of S;

(2) L is positively invariant in N, i.e. given $x \in L$ and $\varphi([0,t],x) \subset N$ then $\varphi([0,t],x) \subset L$;

(3) L is an exit set for N, i.e. given $x \in N$ and $T > 0$ such that $\varphi(T,x) \notin N$, there is a $t \in [0,T]$ such that $\varphi([0,t],x) \subset N$ and $\varphi(t,x) \in L$.

The cohomological Conley index of S is given by

$$CH^*(S) := H^*(N, L).$$

Since this is usually taken to be fixed we shall simplify the notation and write $CH^*(S) = H^*(N, L)$.

Given an isolating neighborhborhood N of S its *immediate exit* and *entrance sets* are defined, respectively, as follows

$$N^- := \{x \in N \mid \varphi([0,t],x) \not\subset N \text{ for all } t > 0\},$$

$$N^+ := \{x \in N \mid \varphi([t,0],x) \not\subset N \text{ for all } t < 0\}.$$

The local stable and unstable sets of S in N are given by

$$\begin{array}{rcl} W^s_N(S) & := & \{x \in N \mid \varphi([0,\infty),x) \subset N\}, \\ W^u_N(S) & := & \{x \in N \mid \varphi((-\infty,0],x) \subset N\}. \end{array}$$

The notion of an index pair as defined above is very general. While this flexibility simplifies some aspects of the index theory, from the point of view of computation it is less than ideal. In particular, given an arbitrary isolating neighborhood N of S it is not in general true that there exists a set L such that (N, L) is an index pair. Furthermore, even if L exists determining it is typically a nontrivial task since in essence one is required to obtain estimates for a global nonlinear problem. On the other hand, these computations can be greatly simplified if very special index pairs, known as isolating blocks, are used.

Definition 2.2 An isolating neighborhood N is an *isolating block* if $\partial N = N^+ \cup N^-$ and N^+ and N^- are subsets of local sections of the flow.

The following theorem indicates that for every isolated invariant set there exists an isolating block.

Theorem 2.3 [2, Theorem 3.4] *Given an isolated invariant set S and its isolating neighborhood N there exists an isolating block $B \subset N$ such that $S = \text{Inv}B$.*

Let N be an isolating block for S. Observe that (N, N^-) is an index pair for S and hence $CH^*(S) \cong H^*(N, N^-)$. A fundamental result is that different choices of index pairs give rise to isomorphic indices. A particular case which we will encounter repeatedly is that of two isolating blocks, N_1 and N_0, where $\text{Inv}(N_1, \varphi) = \text{Inv}(N_0, \varphi)$ and $N_0 \subset N_1$. Observe that for each $x \in N_0 \setminus W^s_{N_0}(S)$ there exists $t_x > 0$ such that $\varphi([0, t_x], x) \subset N_1$ and $\varphi(t_x, x) \in N_1^-$. Let

$$\tau := \sup_{x \in N_0^-} \{t_x\}.$$

Define $\Phi : (N_0, N_0^-) \to (N_1, N_1^-)$ by

$$\Phi := \begin{cases} \varphi(t_x, x) & \text{if } t_x \leq \tau \\ \varphi(\tau, x) & \text{otherwise.} \end{cases} \tag{2.1}$$

Proposition 2.4 [2] *Φ is continuous and*

$$\Phi^* : H^*(N_1, N_1^-) \to H^*(N_0, N_0^-)$$

is an isomorphism.

We shall refer to Φ^* as the *index isomorphism.*

Observe that $W^u_N(S) \cap \partial N \subset N^-$. There is a very strong relation between the topology of the local unstable set of S and the Conley index of S as is indicated by the following result.

Lemma 2.5 [2, Lemma 4.3] *Let N be an isolating block for S and let*

$$c_N : (W^u_N(S), W^u_N(S) \cap N^-) \to (N, N^-)$$

be inclusion. Then

$$c_N^* : H^*(N, N^-) \to H^*(W^u_N(S), W^u_N(S) \cap N^-)$$

is an isomorphism, called the Churchill isomorphism.

We shall also make use the following proposition.

Proposition 2.6 [5, Lemma 5] *Let (N, L) be an index pair for S. Then*

$$H^*(N, L \cup W^u_N(S)) = 0.$$

2.2 Parameterized flows

Recall the discussion and notation of parameterized flows presented in the introduction. Given $K \subset \Lambda$ and $N \subset X \times \Lambda$, let $N_K := N \cap (X \times K)$. The following theorem is at the heart of the continuation theory for the Conley index.

Theorem 2.7 [13, Theorem 6.7] *Let (N, L) be an index pair for an isolated invariant set S under φ_Λ. Then for every $\lambda_0 \in \Lambda$ there is a compact neighborhood $K \subset \Lambda$ of λ_0 such that the natural inclusion map*

$$j_\lambda : N_\lambda / L_\lambda \to N_K / L_K$$

is a homotopy equivalence for every $\lambda \in K$.

Corollary 2.8 *Under assumptions of Theorem 2.7 the map j_λ induces an isomorphism*

$$j_\lambda^* : H^*(N_K, L_K) \to H^*(N_\lambda, L_\lambda)$$

for all $\lambda \in K$.

Corollary 2.9 *Under the assumptions of Theorem 2.7 the map*

$$F^*_{\lambda',\lambda}(S) : H^*(N_{\lambda'}, L_{\lambda'}) \to H^*(N_\lambda, L_\lambda)$$

defined by $F^*_{\lambda',\lambda}(S) := j^*_\lambda \circ (j^*_{\lambda'})^{-1}$, is an isomorphism for all $\lambda, \lambda' \in K$.

2.3 Transition matrices

In this section we review some basic facts about the topological transition matrices which were first introduced in [8]. To simplify the presentation we begin by setting the parameter space $\Lambda = [0, 1]$, and assume that the Morse decomposition $\mathcal{M}(S) := \{M(p) \mid p \in (\mathcal{P}, >)\}$ continues over all of Λ. Since each Morse set $M(p)$ continues over Λ, there are isomorphisms

$$F^*_{1,0}(p) : CH^*(M_1(p)) \to CH^*(M_0(p))$$

(compare Corollary 2.9). Similarly, since S continues over Λ there is an isomorphism

$$F^*_{1,0}(S) : CH^*(S_1) \to CH^*(S_0).$$

If $S_\lambda = \bigcup_{p\in\mathcal{P}} M_\lambda(p)$ i.e. the set of connecting orbits is empty, then there exists an index isomorphism

$$\Phi^*_\lambda : H^*(N_\lambda, L_\lambda) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_\lambda(p)). \tag{2.2}$$

Suppose there are no connections at either $\lambda = 0$ or $\lambda = 1$, then we can construct the following diagram

$$\begin{array}{ccc}
\bigoplus_{p\in\mathcal{P}} CH^*(M_1(p)) & \overset{\bigoplus_{p\in\mathcal{P}} F^*_{1,0}(p)}{\longrightarrow} & \bigoplus_{p\in\mathcal{P}} CH^*(M_0(p)) \\
\Phi^*_1 \uparrow & & \Phi^*_0 \uparrow \\
CH^*(S_1) & \overset{F^*_{1,0}(S)}{\longrightarrow} & CH^*(S_0)
\end{array}$$

Remark 2.10 Even though every map is an isomorphism this diagram is not, in general, commutative. Furthermore, it is the failure to commute which gives information concerning connecting orbits. Therefore, it is important to express this lack of commutativity in the most obvious manner. We shall do this by choosing generators. Fix a set of generators $\mathcal{G}_1$ of $\bigoplus_{p\in\mathcal{P}} CH^*(M_1(p))$. Define the generators of $\bigoplus_{p\in\mathcal{P}} CH^*(M_0(p))$ to be

$$\mathcal{G}_0 := \bigoplus_{p\in\mathcal{P}} F^*_{1,0}(p)(\mathcal{G}_1).$$

With this identification, $\bigoplus_{p\in\mathcal{P}} F^*_{1,0}(p)$ takes the form of the identity matrix.

From this point on we shall assume that this identification has been made and we refer to it as the *natural Morse continuation identification.*

The *topological transition matrix* is

$$T^{1,0} : \bigoplus_{p\in\mathcal{P}} CH^*(M_1(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_0(p)),$$

where the natural Morse continuation identification has been made and is defined by

$$T^{1,0} = \Phi^*_0 \circ F^*_{1,0} \circ (\Phi^*_1)^{-1}.$$

Note that the diagram

$$\begin{array}{ccc} \bigoplus_{p\in\mathcal{P}} CH^*(M_1(p)) & \stackrel{T^{1,0}}{\to} & \bigoplus_{p\in\mathcal{P}} CH^*(M_0(p)) \\ \Phi^*_1 \uparrow & & \Phi^*_0 \uparrow \\ CH^*(S_1) & \stackrel{F^*_{1,0}}{\to} & CH^*(S_0) \end{array} \tag{2.3}$$

commutes by definition. The topological transition matrix is lower triangular, and has the property that its off diagonal nonzero entry implies the existence of connecting orbits between appropriate Morse sets for various $\lambda \in (0,1)$.

Remark 2.11 Following in the spirit of Remark 2.10, observe that each tube $\mathcal{T}(i)$ defines a continuation between the invariant sets $M_{out}(1, i+1)$ and $M_{in}(P_i, i)$. Thus, by Corollary 2.9, $CH^*(M_{out}(1, i+1)) \cong CH^*(M_{in}(P_i, i))$. Since the boxes are disjoint, we can use this continuation to choose basis for these linear spaces in such a way that the matrix representing the continuation is the identity matrix. We shall refer to this choice of bases as the *natural tube continuation identification.*

Remark 2.12 At this point the validity of the definition of Θ (see (1.8)) should be apparent. In particular, given the natural Morse and tube continuation identifications what is missing from (1.8) are identity matrices corresponding to the continuation isomorphisms generated by the tubes and the Morse decompositions.

