THE CONLEY INDEX OVER THE CIRCLE

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September 8, 1997

*Supported by KBN, Grant 0449/P3/94/06.
†Supported by KBN, Grant 2 P03A 040 10.

Abstract

We study the Conley index over a base in the case when the base is the circle. Such an index arises in a natural way when the considered flow admits a Poincaré section. In that case the fiberwise pointed spaces over the circle generated by index pairs are semibundles, i.e. admit a special structure similar to locally trivial bundles. We define a homotopy invariant of semibundles, the monodromy class. We use the monodromy class to prove that the Conley index of the Poincaré map may be expressed in terms of the Conley index over the circle.

Classifications: Primary 34C35; Secondary 58F25, 34A34

Keywords: Conley index, Poincaré map, fiberwise pointed space

1 Introduction

Poincaré section is a frequently used tool in the study of dynamical systems. It lets one translate certain problems in the continuous dynamics of flows to their counterparts in the discrete dynamics of the associated Poincaré map. One of the advantages of such an approach is that the dimension of phase space is reduced by one. The main disadvantage lies in the fact that often very little is known about the Poincaré map beyond its existence. The development of the discrete counterpart of the Conley index (see [9, 5, 13]) raised hopes that the interaction between the dynamics of a flow with a Poincaré section and the dynamics of the Poincaré map may be fruitfully observed and applied on the level of the Conley index theory. It is fairly easy to show that the cohomological Conley index of the Poincaré map completely determines the cohomological Conley index of the flow. This is achieved via the index suspension sequence (see [4]). However the more interesting question goes in the reverse direction: Is it possible to reconstruct the Conley index for the Poincaré map from the Conley index for flow? This is because on one hand the Conley index of the Poincaré map carries a lot of useful information, in particular information about periodic orbits of the flow of arbitrary periods, on the other hand the computation of the Conley index for flows requires only local knowledge of flow and consequently is much easier than the computation of the Conley index of the Poincaré map. Unfortunately it turns out that such a reconstruction is not possible. Perhaps the only thing which may be regained from the classical Conley index for flows is the one-eigenspace of the Conley index of the Poincaré map (see [4]).

In an earlier paper by the same authors a generalization of the Conley index for flows, the Conley index over a base was introduced (see [7]). In the case of flows with a Poincaré section there is a natural choice of the base space: the circle. The aim of the present paper is to show that the Conley index over the circle carries all the information necessary to compute the Conley index of the Poincaré map. Since the computation of the Conley index over the circle requires, as in the case of the classical Conley index for flows, only the local knowledge of flow, replacing the classical Conley index by the Conley index over the circle seems to solve completely the problem.

In order to describe in more detail the contents of this paper let us recall that, in the case of semiflows, the classical index of an isolated invariant set S takes the form of the homotopy type of a pointed topological space. The space is obtained by starting with an isolating neighborhood N of S, and collapsing the exit set L of N to a point. The homotopy type of the quotient is independent of the choice of N (provided the exit set L is closed). In the case of maps, the homotopy type of the quotient is not independent of the choice of N, so one must get rid of some unnecessary information. This is achieved by applying the Leray functor as described in [5] or some its generalizations (see [6] and [13]). In the simplest setting, if f denotes the map which defines the dynamical system, then f induces a map f^* on the cohomology $H^*(N, L)$. If we let CH^* denote the generalized image of f^* mod the generalized kernel of f^* (which corresponds to applying the Leray functor to $H^*(N, L)$), then f^* induces an isomorphism χ^* on CH^*. The resulting pair (CH^*, χ^*) is independent of the choice of N, and is the discrete Conley index of S. In this paper we actually use a universal replacement of the Leray functor due to Szymczak (see [13]) which in particular enables one to study the discrete Conley index on the homotopy level.

Let N be an isolating neighborhood in a semiflow π, which admits Σ, a Poincaré section for N. Then there is an associated discrete dynamical system, namely the Poincaré map Π on Σ. If S denotes the maximal invariant set in N, then $K := S \cap \Sigma$ is an isolated invariant set for Π. To obtain the discrete Conley index of K with respect to Π from looking at the flow in N we first consider the index of S over the base S^1 as introduced in [7]. To briefly recall its definition, assume $\xi : N \to \mathrm{S}^1$ is a fixed continuous map such that the semiflow π is ξ-preserving, i.e. $\xi(\pi(x,t)) = e^{2\pi it}\xi(x)$. Let (P_1, P_2) be a regular index pair for S. Roughly speaking this means that the exit time from the closure of $P_1 \setminus P_2$ is continuous. The pair consisting of an isolating block and its exit set is an example of regular index pair. To define $h_\xi(S)$, the Conley index of S over S^1, we use the adjunction $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$, i.e. we glue P_2 to S^1 via ξ. This results in a fiberwise pointed space (see [3]). If one chooses two regular index pairs, the resulting spaces are not equivalent as fiberwise pointed spaces, but they do have the same *fiberwise deforming homotopy type* (see [7]). By definition, $h_\xi(S)$ is equal to the fiberwise deforming homotopy type of $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$. The index turns out to have all important properties of the classical Conley index, including the continuation (homotopy) property. There is an essential advantage with respect to the classical approach: $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$ is a fiberwise pointed space over S^1 and it carries an additional structure similar in spirit to a locally trivial bundle. We call this the semibundle structure. Roughly speaking, any ξ-preserving semiflow on $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$ defines the same semibundle structure. The homotopy class of the map induced by the semiflow on every fiber leads, via the universal functor defined by Szymczak (see [13]), to an invariant of fiberwise deforming homotopy types of semibundles called the *monodromy class*. All this makes the Conley index over the circle an invariant of isolating neighborhoods which is much stronger then the classical Conley index. An important manifestation of this fact is the main result of this paper, i.e. Theorem 8.3. It states that the Conley index of K with respect to the Poincaré map Π is completely determined by the Conley index of S over S^1, namely it is the monodromy class of $h_\xi(S)$.

We want to emphasize that the Conley index over the circle in principle is not more difficult to compute then the classical Conley index. Moreover, although one way to find the semibundle structure on $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$ is via the semiflow induced from the original flow, any other ξ-preserving semiflow on $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$ defines the same semibundle structure. Therefore in practice only the knowledge of the topology of a regular index pair and the map ξ is needed to compute both the index over S^1 and its monodromy class. Though not in a setting of the Conley index, computations in this spirit were successfully performed for concrete differential equations in [10, 11] and [12]. The reader familiar with [10] will notice that the ideas developed in that paper constitute a natural inspiration to write the present paper and are its main tools.

At least one important question remains unsolved. By using the extension of the classical index to the index over S^1, we are able to determine the index for the Poincaré map. However, we do not know if the correspondence is 1-1, i.e. if different indices over S^1 necessarily give different indices for the Poincaré map.

To keep things as simple as possible in the present paper we restricted our attention to ξ-preserving flows. This is a satisfactory setting for ODE's with periodic forcing but things become technically significantly more complicated if one wants to consider an autonomous differential equation with a Poincaré section. Therefore the machinery needed in that case will be presented in a separate paper.

The paper is organized as follows. Section 2 contains preliminaries, in particular a brief introduction to fiberwise pointed spaces and fiberwise deforming maps. In section 3 we introduce semibundles and study their properties. In the next section we introduce Szymczak category, the main tool to define the monodromy class of a semibundle presented in Section 5. In Section 6 we recall some basic notions and theorems concerning isolated invariant sets, index pairs and the Conley index. Section 7 is devoted to Poincaré maps in the context of our paper. The main result of the paper, Theorem 8.3, is proved in Section 8. In Section 9 we study the suspension operation in the context of the Conley index theory. A result on calculation of the index in the case of the flow determined by the suspension of a homeomorphism is presented in Section 10. That result is illustrated by examples based on Smale's horshoes in Section 11.