If the index set $\mathcal{P}$ is decomposed to two disjoint sets $\mathcal{P}_-$ and $\mathcal{P}_+$, i.e. $\mathcal{P} = \mathcal{P}_- \cup \mathcal{P}_+$, $\mathcal{P}_- \cap \mathcal{P}_+ = \emptyset$, then we can construct the *directional transition matrix*

$$D : \bigoplus_{p\in\mathcal{P}} CH^*(M_{o(p)}(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_{i(p)}(p)),$$

where

$$i(p) = \begin{cases} 0 & p \in \mathcal{P}_+, \\ 1 & p \in \mathcal{P}_-, \end{cases} \qquad \text{and} \qquad o(p) = \begin{cases} 1 & p \in \mathcal{P}_+, \\ 0 & p \in \mathcal{P}_-. \end{cases}$$

Notice that, in this paper, the decomposition is given by the direction of flow on slow manifolds in a box $\mathcal{B}$, namely $\mathcal{P}_- = \{p|\delta(p,\mathcal{B}) < 0\}$ and $\mathcal{P}_+ = \{p|\delta(p,\mathcal{B}) > 0\}$, hence in our case $i(p) = in(p)$ and $o(p) = out(p)$.

The directional transition matrix is defined as follows: If the sets of basis of $CH^*(M_0(p))$ and $CH^*(M_1(p))$ for any $p \in \mathcal{P}$ are given, one obtains the basis of $\bigoplus_{p\in\mathcal{P}} CH^*(M_{i(p)}(p))$ and $\bigoplus_{p\in\mathcal{P}} CH^*(M_{o(p)}(p))$. From the definition, the topological transition matrix determines an invertible correspondence between the basis elements of $\bigoplus_{p\in\mathcal{P}} CH^*(M_0(p))$ and $\bigoplus_{p\in\mathcal{P}} CH^*(M_1(p))$. Since the topological transition matrix is lower-triangular, it also implies an invertible correspondence between the basis elements of $\bigoplus_{p\in\mathcal{P}} CH^*(M_{i(p)}(p))$ and $\bigoplus_{p\in\mathcal{P}} CH^*(M_{o(p)}(p))$, and hence it gives an isomorphism

$$\mathcal{D} : \bigoplus_{p\in\mathcal{P}} CH^*(M_{o(p)}(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_{i(p)}(p)),$$

which is the directional transition matrix.

In this paper, we only need an algebraic map $\mathcal{D}(P,1)$ given as the $(P,1)$-entry of the directional transition matrix. However, in general, off diagonal nonzero entries of the directional transition matrix have similar consequences for the existence of connecting orbits between the Morse sets as topological transition matrix. See [7] for more details.

2.4 Singular isolating neighborhoods and singular index pairs

To simplify the notation we let $y = (x,\lambda) \in \mathbb{R}^{n+1}$ and write

$$\dot{y} = F(y) = F_0(y) + \varepsilon F_1(y) + \ldots + \varepsilon^k F_k(y) + \ldots \tag{2.4}$$

in place of equation (1.2). As will be seen, it is not necessary that F be analytic or C^∞ in ε, only that F have enough derivatives to apply Theorem 2.18 below.

Definition 2.13 A compact set $N \subset \mathbb{R}^{n+1}$ is called a *singular isolating neighborhood* if N is not an isolating neighborhood for φ^0, but there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\bar{\varepsilon}]$, N is an isolating neighborhood for φ^ε.

Definition 2.14 A pair of compact sets (N,L) with $N \subset L$ is a *singular index pair* if $cl(N \setminus L)$ is a singular isolating neighborhood and there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\bar{\varepsilon}]$

$$H^*(N,L) \cong CH^*(\text{Inv}(cl(N \setminus L),\varphi^\varepsilon)).$$

Observe that the last two definitions are most useful if we find a way to construct singular isolating neighborhoods and singular index pairs using primarily the φ^0 flow, along with minimal information about the higher order terms of F. The conditions for the existence of a singular isolating neighborhood were given by Conley [4] and the construction of a singular index pair was done in [11]. We shall follow the latter paper in our exposition.

Let N be a compact set and let $S = \mathrm{Inv}(N, \varphi^0)$. Observe that if N is not an isolating neighborhood for φ^0, then by definition there exists $x \in S \cap \partial N$. If N is to be a singular isolating neighborhood, then such an x has to leave in forward or backward time under φ^ε for all $\varepsilon > 0$. This leads to the following definition.

Definition 2.15 Let N be a compact set and let $x \in S$. x is a *slow exit (entrance) point* if there exists a neighborhood U of x and an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ there exists a time $T(\varepsilon, U) > 0$ ($T(\varepsilon, U) < 0$) satisfying

$$\varphi^\varepsilon(T(\varepsilon, U), U) \cap N = \emptyset.$$

Theorem 2.16 [11, Theorem 1.5] *Let N be a compact set. If $S \cap \partial N$ consists of slow entrance and slow exit points, then N is a singular isolating neighborhood.*

It follows from the last theorem that in order to construct a singular isolating neighborhood it is important to be able to recognize slow exit and slow entrance points. Before we quote a theorem which does just that, we introduce some notation. We let S^- (S^+) denote the set of slow exit (entrance) points. Set $S_\partial := S \cap \partial N$ and $S_\partial^\pm := S_\partial \cap S^\pm$. Given an invariant set K, let $\mathcal{R}(K)$ denote the chain recurrent set of K under φ^0.

Definition 2.17 The average of h on S, $Ave(h, S)$, is the limit as $t \to \infty$ of the set of numbers $\{\frac{1}{t}\int_0^t h(\varphi^0(s, x))ds \mid x \in S\}$. If $Ave(h, S) \subset (0, \infty)$ then h has *strictly positive averages* on S.

Theorem 2.18 [4] *$x \in S$ is a slow exit point if there exists a compact set $K_x \subset S$ invariant under φ^0, a neighborhood U_x of $\mathcal{R}(K_x)$, an $\bar{\varepsilon} > 0$ and a function $l : cl(U_x) \times [0, \bar{\varepsilon}] \to \mathbb{R}$ such that the following conditions are satisfied.*

(1) $\omega(x, \varphi^0) \subset K_x$;

(2) l is of the form
$$l(z, \varepsilon) = l_0(z) + \varepsilon l_1(z) + \ldots + \varepsilon^m l_m(z);$$

(3) If $L_0 = \{z \mid l_0(z) = 0\}$ then
$$K_x \cap cl(U_x) = S \cap L_0 \cap cl(U_x),$$
and furthermore $l_0|_{S \cap cl(U_x)} \leq 0$;

(4) Let
$$h_j(z) = \nabla_z l_0(z) \cdot F_j(z) + \nabla_z l_1(z) \cdot F_{j-1}(z) + \ldots + \nabla_z l_j(z) \cdot F_0(z).$$
Then for some m, $h_j \equiv 0$ if $j < m$, and h_m has strictly positive averages on $\mathcal{R}(K_x)$.

A slow exit point which satisfies the conditions of Theorem 2.18 is called a *C-slow exit point.* If we reverse time we can use the Theorem 2.18 to test for slow entrance points. Slow entrance points of this form will be called *C-slow entrance points.*

Now, given a singular isolating neighborhood N, we want to identify a singular index pair. We need a few definitions. The *immediate exit set* for N is defined by

$$N^- := \{x \in \partial N \mid \varphi^0((0,t),x) \not\subset N \text{ for all } t > 0\}.$$

Given $Y \subset N$ its *push-forward set* in N under the flow φ^0 is defined to be

$$\rho(Y, N, \varphi^0) := \{x \in N \mid \exists z \in Y, t \geq 0 \text{ such that } \varphi^0([0,t],z) \subset N, \varphi^0(t,z) = x\}.$$

Finally, the *unstable set* of an invariant set $Y \subset N$ under φ^0 is

$$W^u_N(Y) := \{x \in N \mid \varphi^0((-\infty,0),x) \subset N \text{ and } \alpha(x,\varphi^0) \subset Y\}.$$

A slow entrance point x is a *strict slow entrance point* if there exists a neighborhood U_x of x and an $\bar{\varepsilon} > 0$ such that if $y \in U_x \cap N$ and $\varepsilon \in (0, \bar{\varepsilon}]$, then there exists $t_y(\varepsilon) > 0$ for which

$$\varphi^\varepsilon([0, t_y(\varepsilon)], y) \subset N.$$

We will let S_∂^{++} denote the strict slow entrance points.

Theorem 2.19 [11, Theorem 1.16] *Let N be a singular isolating neighborhood. Assume*

(1) S_∂^- consists of C-slow exit points;

(2) $S_\partial \subset S_\partial^{++} \cup S_\partial^-$;

(3) $(S_\partial^{++} \setminus S_\partial^-) \cap cl(N^-) = \emptyset$.

For each $x \in S_\partial^-$, let K_x denote a compact invariant set as in Theorem 2.18. Define

$$L := \rho(cl(N^-), N, \varphi^0) \cup W^u_N\Big(\bigcup_{x \in S_\partial^-} \mathcal{R}(K_x)\Big).$$

If L is closed, then (N, L) is a singular index pair for the family of flows φ^ε.

3 Construction of a singular index pair

The point of this section is to construct a singular isolating neighborhood and from that a singular index pair for TB or TBC collections. The construction and proofs are quite similar with or without the caps, thus only the proofs for TB collections are provided. Of course, a few comments concerning TBC collections are included. The actual computation of the index is left to the next section. We will adhere strictly to the notation established in Subsection 2.4 concerning the singular index theory.

We begin by fixing a TB or TBC collection with tubes $\{\mathcal{T}(i) \mid i = 1, \dots, I+1\}$, boxes $\{\mathcal{B}(i) \mid i = 1, \dots, I\}$ and, possibly, caps $\mathcal{C}(R)$ and $\mathcal{C}(A)$. In order to simplify the notation let $M(p,i)$ denote the Morse set $M(p, \mathcal{B}(i))$.

The definition of tubes and boxes was given in terms of isolating neighborhoods. As was mentioned in the introduction these sets need not have much structure. Thus, the first step is to replace isolating neighborhoods with isolating blocks. By Theorem 2.3 for each i we may choose an isolating block $\mathbf{B}(i)$ with the property that

$$\mathbf{B}(i) \subset \mathcal{B}(i) \quad \text{and} \quad \mathrm{Inv}(\mathbf{B}(i), \psi^{\mathcal{B}(i)}) = \mathrm{Inv}(\mathcal{B}(i), \psi^{\mathcal{B}(i)}).$$

Observe that $\mathbf{B}(i)$ is a box where the sets $V(p, \mathbf{B}(i))$ are given by $\mathbf{B}(i) \cap V(p, \mathcal{B}(i))$ for $p \in \mathcal{P}^{\mathcal{B}(i)}$.

Remark 3.1 Recall that by Definition 1.2(2) there are no connecting orbits between the Morse sets in box $\mathbf{B}(i)$ at parameter values $\lambda = c_i, d_i$. From this, a simple argument based on the continuity of the flow shows that one may choose the sets $V(p, \mathbf{B}(i))$ such that if $x \in V_{*_i}(p, \mathbf{B}(i))$ and $\psi_{*_i}([0,t], x) \subset \mathbf{B}_{*_i}(i)$, then $\psi_{*_i}([0,t], x) \cap V_{*_i}(q, \mathbf{B}(i)) = \emptyset$ for $q \neq p$ where $* = c, d$. From now on we assume that the $V(p, \mathbf{B}(i))$ have been chosen to satisfy this property for all p and all i.

Again, using Theorem 2.3 for each $V(p, \mathbf{B}(i))$ we may choose an isolating block $\mathbf{V}(p,i)$ with the property that

$$\mathbf{V}(p,i) \subset V(p, \mathbf{B}(i)) \quad \text{and} \quad \mathrm{Inv}(\mathbf{V}(p,i), \psi^{V(p,\mathbf{B}(i))}) = \mathrm{Inv}(V(p, \mathbf{B}(i)), \psi^{V(p,\mathbf{B}(i))}).$$

Finally, observe that

$$\mathcal{T}_0(i) := (\mathcal{T}(i) \setminus (\mathbf{B}(i) \cup \mathbf{B}(i-1))) \cup V(1, \mathbf{B}(i)) \cup V(P_{i-1}, \mathbf{B}(i-1))$$

is a tube. So again, by Theorem 2.3 for each i we may choose an isolating block $\mathbf{T}(i)$ with the property that

$$\mathbf{T}(i) \subset \mathcal{T}_0(i) \quad \text{and} \quad \mathrm{Inv}(\mathbf{T}(i), \psi^{\mathcal{T}(i)}) = \mathrm{Inv}(\mathcal{T}(i), \psi^{\mathcal{T}(i)}).$$

It is now left to the reader to check that $\{\mathbf{T}(i) \mid i = 1, \dots, I+1\}$ and $\{\mathbf{B}(i) \mid i = 1, \dots, I\}$ form a TB collection.

In the setting of a TBC collection, one should also modify the caps as follows. Choose $\mathbf{C}(R) \subset \mathcal{C}(R)$ and $\mathbf{C}(A) \subset \mathcal{C}(A)$ such that $\mathbf{C}(R) \cup \mathbf{T}(I+1)$ and $\mathbf{C}(A) \cup \mathbf{T}(1)$ form isolating blocks for the appropriate parameterized flows.

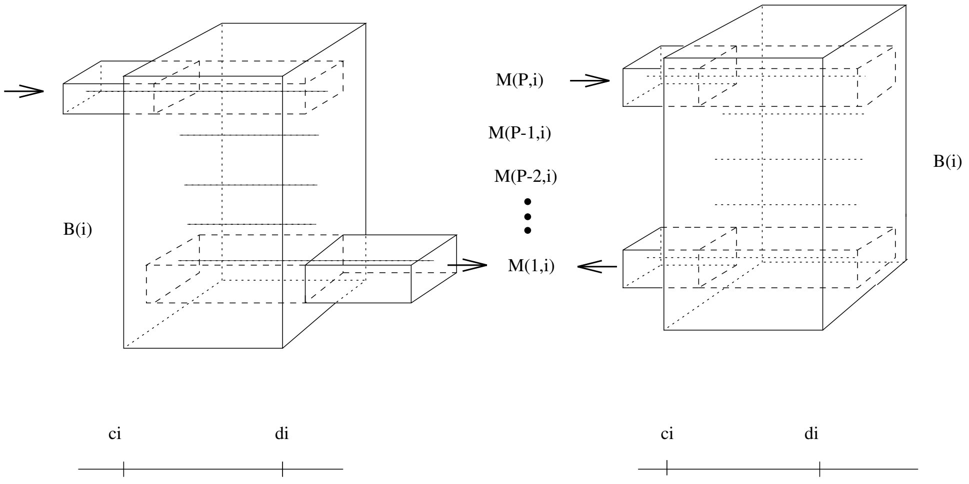

Figure 8: A box with two neighboring tubes attached.

The reason for switching to a construction based on isolating blocks is that the boundary of the sets which have been created consist of strict entrance and exit sets or level sets with respect to the slow variable λ.

In the case of a TB collection, define

$$N := \left(\bigcup_{i=1}^{I} \mathbf{T}(i)\right) \bigcup \left(\bigcup_{i=1}^{I} \mathbf{B}(i)\right)$$

and for a TBC collection, let

$$N := \left(\bigcup_{i=1}^{I+1} \mathbf{T}(i)\right) \bigcup \left(\bigcup_{i=1}^{I} \mathbf{B}(i)\right) \bigcup \left(\mathbf{C}(R) \bigcup \mathbf{C}(A)\right).$$

The goal is now to show that N is a singular isolating neighborhood. Perhaps the first observation that needs to be made is that N is not an isolating neighborhood. To see this let

$$S := \operatorname{Inv}(N, \varphi^0).$$

Now fix a parameter value λ. Observe that $S_\lambda = \operatorname{Inv}(N_\lambda, \psi_\lambda)$. Let $x \in \partial N_\lambda$. Now assume that $\lambda \notin \{c_i, d_i \mid i = 1, \ldots, I\}$. Then, $x \in \partial \mathcal{T}_\lambda(i)$ or $x \in \partial \mathcal{B}_\lambda(i)$ for some i. Because $\mathcal{T}_\lambda(i)$ and $\mathcal{B}_\lambda(i)$ are isolating neighborhoods under ψ_λ, $x \notin S_\lambda$. On the other hand, if $\lambda \in \{c_i, d_i \mid i = 1, \ldots, I\}$, then $x \in cl(\mathcal{B}(i) \setminus (\mathcal{T}(i) \cup \mathcal{T}(i+1)))$. In particular, if $x \in M_\lambda(p, i)$ and $x \notin (\mathcal{T}(i) \cup \mathcal{T}(i+1))$ then $x \in \partial N \cap S$. Therefore, N is not an isolating neighborhood. Theorem 2.16 will be used to show that N is a singular isolating neighborhood. In particular it will be shown that if $x \in \partial N \cap S$, then x is a slow entrance or exit point.

Let

$$\hat{S}^-_\partial(i) := \bigcup_{p=2}^{P_i} M_{out}(p, i) \quad \text{and} \quad \hat{S}^+_\partial(i) := \bigcup_{p=1}^{P_i - 1} M_{in}(p, i).$$

Recall that $M_{out}(p, i)$ and $M_{in}(p, i)$ are defined in (1.6).

Lemma 3.2 *For a TB collection where* $\mathbf{T}(1) = \mathbf{T}(I+1)$*,*

(1) N *is a singular isolating neighborhood;*

(2) The set of slow exit points is

$$S_\partial^- = \bigcup_{i=1}^{I} \hat{S}_\partial^-(i)$$

and, in fact, consists of C-slow exit points;

(3) The set of slow entrance points is

$$S_\partial^+ = \bigcup_{i=1}^{I} \hat{S}_\partial^+(i)$$

and, in fact, consists of strict slow entrance points;

(4) $S_\partial \subset S_\partial^{++} \cup S_\partial^-$.

Proof. We begin with the proof of (2) and (3). Let $S_\partial := S \cap \partial N$. From the previous discussion it is clear that if $(x,\lambda) \in S_\partial$ then $\lambda \in \{c_i, d_i\}$ for some i and $x \in M_\lambda(p,i)$. In fact, because of the tubes,

$$x \in \bigcup_{p=1}^{P_i} M_\lambda(p,i) \setminus (M_{in}(P_i,i) \cup M_{out}(1,i)) \, .$$

Observe now that $l(z,\varepsilon) := \pm\lambda - c_i$ and $l(z,\varepsilon) := \pm\lambda - d_i$ are slow Lyapunov functions defined on the sets $V_{c_i}(p, \mathbf{B}(i))$ and $V_{d_i}(p, \mathbf{B}(i))$, respectively. (2) and (3) now follow.

Since $S_\partial = S_\partial^- \cup S_\partial^+$, N is a singular isolating neighborhood.

Definition 1.2(3) implies that the slow entrance points are strict slow entrance points. □

The proof of the following lemma is similar.

Lemma 3.3 *For a TBC collection:*

(1) N *is a singular isolating neighborhood;*

(2) The set of slow exit points is

$$S_\partial^- = \bigcup_{i=1}^{I} \hat{S}_\partial^-(i) \cup S_a(R)$$

and, in fact, consists of C-slow exit points;

(3) The set of slow entrance points is

$$S_\partial^+ = \bigcup_{i=1}^{I} \hat{S}_\partial^+(i) \cup S_b(A)$$

and, in fact, consists of strict slow entrance points;

(4) $S_\partial \subset S_\partial^{++} \cup S_\partial^-$.

The next step is to construct a set L such that (N, L) is a singular index pair. For this the immediate exit set for N needs to be identified.