2 Preliminaries

We denote the sets of natural numbers by N, the set of all integers by Z and the set of all real numbers by R. For a given topological space X and $A \subset X$, we denote the closure of A by $\operatorname{cl} A$, its interior by $\operatorname{int} A$ and its boundary by $\operatorname{bd} A$. The unit circle in the complex plane will be denoted by S^1. Recall that $R \ni \tau \to e^{2\pi i \tau} \in S^1$ is a covering map.

Lemma 2.1 *Let X be a compact space and $Y \subset X$. If $f \simeq g : X \to S^1$ rel Y then there exists a continuous map $t : X \to R^+$ such that*

$$g(x) = e^{2\pi i t(x)} f(x)$$

for every $x \in X$ and there exists an $n \in \mathrm{N}$ such that $t(y) = n$ for every $y \in Y$.

Proof: Let $h : X \times [0,1] \to \mathrm{S}^1$ be such that $f = h(\cdot,0)$, $g = h(\cdot,1)$ and $h(y,t) = f(y)$ for every $y \in Y$ and $t \in [0,1]$. For $x \in X$ let $\omega_x : [0,1] \to \mathrm{R}$ be the lift of the path $f(x)^{-1}h(x,\cdot) : [0,1] \to \mathrm{S}^1$ such that $\omega_x(0) = 0$. Define

$$s : X \ni x \to \omega_x(1) \in \mathrm{R}$$

It is easy to see that s is continuous. Let n be a nonnegative integer such that $n \geq -\min\{s(x) : x \in X\}$. The map

$$t : X \ni x \to s(x) + n \in \mathrm{R}$$

satisfies the required properties. □

We briefly recall the basic concepts of the theory of fiberwise pointed spaces over a base as considered in [7] in the special case when the base is S^1. A triple $\mathbf{U} = (U, r_U, s_U)$ is called a *fiberwise pointed (topological) space over* S^1 if U is a topological space, $r_U : U \to \mathrm{S}^1$ and $s_U : \mathrm{S}^1 \to U$ are continuous maps, and $r_U \circ s_U = \mathrm{id}_{\mathrm{S}^1}$ (compare also [3]). We refer to r_U as the projection, and to s_U as the section of $\mathbf{U}$. In the sequel, we will often drop the subscript U in r_U and s_U unless it leads to confusion. For $a \in \mathrm{S}^1$ the pointed topological space $U_a := (r_U^{-1}(a), s_U(a))$ is the *fibre* at a.

Let $\mathbf{V} = (V, r_V, s_V)$ be another fiberwise pointed space over S^1. We define a *fiberwise deforming map* (shortly: an *f.d. map*) $f : \mathbf{U} \to \mathbf{V}$ as a continuous map $f : (U, s_U(\mathrm{S}^1)) \to (V, s_V(\mathrm{S}^1))$ such that

$$r_V \circ f \simeq r_U \operatorname{rel} s_U(\mathrm{S}^1).$$

Note that every f.d. map f satisfies

$$r_V \circ f \circ s_U = \mathrm{id}_{\mathrm{S}^1}.$$

One can easily verify that the identity map is an f.d. map and the composition of f.d. maps is an f.d. map. Thus fiberwise pointed spaces over S^1 together with f.d. maps constitute a category. We call it the *fiberwise deforming category* and we denote it by $\mathrm{Fib}_{\mathrm{S}^1}$ or briefly by Fib. If $F_t : \mathbf{U} \to \mathbf{V}$, $t \in [0,1]$ is a collection of f.d. maps such that the map $F : U \times [0,1] \ni (x,t) \to F_t(x) \in V$ is continuous, then both, the map F

and the collection F_t will be referred to as a *homotopy* in the category Fib. We say that two f.d. maps $f, g : \mathbf{U} \to \mathbf{V}$ are *homotopic* in Fib if there exists a homotopy $F_t : \mathbf{U} \to \mathbf{V},\ t \in [0,1]$, in Fib such that $F_0 = f, F_1 = g$. Obviously, the homotopy in the category Fib induces an equivalence relation between the f.d. maps. Taking equivalence classes of f.d. maps as a new collection of morphisms we obtain another category, which we call the *fiberwise deforming homotopy category*. We will denote it by $\mathrm{HtpFib}_{\mathrm{S}^1}$ or briefly by HtpFib. Obviously we have also the homotopy functor $\mathrm{Htp} : \mathrm{Fib} \to \mathrm{HtpFib}$ sending each f.d. map to its equivalence class. An f.d. map f will be called a *fiberwise deforming homotopy equivalence* if $\mathrm{Htp}(f)$ is an isomorphism in HtpFib.

We say that two fiberwise pointed spaces $\mathbf{U}$ and $\mathbf{V}$ over S^1 have the same *fiberwise deforming homotopy type over* S^1 if there exists a fiberwise deforming homotopy equivalence between them. The equivalence class of spaces with the same fiberwise deforming homotopy type as $\mathbf{U}$ is denoted by $[\mathbf{U}]_{\mathrm{S}^1}$.

A very special case of an f.d. map is a *fiberwise pointed map* (see [3]), i.e. an f.d. map $f : \mathbf{U} \to \mathbf{V}$ which satisfies a significantly stronger condition

$$r_V \circ f = r_U.$$

Unlike an f.d. map, for every $a \in \mathrm{S}^1$ a fibrewise pointed map $f : \mathbf{U} \to \mathbf{V}$ induces a map of fibers $f_a : U_a \to V_a$. A homotopy F in Fib such that each F_t is a fiberwise pointed map will be called *fiber preserving*. (The notion of fiber preserving homotopy leads to the notion of *fiberwise pointed homotopy type* (see [3]), which is essentially more restrictive then the fiberwise deforming homotopy type considered in this paper.) Observe that if $F_t : \mathbf{U} \to \mathbf{V}$, $t \in [0,1]$ is a fiber preserving homotopy then the restriction $(F_t)_a : U_a \to V_a$, $t \in [0,1]$ is a homotopy in Top_*.

If π is a semiflow on a topological space X and $t \in \mathrm{R}^+$ then we write $\pi_t := \pi(\cdot, t)$ for the t-translation operator of π. If $\xi : X \to \mathrm{S}^1$ is a continuous map then we say that π is *ξ-preserving* if for all $x \in X$ and all $t \in \mathrm{R}^+$

$$\xi(\pi(x,t)) = e^{2\pi i t}\xi(x).$$

3 Semibundles

A fiberwise pointed space $\mathbf{U} = (U, r, s)$ is called a *semibundle* if there exists an r-preserving semiflow $\varphi : U \times \mathrm{R}^+ \to U$ such that $\varphi(s(a), t) = s(e^{2\pi i t}a)$ for any $a \in \mathrm{S}^1$ and $t \in \mathrm{R}^+$. Such a semiflow is called a *semibundle carrier* on $\mathbf{U}$. The name semibundle is justified by the fact that if $\mathbf{U}$ admits a semibundle carrier which is a flow then $r : U \to \mathrm{S}^1$ is a locally trivial bundle. Note that the semibundle carrier is not unique in general.

The 1-translation operator or the time-one map φ_1 of the semibundle carrier φ on $\mathbf{U}$ plays a special role in the sequel. Let us denote it by Φ.

Proposition 3.1 *The time-one map $\Phi : \mathbf{U} \to \mathbf{U}$ is a fiberwise pointed map. For every $a \in \mathrm{S}^1$ the homotopy class of Φ_a in* Top_* *is independent of the choice of a semibundle carrier φ.*

Proof: The first assertion is obvious. If φ' is another semibundle carrier on $\mathbf{U}$ then the required homotopy is $\varphi'_{1-t}\varphi_t$. $\square$

In the sequel we assume that $\mathbf{U} = (U, r_U, s_V)$ and $\mathbf{V} = (V, r_V, s_V)$ are semibundles. Fix a semibundle carrier ψ on $\mathbf{V}$ and let $\Psi := \psi_1$ be the corresponding time-one map. Let $\alpha : \mathbf{U} \to \mathbf{V}$ be an f.d. map and $t : U \to \mathrm{R}^+$ be a continuous function. Define a map $\bar{\alpha} : U \to V$ by

$$\bar{\alpha}(x) := \psi(\alpha(x), t(x))$$

for $x \in U$.

Proposition 3.2 *The following conditions are equivalent:*

(a) *$\bar{\alpha}$ is an f.d. map.*

(b) *$t|_{s(\mathrm{S}^1)} = n$ for some $n \in \mathrm{N}$.*

Moreover, if (b) is satisfied then $\bar{\alpha}$ is homotopic to $\Psi^n \circ \alpha$ in Fib.

Proof: If $\bar{\alpha}$ is an f.d. map then for any $a \in \mathrm{S}^1$ we have

$$a = r_V(\bar{\alpha}(s_U(a))) = r_V(\psi(\alpha(s_U(a)), t(s_U(a)))) = e^{2\pi i t(s_U(a))}a$$

which implies that $t(s_U(a))$ is an integer. Since t is continuous and $s(\mathrm{S}^1)$ is connected, $t|_{s(\mathrm{S}^1)}$ must be an integer constant.

Assume now that $t|_{s(\mathrm{S}^1)} = n$. For $x \in U$ and $\lambda \in [0,1]$ put

$$H(x,\lambda) = \psi(\alpha(x), \lambda n + (1-\lambda)t(x))$$

One easily verifies that H is a homotopy relative to $s(\mathrm{S}^1)$ joining $\bar{\alpha}$ and $\Psi^n \circ \alpha$, hence the conclusion follows from Proposition 3.2 in [7]. $\square$

If t satisfies the condition (b) in Proposition 3.2 then we say that the map $\bar{\alpha} : U \to V$ is a *t-push-forward* of α of *rank* n or briefly a push-forward of α.

If $\bar{\alpha}$ is also a fiberwise pointed map, then it is called a t-normalization of α or briefly a normalization of α.

Theorem 3.3 *Let U be compact. Then every f.d. map $\mathbf{U} \to \mathbf{V}$ admits at least one normalization.*

Proof: Let $\alpha : \mathbf{U} \to \mathbf{V}$ be an f.d. map. Since $r_U \simeq r_V \circ \alpha \,\mathrm{rel}\; s(\mathrm{S}^1)$, then, by Lemma 2.1 (with $X = U$, $Y = s_U(\mathrm{S}^1)$, $f = r_V \circ \alpha$ and $g = r_U$), there exists a continuous map $t : U \to \mathrm{R}^+$ such that $r_U(u) = e^{2\pi i t(u)} r_V \alpha(u)$ for every $u \in U$ and there exists an $n \in \mathrm{N}$ such that $t(s_U(a)) = n$ for any $a \in \mathrm{S}^1$. The map

$$\bar{\alpha}(u) := \psi(\alpha(u), t(u))$$

has the required properties. $\square$

If $H : \mathbf{U} \times [0,1] \to V$ is a homotopy in Fib, $t : U \times [0,1] \to \mathrm{R}^+$ is a continuous map such that $t|_{s_U(\mathrm{S}^1)\times[0,1]} = n$ for some $n \in \mathrm{N}$ and

$$\bar{H} : U \times [0,1] \ni (x,\lambda) \to \psi(H(x,\lambda), t(x,\lambda)) \in V$$

is a fiber preserving homotopy then $\bar{H}$ is called a t-normalization of the homotopy H or briefly a normalization of H. Notice that if $\bar{H}$ is a normalization of H then for each $\lambda \in [0,1]$ the map $\bar{H}_\lambda := \bar{H}(\cdot,\lambda)$ is a normalization of $H_\lambda := H(\cdot,\lambda)$.

Theorem 3.4 *Let U be compact and let $H : \mathbf{U} \times [0,1] \to \mathbf{V}$ be a homotopy in the category* Fib*. Then there exists a normalization $\bar{H} : \mathbf{U} \times [0,1] \to \mathbf{V}$.*

Proof: Let $R : U \times [0,1] \to \mathrm{S}^1$ be a homotopy joining r_U and $r_V \circ H_0$, i.e. $R(\cdot,0) = r_U$, $R(\cdot,1) = r_V \circ H_0$, and $R(s_U(a),\tau) = a$ for every $a \in \mathrm{S}^1$ and $\tau \in [0,1]$. Define

$$B : U \times [0,1] \times [0,1] \to \mathrm{S}^1$$

by

$$B(u,\lambda,\tau) := \begin{cases} R(u,2\tau) & \text{if } \tau\in[0,\frac{1}{2}],\\ r_V H(u,(2\tau-1)\lambda) & \text{if } \tau\in[\frac{1}{2},1]. \end{cases}$$

It follows that B is continuous and

$$B(\cdot,\lambda,0) = r_U,$$
$$B(\cdot,\lambda,1) = r_V\circ H_\lambda,$$
$$B(s_U(\cdot),\lambda,\tau) = \mathrm{id}_{\mathrm{S}^1}$$

for every $\lambda,\tau\in[0,1]$. By Lemma 2.1 (with $X = U\times[0,1]$, $Y = s_U(\mathrm{S}^1)\times[0,1]$, $f = B(\cdot,\cdot,1)$, and $g = B(\cdot,\cdot,0)$) there exists a continuous map $t : U\times[0,1]\to \mathrm{R}^+$ such that

$$r_U(u) = e^{2\pi i t(u,\lambda)} r_V(H_\lambda(u))$$

for every $u\in U$ and $\lambda\in[0,1]$, and there exists an $n\in\mathrm{N}$ such that $t(s_U(a),\lambda) = n$ for every $\lambda\in[0,1]$ and $a\in\mathrm{S}^1$. Define

$$\bar{H}(u,\lambda) := \psi(H(u,\lambda), t(u,\lambda)).$$

It is easy to verify that $\bar{H}$ satisfies the required properties. □

One can easily check the following proposition.