Lemma 3.4 *For a TB collection the immediate exit and entrance sets of N under φ^0 are:*

$$\begin{aligned} N^- &= \left(\left(\bigcup_{i=1}^{I+1} \mathbf{T}^-(i)\right) \cup \left(\bigcup_{i=1}^{I} \mathbf{B}^-(i)\right)\right) \setminus \left(\bigcup_{i=1}^{I} \left\{\mathbf{B}(i) \cap \left(\mathbf{T}^-(i) \cup \mathbf{T}^-(i+1)\right)\right\}\right), \\ N^+ &= \left(\left(\bigcup_{i=1}^{I+1} \mathbf{T}^+(i)\right) \cup \left(\bigcup_{i=1}^{I} \mathbf{B}^+(i)\right)\right) \setminus \left(\bigcup_{i=1}^{I} \left\{\mathbf{B}(i) \cap \left(\mathbf{T}^+(i) \cup \mathbf{T}^+(i+1)\right)\right\}\right). \end{aligned}$$

Proof. Since we are working with the flow φ^0, it is sufficient to consider ψ_λ for each relevant value of λ.

First consider a tube $\mathbf{T}(i)$. Choose $\lambda \in [a_i, b_i]$ such that $\lambda \notin [c_i, d_i] \cup [c_{i-1}, d_{i-1}]$. Because $\mathbf{T}_\lambda(i)$ is an isolating block, $x \in \mathbf{T}_\lambda(i) \cap N^-$ if and only if $x \in \mathbf{T}_\lambda^-(i)$. On the other hand for $\lambda \in [c_i, d_i] \cup [c_{i-1}, d_{i-1}]$, $\mathbf{T}_\lambda(i) \subset \mathbf{B}_\lambda(i)$ or $\mathbf{T}_\lambda(i) \subset \mathbf{B}_\lambda(i-1)$. In either case x is an immediate exit point for N.

Now consider a box $\mathbf{B}(i)$. Choose $\lambda \in [c_i, d_i]$. Then $x \in \mathbf{B}_\lambda(i)$ is in N^- if and only if $x \in \mathbf{B}_\lambda^-(i)$. □

A similar argument, in which one only needs to consider the caps, in addition, leads to the following lemma.

Lemma 3.5 *For a TBC collection the immediate exit and entrance sets of N under φ^0 are:*

$$\begin{aligned} N^- &= \left(\left(\bigcup_{i=1}^{I+1} \mathbf{T}^-(i)\right) \cup \left(\bigcup_{i=1}^{I} \mathbf{B}^-(i)\right) \cup \mathbf{C}^-(A) \cup \mathbf{C}^-(R)\right) \setminus \left(\bigcup_{i=1}^{I} \left\{\mathbf{B}(i) \cap \left(\mathbf{T}^-(i) \cup \mathbf{T}^-(i+1)\right)\right\}\right), \\ N^+ &= \left(\left(\bigcup_{i=1}^{I+1} \mathbf{T}^+(i)\right) \cup \left(\bigcup_{i=1}^{I} \mathbf{B}^+(i)\right) \cup \mathbf{C}^+(A) \cup \mathbf{C}^+(R)\right) \setminus \left(\bigcup_{i=1}^{I} \left\{\mathbf{B}(i) \cap \left(\mathbf{T}^+(i) \cup \mathbf{T}^+(i+1)\right)\right\}\right). \end{aligned}$$

Using Lemmas 3.2, 3.3, 3.4 and 3.5, it follows that

$$(S_\partial^{++} \setminus S_\partial^-) \cap cl(N^-) = \emptyset. \tag{3.1}$$

Proposition 3.6 *Given a TB collection where $\mathbf{T}(1) = \mathbf{T}(I+1)$, let*

$$L := \rho(cl(N^-), N, \varphi^0) \cup \left(\bigcup_{i=1}^{I} \bigcup_{p=2}^{P_i} W^u_{\mathbf{B}(i)}\left(M_{out}(p, i)\right)\right).$$

Then, (N, L) is a singular index pair.

Proof. By Lemma 3.2 and (3.1) all that remains to be shown is that L is closed.

We begin by considering the set $\rho(cl(N^-), N, \varphi^0)$. Observe that if $(x, \lambda) \in N^-$, then $\rho((x, \lambda), N, \varphi^0) = \{(x, \lambda)\}$. So consider $(x, \lambda) \in cl(N^-) \setminus N^-$. Then, $x \in \mathbf{T}_{c_{i-1}}(i)^-$, $x \in \mathbf{T}_{d_{i-1}}(i)^-$, $x \in \mathbf{T}_{c_i}(i)^-$, or $x \in \mathbf{T}_{d_i}(i)^-$, depending on the sign of $\delta(1, \mathbf{B}(i))$ or $\delta(P_i, \mathbf{B}(i-1))$.

Assume $x \in \mathbf{T}_{c_{i-1}}(i)^-$ (the other cases follow by the same argument). By Definition 1.4(1), $x \in V_{c_{i-1}}(P_i, \mathbf{B}(i))$. However, by Remark 3.1 the forward orbit of x leaves $\mathbf{B}_{c_{i-1}}(i-1)$ in finite time without intersecting $V_{c_{i-1}}(q, \mathbf{B}(i))$ for all $q \neq P_i$. Therefore, $\rho((x,\lambda), N, \varphi^0)$ is closed, which in turn implies that $\rho(cl(N^-), N, \varphi^0)$ is closed.

Observe now that the same argument shows that $W^u_{\mathbf{B}(i)}(M_{out}(p,i))$ is closed for each i and p. Therefore, L is closed. $\square$

A similar argument leads to:

Proposition 3.7 *Given a TBC collection, let*

$$L := \rho(cl(N^-), N, \varphi^0) \cup \left(\bigcup_{i=1}^{I} \bigcup_{p=2}^{P_i} W^u_{\mathbf{B}(i)}\left(M_{out}(p,i)\right) \right) \cup W^u_{\mathbf{C}(R)}(S_*(R)),$$

where $S_*(R) = \mathrm{Inv}(\mathbf{C}(R) \cap \{\lambda = *\}, \varphi^0)$ *and* $*$ *is* e *or* f *depending on how* $\mathcal{T}(I)$ *attaches to* $\mathbf{C}(R)$*. Then,* (N, L) *is a singular index pair.*

4 The Algebra Behind the Index Computation

In the previous section it was shown that (N, L) is a singular index pair. Therefore, to compute $CH^*(\mathrm{Inv}(N, \varphi^\varepsilon))$ it is sufficient to determine $H^*(N, L)$. This is done via a series of Mayer-Vietoris sequences which will be described in this section. We begin by defining the basic units of the decomposition of (N, L).

Let $s_i \in (a_i, b_i)$ such that $\mathbf{T}_{s_i}(i) \cap (\mathbf{B}(i) \cup \mathbf{B}(i-1)) = \emptyset$. If $\delta(\mathbf{T}(i)) > 0$, let

$$\mathbf{T}_{in}(i) := \mathbf{T}(i) \cap (\mathbb{R} \times [s_i, b_i])$$

and

$$\mathbf{T}_{out}(i) := \mathbf{T}(i) \cap (\mathbb{R} \times [a_i, s_i]).$$

If $\delta(\mathbf{T}(i)) < 0$, let

$$\mathbf{T}_{in}(i) := \mathbf{T}(i) \cap (\mathbb{R} \times [a_i, s_i])$$

and

$$\mathbf{T}_{out}(i) := \mathbf{T}(i) \cap (\mathbb{R} \times [s_i, b_i]).$$

Let

$$\begin{aligned} N(i) &:= \mathbf{T}_{out}(i) \cup \mathbf{B}(i) \cup \mathbf{T}_{in}(i+1) \\ L(i) &:= L \cap N(i) \end{aligned}$$

and define, for $l < k$,

$$\begin{aligned} N(l,k) &:= \bigcup_{i=l}^{k} N(i), \\ L(l,k) &:= \bigcup_{i=l}^{k} L(i). \end{aligned}$$

Set

$$\widehat{L}(i) := \bigcup_{p=2}^{P_i} W^u_{\mathbf{B}(i)}(M_{out}(p,i)).$$

One of the goals of this section is to prove the following proposition.

Proposition 4.1 *For each $k = 2, \ldots I$,*

$$H^*(N(1,k), L(1,k)) \cong CH^*(M_{out}(1,1)).$$

The key step is to fully understand the following exact sequence:

$$\begin{aligned} \ldots \to\; & H^*(N(i) \cup N(i+1), L(i) \cup L(i+1)) \\ & \xrightarrow{\iota^*} H^*(N(i+1), L(i+1)) \oplus H^*(N(i), L(i)) \\ & \xrightarrow{\eta^*} H^*(N(i) \cap N(i+1), L(i) \cap L(i+1)) \to \ldots \end{aligned} \tag{4.1}$$

See Figure 9. As will become clear in Section 5, the computation of $H^*(N, L)$ is straightforward, once (4.1) is understood.

Observe that $(N(i) \cap N(i+1), L(i) \cap L(i+1)) = (\mathbf{T}_{s_{i+1}}(i+1), \mathbf{T}^-_{s_{i+1}}(i+1))$.

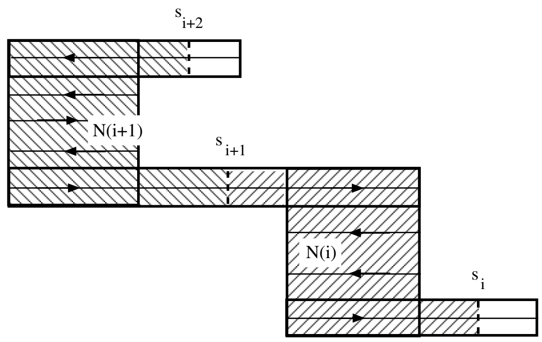

Figure 9: Tubes and boxes collection for the homology computation.