Proposition 3.5 *Let U be compact and $t : U\to\mathrm{R}^+$ be a continuous function. Assume $\alpha : \mathbf{U}\to\mathbf{V}$ is a fiberwise pointed map and $\bar{\alpha}$ is the t-normalization of α. Then $t(U)$ is a finite subset of* N. *In particular, t is constant on connected components of U. If $U_0\subset U$ and $t|_{U_0} = k$ then*

$$\bar{\alpha}|_{U_0} = \Psi^k\circ\alpha|_{U_0}. \qquad \square$$

4 The Szymczak category

Let $\mathcal{E}$ be a category. The category of endomorphisms of $\mathcal{E}$, denoted by $\mathrm{Endo}(\mathcal{E})$, is defined as follows. The objects of $\mathrm{Endo}(\mathcal{E})$ are pairs (A,a), where $A\in\mathcal{E}$ and $a\in\mathcal{E}(A,A)$ is an endomorphism of A. The set of morphisms from $(A,a)\in\mathrm{Endo}(\mathcal{E})$ to $(B,b)\in\mathrm{Endo}(\mathcal{E})$ is the subset of $\mathcal{E}(A,B)$ consisting of exactly those morphisms $f\in\mathcal{E}(A,B)$ for which $b\circ f = f\circ a$. We write $f:(A,a)\to(B,b)$ to denote that f is a morphism from (A,a) to (B,b) in $\mathrm{Endo}(\mathcal{E})$. Note that if $(A,a)\in\mathrm{Endo}(\mathcal{E})$ then $a\in\mathrm{Endo}(\mathcal{E})((A,a),(A,a))$.

The definition of the Szymczak category of $\mathcal{E}$ (see [13]), denoted by $\mathrm{Szym}(\mathcal{E})$, is slightly more complicated. The objects in $\mathrm{Szym}(\mathcal{E})$ coincide with the objects in $\mathrm{Endo}(\mathcal{E})$, i.e. the pairs (A, a) such that a is an endomorphism of A. To define morphisms we first introduce the following relation for $(f, m), (g, n) \in \mathrm{Endo}(\mathcal{E})((A, a), (B, b)) \times \mathrm{N}$

$$(f, m) \equiv (g, n) \;:\Longleftrightarrow\; \exists k \in \mathrm{N}\; f \circ a^{n+k} = g \circ a^{m+k}.$$

One easily verifies that the relation is an equivalence relation and

$$(f, m) \equiv (g, n) \;:\Longleftrightarrow\; \exists k \in \mathrm{N}\; b^{n+k} \circ f = b^{m+k} \circ g.$$

The morphisms in the Szymczak category are now defined as the equivalence classes of the relation $\equiv$:

$$\mathrm{Szym}(\mathcal{E})((A, a), (B, b)) := \Big[\mathrm{Endo}(\mathcal{E})((A, a), (B, b)) \times \mathrm{N}\Big] / \equiv .$$

Let $[f, m] \in \mathrm{Szym}(\mathcal{E})((A, a), (B, b))$ and $[g, n] \in \mathrm{Szym}(\mathcal{E})((B, b), (C, c))$. It is straightforward to check that $[\mathrm{id}_A, 0]$ is the identity on (A, a) in $\mathrm{Szym}(\mathcal{E})$ and $[g \circ f, m + n]$ is a well defined morphism in $\mathrm{Szym}(\mathcal{E})((A, a), (C, c))$, which by definition is the composition of morphisms $[f, m]$ and $[g, n]$ (denoted by $[g, n] \circ [f, m]$). There is a canonical functor from $\mathrm{Endo}(\mathcal{E})$ to $\mathrm{Szym}(\mathcal{E})$ which is equal to the identity on the objects of $\mathrm{Endo}(\mathcal{E})$ and which transforms every morphism f in $\mathrm{Endo}(\mathcal{E})$ to $[f, 0]$ in $\mathrm{Szym}(\mathcal{E})$. We note for future reference the following straightforward proposition.

Proposition 4.1 *If (A, a) is an object in $\mathrm{Szym}(\mathcal{E})$ then*

(a) $[a^n, n] = [\mathrm{id}_A, 0]$ *for each $n \in \mathrm{N}$.*

(b) $[a^p, q]$ *is an isomorphism for every $p, q \in \mathrm{N}$; its inverse is equal to $[\mathrm{id}_A, p - q]$ if $p \geq q$ and $[a^{q-p}, 0]$ if $p \leq q$.*

The reason to introduce the Szymczak category is the following theorem.

Theorem 4.2 *Assume that $(A, a), (B, b) \in \mathrm{Endo}(\mathcal{E})$. Let f and g be morphisms in $\mathcal{E}$ be such that the diagram*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "b"'] \arrow[ur, "g"] & B \end{tikzcd}$$

commutes. Then $f \in \mathrm{Endo}(\mathcal{E})((A,a),(B,b))$, $g \in \mathrm{Endo}(\mathcal{E})((B,b),(A,a))$ *and we have the commutative diagram*

$$\begin{array}{ccc} (A,a) & \xrightarrow{[a,0]} & (A,a) \\ {\scriptstyle [f,0]}\downarrow & \nearrow {\scriptstyle [g,0]} & \downarrow{\scriptstyle [f,0]} \\ (B,b) & \xrightarrow[{[b,0]}]{} & (B,b) \end{array}$$

in $\mathrm{Szym}(\mathcal{E})$, *in which all morphisms are isomorphisms.*

Proof: By Proposition 4.1 $[a,0]$ and $[b,0]$ are isomorphisms. It is now a standard categorial argument to show that also $[f,0]$ and $[g,0]$ are isomorphisms. □

For $A \in \mathcal{E}$ and $a \in \mathcal{E}(A,A)$ the equivalence class

$$[A,a]_{\mathrm{Szym}} := \{\, (B,b) \mid (B,b) \text{ is isomorphic to } (A,a) \text{ in } \mathrm{Szym}(\mathcal{E}) \,\}$$

will be called the *Szymczak class* of (A,a) in $\mathcal{E}$.

5 The monodromy class

Proposition 5.1 *Assume* $\alpha : \mathbf{U} \to \mathbf{V}$ *is an f.d. map. Then* $\alpha \circ \Phi$ *and* $\Psi \circ \alpha$ *are homotopic in* Fib. *Moreover, if* α *is a fiberwise pointed map then for any* $a \in \mathrm{S}^1$ *the maps* $\alpha_a \circ \Phi_a$ *and* $\Psi_a \circ \alpha_a$ *are homotopic in* Top_*.

Proof: One easily checks that the required homotopies are

$$F : U \times [0,1] \ni (x,t) \to \psi(\alpha(\varphi(x,t)), 1-t) \in V$$

and

$$G : U_a \times [0,1] \ni (x,t) \to \psi(\alpha_{a+t}(\varphi(x,t)), 1-t) \in V_a. \qquad \square$$

Proposition 5.2 *For any* $a, b \in \mathrm{S}^1$ *the pairs* $(U_a, [\Phi_a])$, $(U_b, [\Phi_b])$ *are isomorphic in* $\mathrm{Szym}(\mathrm{HtpTop}_*)$.

Proof: Actually, the proposition is true even for the pairs (U_a, Φ_a) and (U_b, Φ_b) in Szym(Top$_*$). It is obvious if $a = b$, thus assume that $a \neq b$. Let $t \in (0,1)$ be such that $b = e^{2\pi i t}a$. Define $\Phi_{ba} : U_a \ni x \to \varphi(x,t) \in U_b$ and $\Phi_{ab} : U_b \ni x \to \varphi(x, 1-t) \in U_a$. Then $\Phi_a = \Phi_{ba} \circ \Phi_{ab}$ and $\Phi_b = \Phi_{ab} \circ \Phi_{ba}$ and the assertion follows immediately from Theorem 4.2. □

The Szymczak class of $(U_a, [\Phi_a])$ in Szym(HtpTop$_*$) does not depend on $a \in \mathrm{S}^1$. By Proposition 3.1, it is also independent of the choice of Φ as the time-one map of some r_U-preserving semiflow. It will be called the *monodromy class* of $\mathbf{U}$ and denoted by $M(\mathbf{U})$.