Proposition 4.2

$$H^*(\mathbf{T}_{s_{i+1}}(i+1), \mathbf{T}^-_{s_{i+1}}(i+1)) \cong CH^*(M_{in}(P,i)) \cong CH^*(M_{out}(1,i+1)).$$

Proof. Since, under the appropriate restrictions $\mathbf{T}(i+1)$ is an isolating block for both $M_{in}(P,i)$ and $M_{out}(1,i+1)$ the result follows from the continuation theorem for the Conley index. □

Remark 4.3 Proposition 4.2 implies that there exists a continuation induced isomorphism

$$F^{i,i+1}(\mathcal{T}(i+1)) : CH^*(M_{out}(1,i+1)) \to CH^*(M_{in}(P,i)).$$

In light of Remark 2.11 we always choose bases such that $F^{i,i+1}(\mathcal{T}(i+1))$ is represented by the identity matrix, and therefore, in an attempt to prevent an overwhelming growth in notation, is ignored.

We now turn to the computation of $H^*(N(i), L(i))$. Again, a Mayer-Vietoris argument is used. Choose $\mu_i \in (c_i, d_i)$ such that $S_{\mu_i} = \cup_{p=1}^{P_i} M_{\mu_i}(p,i)$. Define

$$\begin{aligned} Z(i) &:= N(i) \cap (\mathbb{R}^n \times (-\infty, \mu_i]) \\ Y(i) &:= N(i) \cap (\mathbb{R}^n \times [\mu_i, \infty)) \end{aligned}$$

and

$$\begin{aligned} LZ(i) &:= L \cap Z(i) \\ LY(i) &:= L \cap Y(i). \end{aligned}$$

Note that $Z(i) \cup Y(i) = N(i)$.

At this point we are focussing on the computation of $H^*(N(i), L(i))$ for a fixed i. Therefore, to simplify the notation we shall, for the most part, ignore the indexing argument i, i.e. $\mathbf{B} = \mathbf{B}(i)$, $Z = Z(i)$, $P = P_i$, $c = c_i$, etc. This simplification is not always possible since both $\mathbf{T}(i)$ and $\mathbf{T}(i+1)$ are involved in the definition of $N(i)$, so we will include the index i when it might otherwise be ambiguous. With this in mind the following sequence needs to be understood:

$$\ldots \to H^*(N(i), L(i)) \xrightarrow{\varrho^*} H^*(Z, LZ) \oplus H^*(Y, LY) \xrightarrow{\kappa^*} H^*(Z \cap Y, LZ \cap LY) \to \ldots \tag{4.2}$$

Observe that $(Z \cap Y, LZ \cap LY) = (\mathbf{B}_\mu, \mathbf{B}_\mu^-)$. Thus, to understand (4.2), we need to compute $H^*(Z, LZ)$, $H^*(Y, LY)$ and the maps ϱ^* and κ^* which are induced homomorphisms from natural inclusion maps. We begin with the computation of $H^*(Z, LZ)$ which, yet again, is done using a set of Mayer-Vietoris arguments.

First we establish some notation. Let

$$\begin{aligned} TZ &:= cl\,(Z \setminus \mathbf{B}) \\ TZ^- &:= (\mathbf{T}^-(i) \cup \mathbf{T}^-(i+1)) \cap TZ \end{aligned}$$

and

$$\begin{aligned} BZ &:= \mathbf{B} \cap Z \\ BZ^- &:= \mathbf{B}^- \cap BZ. \end{aligned}$$

Finally, set

$$\begin{aligned} \widehat{L}Z &:= \widehat{L} \cap Z \\ LZB &:= LZ \cap \mathbf{B}. \end{aligned}$$

The definitions of TY, TY^-, BY, BY^-, $\widehat{L}Y$ and LYB are similar.

Observe that $TZ \cap BZ = \mathbf{T}_c$ and that $\widehat{L}Z \subset \mathbf{T}_c^-$. This leads to the following Mayer-Vietoris sequence.

$$\ldots \to H^*(Z, LZ) \to H^*(TZ, TZ^-) \oplus H^*(BZ, LZB) \to H^*(\mathbf{T}_c, \mathbf{T}_c^-) \to \ldots \tag{4.3}$$

Also observe that there is a flow defined homotopy equivalence between (BZ, LZB) and $(BZ, BZ^- \cup \widehat{L}Z)$. Thus, (4.3) can be reduced to

$$\ldots \to H^*(Z, LZ) \xrightarrow{\alpha^*} H^*(TZ, TZ^-) \oplus H^*(BZ, BZ^- \cup \widehat{L}Z) \xrightarrow{\beta^*} H^*(\mathbf{T}_c, \mathbf{T}_c^-) \to \ldots$$

Recall from Corollary 2.8 that the inclusion induced map $j_c^* : H^*(TZ, TZ^-) \to H^*(\mathbf{T}_c, \mathbf{T}_c^-)$ is an isomorphism. Therefore, β^* is an epimorphism and α^* is a monomorphism. This and similar arguments in the other settings lead to the following result.

Proposition 4.4 *For the inclusion maps $\alpha_z : (BZ, BZ^- \cup \widehat{L}Z) \hookrightarrow (Z, LZ)$ and $\alpha_y : (BY, BY^- \cup \widehat{L}Y) \hookrightarrow (Y, LY)$, the induced homomorphisms*

$$\begin{aligned} \alpha_z^* &: H^*(Z, LZ) \to H^*(BZ, BZ^- \cup \widehat{L}Z) \\ \alpha_y^* &: H^*(Y, LY) \to H^*(BY, BY^- \cup \widehat{L}Y) \end{aligned}$$

are in fact isomorphisms.

Proof. Since β^* is an epimorphism, α^* is a monomorphism. Therefore $H^*(Z, LZ)$ is isomorphic to $\mathrm{Im}\ \alpha^* = \mathrm{Ker}\ \beta^*$ which is isomorphic to $H^*(BZ, BZ^- \cup \widehat{L}Z)$, because

$H^*(TZ, TZ^-)$ and $H^*(\mathbf{T}_c, \mathbf{T}_c^-)$ are isomorphic. The claim follows since α_z^* is the composition of α^* and the projection to the second component of $H^*(TZ, TZ^-) \oplus H^*(BZ, BZ^- \cup \widehat{L}Z)$. The same argument shows that α_y^* is an isomorphism. $\square$

Via the isomorphism α_z^*, we have reduced the problem of determining $H^*(Z, LZ)$ to the computation of $H^*(BZ, BZ^- \cup \widehat{L}Z)$. With this in mind consider the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & & \\
{} \arrow[r] & H^*(BZ, BZ^- \cup \widehat{L}Z) \arrow[r, "\nu_z^*"] & H^*(BZ, BZ^-) \oplus H^*(\widehat{L}Z, \widehat{L}Z) \arrow[r] \arrow[dl, "j_c^*"] & H^*(\widehat{L}Z, BZ^- \cap \widehat{L}Z) \arrow[r] \arrow[d, equal] & {} \\
 & H^*(\mathbf{B}_c, \mathbf{B}_c^-) \arrow[d, "\Phi_c^*"] & & H^*(\widehat{L}_c, \mathbf{B}_c^- \cap \widehat{L}_c) \arrow[d, equal] & \\
 & \bigoplus_{p=1}^{P} H^*(\mathbf{V}_c(p), \mathbf{V}_c^-(p)) \arrow[d, equal] & & \bigoplus_{\substack{out=c \\ p>1}} H^*\left(W^u_{\mathbf{B}_c}(M_{out}(p)), \mathbf{B}_c^- \cap W^u_{\mathbf{B}_c}(M_{out}(p))\right) \arrow[d, "\oplus_{p=1}^{P} c^*_{\mathbf{V}_c(p)}"] & \\
 & \bigoplus_{p=1}^{P} CH^*(M_c(p)) \arrow[rr, "proj"'] & & \bigoplus_{\substack{out=c \\ p>1}} CH^*(M_{out}(p)) &
\end{tikzcd}$$

Observe that along the top we have a Mayer-Vietoris sequence. Furthermore, Φ_c^* is an index isomorphism, j_c^* is a continuation isomorphism, and $c^*_{\mathbf{V}_c(p)}$ is the Churchill isomorphism of Lemma 2.5. Therefore, the diagram commutes.

Since *proj* is an epimorphism and all the vertical arrows represent isomorphisms, ν_z^* is a monomorphism. Thus, by the same reasoning as in the proof of Proposition 4.4, we have the following proposition.

Proposition 4.5 *The maps*

$$\Phi_c^* \circ j_c^* \circ \nu_z^* : H^*(BZ, BZ^- \cup \widehat{L}Z) \to CH^*(M_c(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_{in}(p))$$

and

$$\Phi_d^* \circ j_d^* \circ \nu_y^* : H^*(BY, BY^- \cup \widehat{L}Y) \to CH^*(M_d(1)) \oplus \bigoplus_{\substack{in=d \\ p>1}} CH^*(M_{in}(p))$$

are isomorphisms.

We now turn to our attention to the map κ^* in (4.2). From the definition of a Mayer-Vietoris sequence

$$\kappa^* : H^*(Z, LZ) \oplus H^*(Y, LY) \to H^*(\mathbf{B}_\mu, \mathbf{B}_\mu^-)$$

is given by $\kappa^* = \left(\kappa_z^*, -\kappa_y^*\right)$, where

$$\kappa_z^* : H^*(Z, LZ) \to H^*(\mathbf{B}_\mu, \mathbf{B}_\mu^-) \quad \text{and} \quad \kappa_y^* : H^*(Y, LY) \to H^*(\mathbf{B}_\mu, \mathbf{B}_\mu^-).$$

Furthermore, κ_z^* and κ_y^* are induced by the inclusion maps κ_z and κ_y, respectively. We may view these maps as being induced by the following inclusion maps

$$\alpha_z \circ \nu_z \circ j_{\mu,z} : (\mathbf{B}_\mu, \mathbf{B}_\mu^-) \hookrightarrow (BZ, BZ^-) \hookrightarrow (BZ, BZ^- \cup \widehat{L}Z) \hookrightarrow (Z, LZ),$$

and

$$\alpha_y \circ \nu_y \circ j_{\mu,y} : (\mathbf{B}_\mu, \mathbf{B}_\mu^-) \hookrightarrow (BY, BY^-) \hookrightarrow (BY, BY^- \cup \widehat{L}Y) \hookrightarrow (Y, LY),$$

respectively. Therefore, $\kappa_z^* = j_{\mu,z}^* \circ \nu_z^* \circ \alpha_z^*$ and $\kappa_y^* = j_{\mu,y}^* \circ \nu_y^* \circ \alpha_y^*$.