The crucial role in this paper is played by the following result:

Theorem 5.3 *Assume* $\mathbf{U}$, $\mathbf{V}$ *are two compact semibundles over* S^1. *If* $\mathbf{U}$ *and* $\mathbf{V}$ *have the same fiberwise deforming homotopy type then their monodromy classes are equal.*

Proof: It is enough to show that for a fixed $a \in \mathrm{S}^1$ the objects $(U_a, [\Phi_a])$ and $(V_a, [\Psi_a])$ are isomorphic in the category Szym(HtpTop$_*$). Let $\alpha : \mathbf{U} \to \mathbf{V}$ and $\beta : \mathbf{V} \to \mathbf{U}$ be mutually inverse fiberwise deforming homotopy equivalencies. Let $\bar{\alpha}$ and $\bar{\beta}$ be normalizations of α and β of rank m and n respectively. It follows from Proposition 5.1 that $[\bar{\alpha}_a]$ and $[\bar{\beta}_a]$ commute with $[\Phi_a]$ and $[\Psi_a]$. We will show that $[[\bar{\alpha}_a], 0]$ and $[[\bar{\beta}_a], 0]$ are isomorphisms in Szym(HtpTop$_*$). First observe that by Proposition 3.2 $\bar{\alpha}$ is homotopic to $\Psi^n \circ \alpha$ and $\bar{\beta}$ is homotopic to $\Phi^m \circ \beta$ in Fib, hence, by Proposition 5.1, the fiberwise pointed map $\gamma := \bar{\beta} \circ \bar{\alpha}$ is homotopic to Φ^{n+m} in Fib. Let $H : U \times [0,1] \to U$ be a homotopy in Fib such that $H_0 = \gamma$ and $H_1 = \Phi^{n+m}$. By Theorem 3.4 we can take $\bar{H} : U \times [0,1] \to U$, a t-normalization of H, where $t : U \times [0,1] \to \mathrm{R}^+$ is a continuous map. Let $t_i := t(\cdot, i)$ for $i = 0, 1$. It follows from Proposition 3.5 that there exists a partition $U_1 \cup U_2 \cup \ldots \cup U_d$ of U consisting of closed and open subsets of U such that $t_i|_{U_j} = \mathrm{const} \in \mathrm{N}$ for $i = 0, 1$, and $j = 1, 2, \ldots, d$. We assume that U_1 contains $s_U(\mathrm{S}^1)$ (which implies, for example, that $t_0|_{U_1} = t_1|_{U_1}$). One easily verifies that Φ, and consequently also γ, map each connected component of U into itself. Define $\Phi_i, \gamma_i : U_i \to U_i$ as the restrictions of Φ and, respectively, γ. Then

$$\Phi = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_d, \quad \gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_d.$$

Let $k_j^i \in \mathrm{N}$ be the value of the constant map $t_i|_{U_j}$. It follows from Proposition 3.5 that

$$\bar{H}_0|_{U_j} = \Phi_j^{k_j^0} \circ \gamma_j, \quad \bar{H}_1|_{U_j} = \Phi_j^{k_j^1 + m + n}.$$

Since $\bar{H}$ is fibre preserving, we conclude that

$$[(\Phi_j^{k_j^0})_a \circ (\gamma_j)_a] = [(\Phi_j^{k_j^1+m+n})_a], \tag{1}$$

where the homotopy classes are taken in Top_* if $j = 1$ and in Top in the other cases. Put

$$p := m + n + \max\{\, k_j^1 \mid j = 1, 2, \ldots, d\,\}.$$

Then

$$q_j := p + k_j^0 - k_j^1 - m - n \geq k_j^0.$$

Let $\gamma^* := \Phi_1^{q_1} \cup \Phi_2^{q_2} \cup \ldots \cup \Phi_d^{q_d}$. It follows from (1) that

$$\begin{aligned}
&[[\gamma_a^*], p] \circ [[\gamma_a], 0] = \\
&\quad [[(\Phi_1^{q_1-k_1^0})_a \circ (\Phi_1^{k_1^0})_a \circ (\gamma_1)_a \cup \ldots \cup (\Phi_d^{q_d-k_d^0})_a \circ (\Phi_d^{k_d^0})_a \circ (\gamma_d)_a], p] = \\
&\qquad [[(\Phi_1^p)_a \cup (\Phi_2^p)_a \cup \ldots \cup (\Phi_d^p)_a], p] = [[\Phi_a^p], p] = [[\mathrm{id}_{U_a}], 0].
\end{aligned}$$

Notice that by Proposition 5.1, we have the following sequence of homotopies in Top_*

$$\gamma_a \circ \Phi_a = \bar{\beta}_a \circ \bar{\alpha}_a \circ \Phi_a \simeq \bar{\alpha}_a \circ \Psi_a \circ \bar{\beta}_a \simeq \Phi_a \circ \bar{\beta}_a \circ \bar{\alpha}_a = \Phi_a \circ \gamma_a$$

and obviously all homotopies break down to $(U_j)_a$ so that

$$[(\gamma_j)_a \circ (\Phi_j)_a] = [(\Phi_j)_a \circ (\gamma_j)_a]$$

for $j = 1, 2, \ldots, d$. Hence we also get

$$[[\gamma_a], 0] \circ [[\gamma_a^*], p] = [[\mathrm{id}_{U_a}], 0].$$

It follows that the composition $[[\bar{\beta}_a], 0] \circ [[\bar{\alpha}_a], 0] = [[\gamma_a], 0]$ is an isomorphism in $\mathrm{Szym}(\mathrm{HtpTop}_*)$. An analogous argument shows that also $[[\bar{\alpha}_a], 0] \circ [[\bar{\beta}_a], 0]$ is an isomorphism in $\mathrm{Szym}(\mathrm{HtpTop}_*)$. Thus, by a standard categorial argument, both $[[\bar{\alpha}_a], 0]$ and $[[\bar{\beta}_a], 0]$ establish the required isomorphism. □

6 Isolating neighborhoods and isolated invariant sets

Assume X is a locally compact metric space. Let $\pi : X \times \mathrm{R} \to X$ be a flow on X. In the sequel we will usually use the group action notation $x \cdot t$ for $\pi(x, t)$.

Let $N \subset X$ be a compact set. The *invariant part* of N with respect to π is defined by

$$\mathrm{Inv}(N, \pi) := \mathrm{Inv}\, N := \{\, x \in N \mid x \cdot t \in N \text{ for all } t \in \mathrm{R} \,\}$$

N is called an *isolating neighborhood* for π if $\mathrm{Inv}\, N \subset \mathrm{int}\, N$. For an isolating neighborhood N we define the function

$$\sigma_N : N \ni x \to \sup \{\, t \geq 0 \mid x \cdot [0, t] \subset N \,\}. \tag{2}$$

Proposition 6.1 *For every $T > 0$ there exists an open neighborhood U of* $\mathrm{Inv}\, N$ *such that $x \in U$ implies $\sigma_N(x) \geq T$.*

Proof: Assume the contrary. Then there exists a sequence $\{x_n\} \subset N$ such that $x_n \to x \in \mathrm{Inv}\, N$ and $\sigma_N(x_n) =: t_n \to t \in [0, T]$. Since $x_n \cdot t_n \in \mathrm{bd}\, N$, it follows that $x \cdot t \in \mathrm{bd}\, N$, which contradicts the fact that $x \cdot t \in \mathrm{Inv}\, N \subset \mathrm{int}\, N$. $\square$

Let $S \subset X$ be compact. S is called invariant if $S = \mathrm{Inv}\, S$. S is called an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \mathrm{Inv}\, N$. The set N is then called an *isolating neighborhood* for S.

Let N be an isolating neighborhood for π. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair in* N if the following conditions are satisfied:

$$x \in P_i, t > 0, x \cdot [0, t] \subset N \;\Rightarrow\; x \cdot [0, t] \subset P_i \tag{3}$$

$$x \in P_1, t > 0, x \cdot t \notin N \;\Rightarrow\; \exists t' \in [0, t]\; x \cdot t' \in P_2, x \cdot [0, t'] \subset N \tag{4}$$

$$\mathrm{Inv}\, N \subset \mathrm{int}(P_1 \setminus P_2) \tag{5}$$

Note that it is possible to have $P_1 = N$ in the above definition. Therefore whenever $P = (P_1, P_2)$ is mentioned in the sequel without indicating what is the corresponding isolating neighborhood, this means that we consider P as an index pair in P_1.

We call the index pair P *regular* if $\sigma_{\mathrm{cl}(P_1 \setminus P_2)}$ is continuous (note that in case of a semiflow the definition is more complicated , cf [7]).

Theorem 6.2 *([7]) For every open neighborhood U of* $\operatorname{Inv} N$ *there exists a regular index pair $P = (P_1, P_2)$ in N which satisfies $P_1 \backslash P_2 \subset U$.*

Let B be a compact subset of X. Define

$$B^+ =: \{x \in B : \exists \epsilon_n > 0, \epsilon_n \to 0 : x.(-\epsilon_n) \notin B\},$$
$$B^- =: \{x \in B : \exists \epsilon_n > 0, \epsilon_n \to 0 : x.\epsilon_n \notin B\}.$$

We call B an *isolating block* for the flow π if both B^+ and B^- are compact and $\partial B = B^+ \cup B^-$.

The sets B^+ and B^- are called the entrance set and the exit set, respectively. As a consequence of the proof of Ważewski Theorem in [1, section II.2], the pair (B, B^-) consisting of an isolating block and its exit set is an example of a regular index pair.

Let $\xi : X \to \mathrm{S}^1$ be a fixed continuous map. In the sequel we assume that $\pi : X \times \mathrm{R} \to X$ is a ξ-preserving flow on X. Note that in this situation ξ is necessarily a surjection.

We recall (see [7]) that for $P = (P_1, P_2)$ a pair of closed spaces, $U_\xi(P)$ is defined as the adjunction $P_1 \cup_{\xi|_{P_2}} \mathrm{S}^1$, i.e.

$$U_\xi(P) := \mathrm{S}^1 \times 0 \cup P_1 \times 1/\sim,$$

where $\sim$ denotes the minimal equivalence relation such that $(u, 1) \sim (\xi(u), 0)$ for every $u \in P_2$. For $(u, i) \in \mathrm{S}^1 \times 0 \cup P_1 \times 1$ denote by $[u, i]$ the equivalence class of (u, i) in the relation $\sim$. One easily checks that

$$\mathbf{U}_\xi(P) := (U_\xi(P), r_{U_\xi(P)}, s_{U_\xi(P)}),$$

where

$$r_{U_\xi(P)}([x, i]) := \xi(x), \quad s_{U_\xi(P)}(a) := [a, 0],$$

is a fiberwise pointed space.

Theorem 6.3 *([7]) Let S be an isolated invariant set. Then $h_\xi(S, \pi) := [\mathbf{U}_\xi(P)]_{\mathrm{S}^1}$ (i.e. the fiberwise deforming homotopy type of $\mathbf{U}_\xi(P)$ over S^1) is independent of the choice of an isolating neighborhood N of S and of a regular index pair P in N.*

The homotopy type $h_\xi(S,\pi)$ given by the above theorem (denoted shortly by $h_\xi(S)$ if π is clear from the context) is called the *Conley index of S over base* S^1 *with base map* ξ or just the *Conley index of S over* S^1 if ξ is clear from context.

Now we briefly recall the Conley index theory in the discrete case (see [13], [5]). By a discrete dynamical system on X we mean a homeomorphism $f : X \to X$. For a compact set $N \subset X$ we define the invariant part of N (with respect to f) by

$$\operatorname{Inv}(N,f) := \operatorname{Inv} N := \{\, x \in N \mid f^n(x) \in N \text{ for all } n \in \mathrm{Z} \,\}$$

The set K is said to be invariant if $f(K) = K$. This is easily seen to be equivalent to $K = \operatorname{Inv}(K,f)$.

A closed set $N \subset X$ is called an *isolating neighborhood for f* if $\operatorname{Inv}(N,f) \subset \operatorname{int} N$.

A compact set K is called an *isolated invariant set for f*, if there exists an isolating neighborhood N such that $K = \operatorname{Inv} N$. The neighborhood N is then called an isolating neighborhood of K.

The pair $P = (P_1, P_2)$ of closed subsets of an isolating neighborhood N will be called an *index pair* of K in N (with respect to f) if the following three conditions are satisfied.

$$P_i \cap f^{-1}(N) \subset f^{-1}(P_i) \text{ for } i = 0,1, \tag{6}$$

$$P_1 \backslash P_2 \subset f^{-1}(N), \tag{7}$$

$$K \subset \operatorname{int}(P_1 \backslash P_2). \tag{8}$$

With every index pair P one associates an object $(P_*, [f_P]) \in \operatorname{Endo}(\mathrm{HtpTop}_*)$ as follows. P_* is a pointed topological space P_1/P_2 with the distinguished point $[P_2]$ being the set P_2 collapsed to a point. The morphism $[f_P]$ is the pointed homotopy class of the induced map $f_P : P_1/P_2 \to P_1/P_2$ given by

$$f_P([x]) := \begin{cases} [f(x)] & \text{if } f(x) \in P_1 \\ [P_2] & \text{otherwise.} \end{cases}$$

The following two theorems let us define the Conley index in the discrete case.

Theorem 6.4 *([8]) Every isolating neighborhood for f admits an index pair.*

Theorem 6.5 *([13]) Assume K is an isolated invariant set with respect to f. Let N be an isolating neighborhood for K and P an index pair in N. Then the Szymczak class of $(P_*, [f_P])$ in $\mathrm{Szym}(\mathrm{HtpTop}_*)$ does not depend on the particular choice of N and P.*

Now we define *the Conley index of K* by

$$h(K,f) := h(K) := [P_*, [f_P]]_{\mathrm{Szym}}$$

for an arbitrarily chosen index pair P for K.

7 Poincaré maps

Again we assume that π is a flow on a locally compact space. Let $\Pi := \pi_1$ be the time-one map of π. For $a \in \mathrm{S}^1$ we put $X_a := \xi^{-1}(a)$ and $\Pi_a : X_a \ni x \to \Pi(x) \in X_a$. Recall that the map Π_a is called the Poincaré map.

Theorem 7.1 *Any isolated invariant set S for π admits an isolating neighborhood N and a regular index pair P in N such that the pair $Q := X_a \cap P$ is an index pair in $N_a := X_a \cap N$ for Π_a.*

Proof: Let $\bar{N}$ be an isolating neighborhood for S. First observe that S admits another isolating neighborhood N such that

$$x, x\cdot 1 \in N \;\Rightarrow\; x \cdot [0,1] \subset N. \tag{9}$$

Indeed, by Proposition 6.1 there exists an open set $U'' \supset S$ such that $x \in U''$ implies $x \cdot [0,1] \subset \bar{N}$. Let U' be an open neighborhood of S such that $\mathrm{cl}\, U' \subset U''$. By Theorem 6.2 we can find a regular index pair $\bar{P}$ in $\bar{N}$ such that $\bar{P}_1 \setminus \bar{P}_2 \subset U'$. One easily verifies that $N := \mathrm{cl}(\bar{P}_1 \setminus \bar{P}_2)$ satisfies (9). Now select again an open set $U \supset S$ such that

$$x \in U \;\Rightarrow\; x \cdot [0,1] \subset N \tag{10}$$

and an index pair P in N such that $P_1 \setminus P_2 \subset U$. We will show that $Q := P \cap X_a$ is an index pair for Π_a in N_a.