Combining Propositions 4.4 and 4.5 and making the appropriate identifications, the exact sequence (4.2) reduces to

$$\begin{aligned}
&\ldots \to H^*(N(i), L(i)) \\
&\quad \xrightarrow{\tilde{\delta}} \left[CH^*(M_c(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_{in}(p))\right] \oplus \left[CH^*(M_d(1)) \oplus \bigoplus_{\substack{in=d \\ p>1}} CH^*(M_{in}(p))\right] \\
&\quad \xrightarrow{\tilde{\kappa}} CH^*(M_\mu(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_\mu(p)) \oplus \bigoplus_{\substack{in=d \\ p>1}} CH^*(M_\mu(p)) \to \ldots
\end{aligned} \tag{4.4}$$

In this exact sequence $\tilde{\kappa} = (\tilde{\kappa}_z, -\tilde{\kappa}_y)$ is given by

$$\tilde{\kappa}_z = \Phi_\mu^* \circ \kappa_z^* \circ (\alpha_z^*)^{-1} \circ (\nu_z^*)^{-1} \circ (j_c^*)^{-1} \circ (\Phi_c^*)^{-1}$$

and

$$\tilde{\kappa}_y = \Phi_\mu^* \circ \kappa_y^* \circ (\alpha_y^*)^{-1} \circ (\nu_y^*)^{-1} \circ (j_d^*)^{-1} \circ (\Phi_d^*)^{-1}.$$

Notice that $(\nu_z^*)^{-1}$ is well-defined if $\tilde{\kappa}_z$ is restricted to $CH^*(M_c(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_{in}(p))$. Using the above representation of κ_z^*, we have

$$\tilde{\kappa}_z = \Phi_\mu^* \circ j_{\mu,z}^* \circ (j_c^*)^{-1} \circ (\Phi_c^*)^{-1},$$

which is nothing but the topological transition matrix $T^{\mu c}$ restricted to $CH^*(M_c(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_{in}(p))$. Since the topological transition matrix is lower triangular, its restriction $\tilde{\kappa}_z$ is a monomorphism whose image is

$$CH^*(M_\mu(1)) \oplus \bigoplus_{\substack{in=c \\ p>1}} CH^*(M_\mu(p)).$$

Similarly, $\tilde{\kappa}_y$ is also a restriction of the transition matrix $T^{\mu d}$, and hence it is a monomorphism whose image is

$$CH^*(M_\mu(1)) \oplus \bigoplus_{\substack{in=d \\ p>1}} CH^*(M_\mu(p)).$$

Furthermore, both $\tilde{\kappa}_z$ restricted to $CH^*(M_c(1))$ and $\tilde{\kappa}_y$ restricted to $CH^*(M_d(1))$ are identity matrices given the generators defined by the natural Morse continuation identifications. Therefore the map $\tilde{\kappa} = (\tilde{\kappa}_z, -\tilde{\kappa}_y)$ is an epimorphism, and hence, in view of the exact sequence (4.4), $\tilde{\varrho}$ is a monomorphism. By the same argument as in the proof of Propositions 4.4 and 4.5, we obtain

$$H^*(N(i), L(i)) \cong \text{Im } \tilde{\varrho} \cong \text{Ker } \tilde{\kappa},$$

which is isomorphic to the diagonal $\Delta \subset CH^*(M_c(1)) \oplus CH^*(M_d(1))$. This is also isomorphic to $CH^*(M_{out}(1))$ by the projection $\pi : CH^*(M_c(1)) \oplus CH^*(M_d(1)) \to CH^*(M_{out}(1))$ applied to Δ, where $out = d$ if $\delta(1) > 0$, otherwise $out = c$. The isomorphism $\psi(i) : H^*(N(i), L(i)) \cong CH^*(M_{out}(1))$ is thus given by

$$\psi(i) = \pi \circ \tilde{\varrho}_{out} = \pi \circ \Phi^*_{out}(i) \circ \theta^*_{out}(i)$$

where $\theta_{out}(i) : (\mathbf{B}_{out}(i), \mathbf{B}^-_{out}(i)) \hookrightarrow (N(i), L(i))$ is the inclusion map. Up to the natural tube continuation identification, the map $\psi(i)$ is in fact induced from the inclusion

$$\xi_{out}(i) : (\mathbf{T}_{s_i}(i), \mathbf{T}^-_{s_i}(i)) \hookrightarrow (N(i), L(i)).$$

Therefore, we have shown

Proposition 4.6 *For each i,*

$$\psi(i) : H^*(N(i), L(i)) \cong CH^*(M_{out}(1, i)).$$

The isomorphism $\psi(i)$ is the inclusion-induced map $\xi^*_{out}(i)$ up to the natural tube continuation identification.

Recall the original Mayer-Vietoris sequence (4.1):

$$\begin{aligned} \ldots \to H^*(N(i, i+1), L(i, i+1)) \;&\xrightarrow{\iota^*}\; H^*(N(i+1), L(i+1)) \oplus H^*(N(i), L(i)) \\ &\xrightarrow{\eta^*}\; CH^*(M_{in}(P_i, i)) \to \ldots \end{aligned}$$

Here the maps $\iota^* = \iota_1^* \oplus \iota_2^*$ and $\eta^* = (\eta_1^*, -\eta_2^*)$ are induced from inclusion maps as follows:

$$\begin{aligned} \iota_1 &: (N(i+1), L(i+1)) \hookrightarrow (N(i, i+1), L(i, i+1)), \\ \iota_2 &: (N(i), L(i)) \hookrightarrow (N(i, i+1), L(i, i+1)), \\ \eta_1 &: (\mathbf{T}_{s_{i+1}}(i+1), \mathbf{T}^-_{s_{i+1}}(i+1)) \hookrightarrow (N(i+1), L(i+1)), \\ \eta_2 &: (\mathbf{T}_{s_{i+1}}(i+1), \mathbf{T}^-_{s_{i+1}}(i+1)) \hookrightarrow (N(i), L(i)). \end{aligned}$$

Note that $\eta_1^* = \xi^*_{out}(i+1) = \psi(i+1)$. Using Proposition 4.6 and natural Morse and tube continuation identifications (see Remarks 2.10 and 2.11), this sequence can be rewritten as

$$\begin{aligned} \ldots \to\; & H^*(N(i, i+1), L(i, i+1)) \\ &\xrightarrow{\tilde{\iota}} CH^*(M_{in}(P_i, i)) \oplus CH^*(M_{out}(1, i)) \\ &\xrightarrow{\tilde{\eta}} CH^*(M_{in}(P_i, i)) \to \ldots \end{aligned} \tag{4.5}$$

where

$$\tilde{\iota} = \tilde{\iota}_1 \oplus \tilde{\iota}_2 \quad \text{and} \quad \tilde{\eta} = (\tilde{\eta}_1, -\tilde{\eta}_2)$$

are given by

$$\begin{aligned}
\tilde{\iota}_1 &= \psi(i+1) \circ \iota_1^*, \\
\tilde{\iota}_2 &= \psi(i) \circ \iota_2^*, \\
\tilde{\eta}_1 &= \eta_1^* \circ (\psi(i+1))^{-1} = \mathrm{id}, \\
\tilde{\eta}_2 &= \eta_2^* \circ (\psi(i))^{-1}.
\end{aligned}$$

This implies that $\tilde{\eta}$ is an epimorphism and thus $\tilde{\iota}$ is a monomorphism. In particular, we obtain the following result.

Proposition 4.7

$$\tilde{\iota}_2 : H^*(N(i,i+1), L(i,i+1)) \to CH^*(M_{out}(1,i))$$

is an isomorphism. Furthermore $\tilde{\iota}_2 = \psi(i) \circ \iota_2^*$ *where* $\iota_2 : (N(i), L(i)) \hookrightarrow (N(i,i+1), L(i,i+1))$ *is the inclusion map.*

Proof of Proposition 4.1. The proof is by induction. Observe that for $k = 2$, the result follows from Proposition 4.7 with $i = 1$. Therefore, we make the following induction assumption:

$$\Psi(k) = \psi(1) \circ l^*(k) : H^*(N(1,k), L(1,k)) \to CH^*(M_{out}(1,1)) \tag{4.6}$$

is an isomorphism, where $l(k) : (N(1), L(1)) \hookrightarrow (N(1,k), L(1,k))$ is the inclusion.

Consider now the following Mayer-Vietoris sequence,

$$\begin{aligned}
\ldots \to H^*(N(1,k+1), L(1,k+1)) & \\
\xrightarrow{\iota^*} H^*(N(k+1), L(k+1)) \oplus H^*(N(1,k), L(1,k)) & \\
\xrightarrow{\eta^*} H^*(N(k+1) \cap N(1,k), L(k+1) \cap L(1,k)) \to \ldots &
\end{aligned} \tag{4.7}$$

Proceeding by the same arguments that follow the Mayer-Vietoris sequence (4.1), we can replace (4.7) with

$$\begin{aligned}
\ldots \to H^*(N(1,k+1), L(1,k+1)) &\to CH^*(M_{in}(P_k,k)) \oplus H^*(N(1,k), L(1,k)) \\
&\to CH^*(M_{in}(P_k,k)) \to \ldots
\end{aligned}$$

Now, using the induction step, this becomes

$$\begin{aligned}
\to H^*(N(1,k+1), L(1,k+1)) &\xrightarrow{\tilde{\iota}} CH^*(M_{in}(P_k,k)) \oplus CH^*(M_{out}(1,1)) \\
&\xrightarrow{\tilde{\eta}} CH^*(M_{in}(P_k,k)) \to .
\end{aligned} \tag{4.8}$$

As before, $\tilde{\eta}$ is an epimorphism and $\tilde{\iota}$ is a monomorphism, and hence $H^*(N(1,k+1), L(1,k+1))$ is isomorphic to Ker $\tilde{\eta}$, which is in fact $CH^*(M_{out}(1,1))$. The isomorphism is given by the second component of $\tilde{\iota}$ which is the composition of $\psi(1) \circ l^*(k)$ followed by the second component of ι^*. Since the second component of ι is the inclusion $(N(1,k), L(1,k)) \hookrightarrow$

$(N(1,k+1), L(1,k+1))$, the desired isomorphism is $\psi(1) \circ l^*(k+1) = \Psi(k+1)$. Therefore, the result follows. $\square$

Finally we relate the isomorphism $\Psi(k) : H^*(N(1,k), L(1,k)) \to CH^*(M_{out}(1,1))$ to the map $\Theta(k)$ introduced in (1.8).