Let $x \in Q_i \cap \Pi_a^{-1}(N_a)$. Then $x, x\cdot 1 \in N$ and by (9) $x \cdot [0,1] \subset N$. Since $x \in P_i$, we get $\Pi_a(x) = x \cdot 1 \in P_i \cap N_a$, hence $\Pi_a(x) \in Q_i$. This proves (6). To prove (7) assume the contrary. Then there exists an $x \in Q_1 \setminus Q_2 \setminus \Pi_a^{-1}(N_a)$. In particular $\Pi_a(x) = x \cdot 1 \notin N$. However $x \in Q_1 \setminus Q_2 = (P_1 \setminus P_2) \cap X_a \subset U$ and property (10) implies $x \cdot [0,1] \subset N$, a contradiction.

Finally we have

$$\begin{aligned}\mathrm{Inv}(N_a, \Pi_a) = \mathrm{Inv}(N, \pi) \cap X_a \subset \mathrm{int}(P_1 \setminus P_2) \cap X_a = \\ \mathrm{int}_{X_a}((P_1 \setminus P_2) \cap X_a) = \mathrm{int}_{X_a}(Q_1 \setminus Q_2),\end{aligned}$$

which proves (8). □

8 The monodromy class of the Conley index over S^1 and the Conley index of the Poincaré map

Let N be an isolating neighborhood for π and let P be a regular index pair in N. For $[x,i] \in U_\xi(P)$ and $t \in \mathrm{R}^+$ put

$$\pi^P([x,i],t) := \begin{cases} [e^{2\pi i t}x, 0] & \text{if } i = 0 \\ [e^{2\pi i t}\xi(x), 0] & \text{if } i = 1 \text{ and } t \geq \sigma_{\mathrm{cl}(P_1 \setminus P_2)}(x) \\ [\pi(x,t), 1] & \text{if } i = 1 \text{ and } t \leq \sigma_{\mathrm{cl}(P_1 \setminus P_2)}(x) \end{cases} \tag{11}$$

One easily verifies the following proposition.

Proposition 8.1 *π^P given by formula (11) is an $r_{U_\xi(P)}$-preserving semiflow on $\mathbf{U}_\xi(P)$.*

We call it *a semiflow on $\mathbf{U}_\xi(P)$ induced by π*. The semiflow turns $\mathbf{U}_\xi(P)$ into a semibundle.

Theorem 8.2 *The monodromy class of $\mathbf{U}_\xi(P)$ does not depend on the choice of a particular regular index pair P for a given isolated invariant set S.*

Proof: Let P and Q be two regular index pairs for S. By Theorem 6.3 in [7] $\mathbf{U}_\xi(P)$ and $\mathbf{U}_\xi(Q)$ have the same fiberwise deforming homotopy type. Moreover, the spaces $U_\xi(P)$ and $U_\xi(Q)$ are compact. Thus the conclusion follows from Theorem 5.3. □

The above theorem lets us introduce the notation

$$M(h_\xi(S,\pi)) := M(\mathbf{U}_\xi(P))$$

for any regular index pair of S.

Theorem 8.3 *For any isolated invariant set S with respect to π we have*

$$M(h_\xi(S,\pi)) = h(S_a, \Pi_a)$$

for any $a \in \mathrm{S}^1$.

Proof: Fix an $a \in \mathrm{S}^1$. By Theorem 7.1 we can select an isolating neighborhood N for π and a regular index pair P in N such that $Q := X_a \cap P$ is an index pair for Π_a in $N_a := X_a \cap N$. Let Π^P denote the time-one map of π^P. It follows easily from (11) that the its restriction to the fiber $\mathbf{U}_\xi(P)_a$ is given by

$$\Pi_a^P([x,i]) := \begin{cases} [a,0] & \text{if } i = 0 \text{ or } \sigma_{\mathrm{cl}(P_1 \setminus P_2)}(x) \leq 1 \\ [\Pi(x),1] & \text{if } \sigma_{\mathrm{cl}(P_1 \setminus P_2)}(x) \geq 1. \end{cases} \tag{12}$$

Now it is a straightforward observation that the map

$$\iota : (\mathbf{U}_\xi(P)_a, s_{U_\xi(P)}(a)) \to (Q_1/Q_2, [Q_2])$$

given by

$$\iota([x,i]) := \begin{cases} [Q_2] & \text{if } i = 0 \text{ or } x \in Q_2 \\ [x] & \text{otherwise} \end{cases}$$

is a homeomorphism which conjugates Π_a^P with the index map of Π_a in Q. Thus we obtain

$$M(h_\xi(S,\pi)) = [((U_\xi(P))_a, \Pi_a^P)]_{\mathrm{Szym}} = [(Q_1/Q_2, (\Pi_a)_Q)]_{\mathrm{Szym}} = h(S_a, \Pi_a)$$

□

9 Suspension and the discrete Conley index

Let X be a locally compact space and let $f : X \to X$ be a proper map of a pointed topological space into itself. The suspension of f is a topological space Σ_f defined by $\Sigma_f := X \times \mathrm{R}^+/\equiv$, where $\equiv$ is the smallest equivalence relation such that for any $x \in X$ and any $\tau \in \mathrm{R}^+$

$$(x, \tau + 1) \equiv (f(x), \tau).$$

(Equivalently, Σ_f can be defined as the quotient space of $X \times [0,1]$ by the relation identifying $(x,1)$ with $(f(x),0)$. In the sequel we will also use that approach.) There is an associated projection $r_f : \Sigma_f \ni [x,b] \to e^{2\pi ib} \in \mathrm{S}^1$ and an r_f-preserving semiflow $\varphi_f : \Sigma_f \times \mathrm{R}^+ \to \Sigma_f$ given by

$$\varphi_f([x,\tau],t) := [x, \tau + t].$$

If f preserves a point $x_0 \in X$, i.e. f is a map of the pointed space (X, x_0) into itself, then there is a natural embedding $s_f : \mathrm{S}^1 \ni e^{2\pi ia} \to [x_0, a] \in \Sigma_f$, where $a \geq 0$. This makes (Σ_f, r_f, s_f) a fiberwise pointed space over S^1 and consequently (Σ_f, r_f, s_f) is a semibundle. The set

$$\Sigma_f^0 := \{\, [x_0, b] \mid b \in [0,1] \,\}$$

is obviously invariant under φ_f.

For $A \subset X$ put

$$\Sigma_f(A) := \{\, [x,b] \mid x \in A,\ \ b \in [0,1] \,\}.$$

Let K be an isolated invariant set for f. The set $\Sigma_f(K)$ is an isolated invariant set of φ_f hence we can consider $h_{r_f}(\Sigma_f(K), \varphi_f)$ as the Conley-type index of K. As a trivial consequence of Theorem 8.3 we get:

Theorem 9.1 *If K is an isolated invariant set for the homeomorphism f then*

$$h(K,f) = M(h_{r_f}(\Sigma_f(K), \varphi_f)).$$

It follows that $h_{r_f}(\Sigma_f(K), \varphi_f)$ delivers no less information than the discrete Conley index $h(K,f)$.