Proposition 4.8 *Let $\xi_{in}(k) : (\mathbf{T}_{s_{k+1}}(k+1), \mathbf{T}^-_{s_{k+1}}(k+1)) \hookrightarrow (N(1,k), L(1,k))$ be the inclusion. Then*

$$\xi^*_{in}(k) : H^*(N(1,k), L(1,k)) \to CH^*(M_{in}(P_k,k))$$

is equal to the composition map

$$\Theta(k) \circ \Psi(k) : H^*(N(1,k), L(1,k)) \to CH_{out}(1,1) \to CH^*(M_{in}(P_k,k)),$$

where $\Psi(k)$ is defined in (4.6).

Proof. The proof is again by induction. Recall $\Theta(k)$ is the composition

$$\Theta(k) = D^k(P_k,1) \circ D^{k-1}(P_{k-1},1) \circ \ldots \circ D^2(P_2,1) \circ D^1(P_1,1)$$

where

$$D^i = D^i(P_i,1) : CH^*(M_{out}(1,i)) \to CH^*(M_{in}(P_i,i))$$

is the $(P_i,1)$-entry of the directional transition matrix. The case $k=2$ follows from the commutative diagram:

$$\begin{tikzcd}
H^*(N(1,2),L(1,2)) \arrow[r, "l^*(1)"] \arrow[d] & H^*(N(1),L(1)) \arrow[d, "\psi(1)"] \arrow[dl] \\
CH^*(M_{in}(P_1,1)) & CH^*(M_{out}(1,1)) \arrow[l, "D^1"']
\end{tikzcd}$$

where arrows without name are inclusion-induced maps. Supposing that the claim is true up to k, we consider the commutative diagram:

$$\begin{tikzcd}
H^*(N(1,k+1),L(1,k+1)) \arrow[r] \arrow[ddd] & H^*(N(1,k),L(1,k)) \arrow[d] \arrow[dr, "\Psi(k)"] & \\
& CH^*(M_{in}(P_k,k)) \arrow[d, equal] & CH^*(M_{out}(1,1)) \arrow[l, "\Theta(k)"'] \\
& & \\
CH^*(M_{in}(P_{k+1},k+1)) & CH^*(M_{out}(1,k+1)) \arrow[l, "D^{k+1}"'] &
\end{tikzcd}$$

where all arrows without name are inclusion-induced maps. This clearly shows the assertion, since $D^{k+1} \circ \Theta(k) = \Theta(k+1)$. $\square$

5 Computation of the Index

The exact sequences of Section 4 are true in complete generality. In this section we will specialize and prove Theorems 1.6 and 1.10.

Proof of Theorem 1.6. We begin with the proof that

$$\mathcal{N} = \bigcup_{i=1}^{I} \mathcal{B}(i) \cup \bigcup_{i=1}^{I} \mathcal{T}(i)$$

is an isolating neighborhood for φ^{ε}. By Lemma 3.2.1, N is an isolating neighborhood for φ^{ε}. Observe that $\mathrm{Inv}(N, \varphi^0) = \mathrm{Inv}(\mathcal{N}, \varphi^0)$. Let $x \in cl(\mathcal{N} \setminus N)$. Then, $x \notin \mathrm{Inv}(\mathcal{N}, \varphi^0)$, and hence, leaves $\mathcal{N}$ in a finite time t_x under φ^0. Since $cl(\mathcal{N} \setminus N)$ is compact, there is a uniform $T > 0$ such that $|t_x| < T$ for all $x \in cl(\mathcal{N} \setminus N)$. Therefore, for sufficiently small $\varepsilon > 0$, $x \notin \mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon})$. This implies that $\mathcal{N}$ is a singular isolating neighborhood, and furthermore, that $\mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon}) = \mathrm{Inv}(N, \varphi^{\varepsilon})$.

Since the Conley index of an isolated invariant set is independent of the isolating neighborhood used to compute it, the fact that $\mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon}) = \mathrm{Inv}(N, \varphi^{\varepsilon})$ implies that

$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^{\varepsilon}), \mathbb{Z}_2)) \cong CH^*(\mathrm{Inv}(N, \varphi^{\varepsilon}), \mathbb{Z}_2)).$$

Since (N, L) is a singular index pair it is sufficient to compute $H^*(N, L)$. Now recall that $\mathcal{T}(1) = \mathcal{T}(I+1)$. Thus, we can, without loss of generality, assume that $\mathbf{T}(1) = \mathbf{T}(I+1)$. At the same time, however, let us assume that s_1 and s_{I+1} have been chosen such that $N(I) \cap N(1) \cap \mathbf{T}(1) = \emptyset$. In this case there exists $a_1 < q_1 < q_2 < b_1$ such that

$$N \setminus N(1, I) = \bigcup_{q_1 < \lambda < q_2} \mathbf{T}_{\lambda}(1).$$

Let

$$Q := \bigcup_{q_1 \le \lambda \le q_2} \mathbf{T}_{\lambda}(1) \quad \text{and} \quad Q^- := \bigcup_{q_1 \le \lambda \le q_2} \mathbf{T}_{\lambda}^-(1)$$

This sets up yet another Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to H^*(N, L) &\to H^*(Q, Q^-) \oplus H^*(N(1, I), L(1, I)) \\ &\to H^*(\mathbf{T}_{q_1}(1), \mathbf{T}_{q_1}^-(1)) \oplus H^*(\mathbf{T}_{q_2}(1), \mathbf{T}_{q_2}^-(1)) \to \ldots \end{aligned} \tag{5.1}$$

By Corollaries 2.8 and 2.9

$$H^j(Q, Q^-; \mathbb{Z}_2) \cong H^j(\mathbf{T}_{q_i}(1), \mathbf{T}_{q_i}^-(1); \mathbb{Z}_2) \cong CH^j(M_{out}(1,1); \mathbb{Z}_2) \cong CH^j(M_{in}(P_I, I); \mathbb{Z}_2).$$

By Proposition 4.1

$$H^j(N(1, I), L(1, I)) \cong CH^j(M_{out}(1,1); \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, using the isomorphisms and applying Corollary 2.9 to $H^j(\mathbf{T}_{q_i}(1), \mathbf{T}_{q_i}^-(1); \mathbb{Z}_2)$, (5.1) reduces to

$$\begin{aligned} 0 \;\to\; & H^k(N, L; \mathbb{Z}_2) \xrightarrow{W} H^k(Q, Q^-; \mathbb{Z}_2) \oplus H^k(N(1, I), L(1, I); \mathbb{Z}_2) \\ \xrightarrow{X}\; & CH^k(M_{in}(P_I, I); \mathbb{Z}_2) \oplus CH^k(M_{out}(1,1); \mathbb{Z}_2) \xrightarrow{\delta^*} H^{k+1}(N, L; \mathbb{Z}_2) \to 0 \end{aligned} \tag{5.2}$$

To determine the map X observe that Corollary 2.9 implies that the matrix representing

$$H^k(Q,Q^-;\mathbb{Z}_2) \to CH^k(M_{in}(P_I,I);\mathbb{Z}_2) \oplus CH^k(M_{out}(1,1);\mathbb{Z}_2)$$

is the transpose of $(1,1)$. Similarly, Proposition 4.1 implies that

$$H^k(N(1,I),L(1,I);\mathbb{Z}_2) \to CH^k(M_{out}(1,1);\mathbb{Z}_2)$$

is the isomorphism $\Psi(I)$. Therefore, we only need to consider

$$H^k(N(1,I),L(1,I);\mathbb{Z}_2) \cong CH^k(M_{out}(1,1);\mathbb{Z}_2) \to CH^k(M_{in}(P_I,I);\mathbb{Z}_2).$$

However, from Proposition 4.8, this is clearly $\Theta \circ \Psi(I)$. Since X is an isomorphism if and only if $\Theta = 0$, the result follows. □

Remark 5.1 The hypotheses of Theorem 1.6 are rather restricitive compared with those of Theorem 1.10 which will be proven next. This deserves a brief comment. First, it should be obvious that all the information needed to compute $H^*(N,L)$ is contained in the sequence (5.1) and that X is completely determined by the directional transition matrices. However, in the final analysis the usefulness of Theorem 1.6 is via Corollary 1.8. In particular, one needs a theorem which translates the Conley index information into useful information concerning the dynamics. The crucial result [10, Theorem 1.3] allows for more general Conley indices than those computed by Theorem 1.6. Unfortunately, it is not clear what are the most general form of indices of Morse sets and conditions on the directional transition matrices that lead to the hypothesis of [10, Theorem 1.3]. A more important question, however, is the need for further theorems which translate Conley index information into structural information about the dynamics.

Proof of Theorem 1.10.

Proof of (1): The proof that $\mathcal{N}$ is a singular isolating neighborhood proceeds as in the proof of Theorem 1.6.

Proof of (2): From Definitions 1.1(2), 1.2(2), and 1.2(3), it follows that

$$\mathrm{Inv}(N(1,I),\varphi^\varepsilon) = \emptyset$$

for all $\varepsilon > 0$ sufficiently small. Furthermore, from Definitions 1.1(2) and 1.3(2), it follows that, under the flow φ^ε for any $\varepsilon > 0$, points in $\mathrm{Inv}(N,\varphi^\varepsilon)$ which leave $\mathbf{C}(R)$ flow into $\mathbf{B}(I)$ and points leave $\mathbf{B}(1)$ into $\mathbf{C}(A)$. Hence $(\mathrm{Inv}(\mathbf{C}(R),\varphi^\varepsilon), \mathrm{Inv}(\mathbf{C}(A),\varphi^\varepsilon))$ is an attractor-repeller decomposition of $\mathrm{Inv}(N,\varphi^\varepsilon)$ for all $\varepsilon > 0$.

Proof of (3): Let $N_A := N(1,I) \cup \mathbf{C}(A)$ and $L_A := L(1,I) \cup \mathbf{C}^-(A)$. This leads to the following Mayer-Vietoris sequence

$$\begin{array}{rcl} \ldots \to H^*(N_A,L_A) & \to & H^*(\mathbf{C}(A),\mathbf{C}^-(A)) \oplus H^*(N(1,I),L(1,I)) \\ & \to & H^*(\mathbf{C}(A)\cap N(1,I), \mathbf{C}^-(A) \cap L(1,I)) \to \ldots \end{array}$$

It is left to the reader to check that this implies that

$$H^*(N_A,L_A) \cong H^*(N(1,I),L(1,I)).$$

The final Mayer-Vietoris sequence that we need to consider is

$$\begin{aligned} \ldots \to\ & H^*(N, L) \\ & \to H^*(\mathbf{C}(R), \mathbf{C}^-(R) \cup W^u_{\mathbf{C}(R)}(S_*(R))) \oplus H^*(N_A, L_A) \\ & \to H^*(\mathbf{C}(R) \cap N_A, (\mathbf{C}^-(R) \cup W^u_{\mathbf{C}(R)}(S_*(R))) \cap L_A) \to \ldots \end{aligned} \tag{5.3}$$

By Corollary 2.9 and Proposition 2.6, $H^*(\mathbf{C}(R), \mathbf{C}^-(R) \cup W^u_{\mathbf{C}(R)}(S_*(R))) = 0$. Obviously,

$$H^*(\mathbf{C}(R) \cap N_A, (\mathbf{C}^-(R) \cup W^u_{\mathbf{C}(R)}(S_*(R))) \cap L_A) \cong CH^*(M_{in}(P_I, I)).$$

Therefore, by Proposition 4.1, (5.3) reduces to

$$\ldots \to H^*(N, L) \to CH^*(M_{out}(1,1)) \xrightarrow{X} CH^*(M_{in}(P_I, I)) \to \ldots$$

From Proposition 4.8, the map X is equal to Θ. However, $CH^*(\mathrm{Inv}(\mathbf{C}(A), \varphi^\varepsilon)) \cong CH^*(M_{out}(1,1))$, $CH^{*+1}(\mathrm{Inv}(\mathbf{C}(R), \varphi^\varepsilon)) \cong CH^*(M_{in}(P_I, I))$ and $CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) \cong H^*(N, L)$. Therefore, the necessary condition for

$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) \cong CH^*(\mathrm{Inv}(\mathbf{C}(A), \varphi^\varepsilon)) \oplus CH^*(\mathrm{Inv}(\mathbf{C}(R), \varphi^\varepsilon))$$

is $X = \Theta = 0$. Indeed, one can show the following lemma:

Lemma 5.2 *For an exact sequence of finitely generated Abelian groups:*

$$\xrightarrow{X} C \xrightarrow{\alpha} A \xrightarrow{\beta} B \to,$$

if $X \neq 0$, then A cannot be isomorphic to the direct sum $B \oplus C$.

Proof. Let $T(A)$ and $F(A)$ denote the torsion and free part of A, respectively. Since $0 \to T(A) \to A \to F(A) \cong A/T(A) \to 0$ is an exact sequence, one has the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T(C) & \xrightarrow{\alpha'} & T(A) & \xrightarrow{\beta'} & T(B) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(C) & \xrightarrow{\bar{\alpha}} & F(A) & \xrightarrow{\bar{\beta}} & F(B) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the vertical arrows are exact, whereas the horizontal arrows are not in general exact but are merely chain maps, hence one has the homology long exact sequence as follows:

$$\begin{aligned} 0 \to \mathrm{Ker}\ \alpha' \to \mathrm{Ker}\ \alpha \to \mathrm{Ker}\ \bar{\alpha} & \\ \to \mathrm{Ker}\ \beta'/\mathrm{Im}\ \alpha' \to \mathrm{Ker}\ \beta/\mathrm{Im}\ \alpha \to \mathrm{Ker}\ \bar{\beta}/\mathrm{Im}\ \bar{\alpha} & \\ \to \mathrm{Coker}\ \beta' \to \mathrm{Coker}\ \beta \to \mathrm{Coker}\ \beta \to 0. & \end{aligned}$$

Suppose $A \cong B \oplus C$ were true, then $F(A) \cong F(B) \oplus F(C)$, which implies Ker $\bar{\alpha} = 0$ and therefore Ker $\alpha' \cong$ Ker α. Since $C \to A \to B$ is exact, Ker $\beta/$Im $\alpha = 0$, and hence Ker $\beta'/$Im $\alpha' = 0$ from the above long exact sequence. This means that, under the assumption $A \cong B \oplus C$, the sequence $T(C) \to T(A) \to T(B)$ is exact, and therefore Ker $\beta' \cong$ Im α'. Since $T(A), T(B), T(C)$ are all finite groups, their orders $|T(A)|, |T(B)|, |T(C)|$ satisfy

$$|T(B)| \geq |\text{Im } \beta'| = |T(A)|/|\text{Ker } \beta'| = |T(A)|/|\text{Im } \alpha'|$$

and

$$|\text{Im } \alpha'| = |T(C)|/|\text{Ker } \alpha'|,$$

and hence

$$|T(A)| \leq |T(B)| \cdot |T(C)|/|\text{Ker } \alpha'|.$$

On the other hand, $A \cong B \oplus C$ implies $T(A) \cong T(B) \oplus T(C)$ as well, thus $|T(A)| = |T(B)| \cdot |T(C)|$. Combining this with the above, one obtains

$$|\text{Ker } \alpha'| \leq 1,$$

and therefore $0 = \text{Ker } \alpha' \cong \text{Ker } \alpha$. This shows the injectivity of α, and hence $X = 0$. This completes the proof of Lemma and hence of Theorem 1.10. $\square$

6 Existence of a Poincaré Section

Proof of Theorem 1.7. Since N is an isolating neighborhood, by Theorem 2.3 there is an isolating block B with $\mathrm{Inv}(B, \varphi^\varepsilon) = \mathrm{Inv}(N, \varphi^\varepsilon)$. If $\mathrm{Inv}(B, \varphi^\varepsilon)$ were empty then B^- would be the required section. We need to add something to B^- to take care of the points which stay in B.

For any time $T > 0$ there is an isolating block B' such that $B' \subset \mathrm{int}(B)$, $x \in B'$ implies $\varphi^\varepsilon([0, T], x) \subset B$, and $\mathrm{Inv}(B', \varphi^\varepsilon) = \mathrm{Inv}(N, \varphi^\varepsilon)$ (see [13]). So fix such a $T > 0$ and a B'. Let K' be a piece of the tube $\mathcal{T}_1$ with K' homeomorphic to $[0,1]^n \times (q_1, q_2)$, where $a_1 < q_1 < q_2 < b_1$. We have already used this construction in the proof of Theorem 1.6. By making $q_2 - q_1$ sufficiently small, we may assume that $x \in B' \cap K'$ implies that either $\varphi^\varepsilon([0, \infty), x) \subset B'$ or that $\varphi^\varepsilon([0, \infty), x) \cap (B \cap [0,1]^n \times \{q_1, q_2\}) \neq \emptyset$. Using the identification of K' with $[0,1]^n \times (q_1, q_2)$, let $\Pi = [0,1]^n \times \{(q_1 + q_2)/2\}$ and let

$$\Xi = \Pi \cup cl(B^- \setminus (B^- \cap K') \cup cl((B')^- \setminus ((B')^- \cap K').$$

We claim that Ξ is the required Poincaré section for B'. It is clearly closed, and it is transverse to the flow because the disjoint pieces $B^-, (B')^-$ and Π are. Finally, we must show that the forward orbit of every point in B' intersects Ξ. Clearly if x stays in B' for all forward time under φ^ε its forward orbit intersects Π. Without loss of generality, assume the direction of flow in the tube $\mathcal{T}_1$ is from q_1 to q_2. If $x \in B' \cap \setminus ((B')^- \cap K')$, then follow the orbit forward. Either the orbit stays in B' until it reaches $B' \cap ([0,1]^n \times \{q_1\})$, or it leaves via $(B')^- \subset \Xi$. If the orbit reaches $B' \cap [0,1]^n \times \{q_1\}$, then by the choice of B', the orbit must continue and intersect $\Pi \subset \Xi$ since $B' \cap [0,1]^n \times \{(q_1 + q_2)/2\} \subset B$. In either case, the forward orbit of x intersects Ξ, so Ξ is a Poincaré section for B'. $\square$

Acknowledgements. We would like to thank Akira Kono for his help for the proof of Lemma 5.2. The first author was partially supported by NSF Grant DMS-291222 and MONTS Grant (No. 219627 and 291725). The second, third and fourth authors were partially supported by NSF Grant INT-9315117 and JSPS US-Japan Cooperative Research Grant MPCR-294. The second author was partially supported by Grant-in-Aid for Scientific Research (No. 06740150, 07740150 and 08740139), Ministry of Education, Science and Culture, Japan. The third author was partially supported by NSF Grant DMS-9101412 and DMS-9505116. The fourth author was partially supported by Science and Technology Fund for Research Grants of Ryukoku University, and by Grant-in-Aid for Scientific Research(No. 07640338 and 0840143), Ministry of Education, Science and Culture, Japan.

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