We conclude this section by a few results which facilitate calculation of the fiberwise deforming homotopy type of suspensions. Let (X, x_0) be a well-pointed space, i.e. $X \times 0 \cup \{x_0\} \times I$ is a retract of $X \times I$ or, equivalently, if (X, x_0) admits a *Strøm structure*, i.e. a pair (G, a) of continuous maps $a : X \to I$ and $G : X \times I \ni (x, t) \to G_t(x) \in X$ such that $a^{-1}(0) = \{x_0\}$ and

$$G_0 = \mathrm{id}_X, \quad G_t(x_0) = x_0 \text{ if } t \in I, \quad G_1(x) = x_0 \text{ if } a(x) < 1$$

(compare [2, Ch. 6]). In the sequel we assume that (G, a) is a fixed Strøm structure for (X, x_0).

Proposition 9.2 *If $f, g : (X, x_0) \to (X, x_0)$, $f \simeq g \operatorname{rel} x_0$ then*

$$[\Sigma_f]_{\mathrm{S}^1} = [\Sigma_g]_{\mathrm{S}^1}$$

Proof: Let $F : (X, x_0) \times [0, 1] \to (X, x_0)$, a homotopy such that $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$. The maps $A : (\Sigma_f, \Sigma_f^0) \to (\Sigma_g, \Sigma_g^0)$ and $B : (\Sigma_g, \Sigma_g^0) \to (\Sigma_f, \Sigma_f^0)$ given by

$$A([x,t]_{\Sigma_f}) := \begin{cases} [x, (1 - \frac{a(x)}{2})^{-1} t]_{\Sigma_g} & \text{if} \quad t \in [0, 1 - \frac{a(x)}{2}], \\ [F(x, \frac{2(1-t)}{a(x)}), 0]_{\Sigma_g} & \text{if} \quad t \in [1 - \frac{a(x)}{2}, 1], x \neq x_0, \\ [x_0, 0]_{\Sigma_g} & \text{if} \quad x = x_0, \end{cases}$$

$$B([x,t]_{\Sigma_g}) := \begin{cases} [x, (1 - \frac{a(x)}{2})^{-1} t]_{\Sigma_f} & \text{if} \quad t \in [0, 1 - \frac{a(x)}{2}], \\ [F(x, \frac{2(t-1)}{a(x)} + 1), 0]_{\Sigma_f} & \text{if} \quad t \in [1 - \frac{a(x)}{2}, 1], x \neq x_0, \\ [x_0, 0]_{\Sigma_f} & \text{if} \quad x = x_0. \end{cases}$$

are mutually inverse fiberwise deforming homotopy equivalencies. □

Let c_{x_0} denote the constant map $x \to x_0$. Recall that the fiberwise deforming homotopy type of the triple $(\mathrm{S}^1, \mathrm{id}_{\mathrm{S}^1}, \mathrm{id}_{\mathrm{S}^1})$ is called *trivial.*

Proposition 9.3 *$\Sigma_{c_{x_0}}$ has the trivial fiberwise deforming homotopy type over S^1.*

Proof: Put $r := r_{c_{x_0}}$ and $s := s_{c_{x_0}}$. We will prove that $r : \Sigma_{c_{x_0}} \to \mathrm{S}^1$ and $s : \mathrm{S}^1 \to \Sigma_{c_{x_0}}$ are mutually inverse fiberwise deforming homotopy equivalencies. Since $r \circ s = \mathrm{id}_{\mathrm{S}^1}$, it suffices to prove that

$$s \circ r : \Sigma_{c_{x_0}} \ni [x, t] \longrightarrow [x_0, t] \in \Sigma_{c_{x_0}}$$

is homotopic to the identity relative to $\Sigma^0_{c_{x_0}}$. The required homotopy is composed from the following two homotopies H and J:

$$H : \Sigma_{c_{x_0}} \times [0,1] \ni ([x,t],s) \to [G_s(x), t + sa(x)(1-t)] \in \Sigma_{c_{x_0}},$$

$$J : \Sigma_{c_{x_0}} \times [0,1] \ni ([x,t],s) \to [x_0, t + sa(x)(1-t)] \in \Sigma_{c_{x_0}}.$$

Since $H(\cdot,0) = \mathrm{id}$, $H(\cdot,1) = J(\cdot,1)$, and $J(\cdot,0) = s \circ r$, the result follows. $\Box$

Proposition 9.4 *Let $x_0 \in A \subset X$. Assume that A is a strong deformation retract of X. Then*

$$[\Sigma_f]_{\mathrm{S}^1} = [\Sigma_{r\circ f\circ i}]_{\mathrm{S}^1}$$

where $r : X \to A$ is a strong deformation retraction and $i : A \hookrightarrow X$ is the inclusion.

Proof: By Proposition 9.2 the fiberwise deforming homotopy types of the suspensions of f and $i \circ r \circ f$ are equal. It is elementary to prove that the suspensions of $i \circ r \circ f$ and $r \circ f \circ i$ have the same fiberwise pointed (hence also deforming) homotopy type. $\Box$

10 On calculation of the index

Let $f : X \to X$ be a homeomorphism. Our aim is to present a possible way of calculation of the index $h_{r_f}(\Sigma_f(K), \varphi_f)$ (see Section 9) for some isolated invariant set K for f. Recall that the definition of isolating block was given in Section 6. Note that if B is an isolating block with the exit set B^- then $(B/B^-, [B^-])$ is a well-pointed space (see [7, Prop. 6.1]). For a reason which will be made clear in the proof of the following theorem, we impose an additional condition for isolating blocks considered in the sequel. An isolating block B is called *proper* if there exists a $\delta > 0$ such that $x.[-\delta, 0) \subset X \setminus B$ for every $x \in B^+$ (or, equivalently, $x.(0,\delta] \subset X \setminus B$ for every $x \in B^-$).

Theorem 10.1 *Let B be a proper isolating block for some (continuous-time) flow on X, B^+ and B^- denote its entrance and exit sets, respectively. If*

$$f(B) \cap B^+ = \emptyset, \qquad f(B^-) \cap B = \emptyset \tag{13}$$

then $K := \{x \in X : f^n(x) \in B \ \forall n \in \mathrm{Z}\}$ *is an isolated invariant set for* f *and*

$$h_{r_f}(\Sigma_f(K), \varphi_f) = [\Sigma_{f\#}]_{\mathrm{S}^1},$$

where $f_\# : (B/B^-, [B^-]) \to (B/B^-, [B^-])$ *is given by*

$$f_\#([x]) := \begin{cases} [f(x)] & \text{if} \quad f(x) \in B, \\ [B^-] & \text{if} \quad f(x) \notin B. \end{cases}$$

Proof: K is an isolated invariant set by the assumption (13) and the map $f_\#$ is continuous because $\mathrm{cl}(f(B) \setminus B) \cap B \subset B^-$. Let ψ denote an auxiliary flow on X for which B is an isolating block. Let $\epsilon > 0$ and $t \in [0, 1]$. Put

$$B_t := \psi_{-t\epsilon}(B), \qquad B_t^\pm := \psi_{-t\epsilon}(B^\pm),$$

Define

$$W := \{[x, t] \in \Sigma_f : x \in B_t\}.$$

Let ϵ be so small that

$$f(B_t) \cap B^+ = \emptyset, \qquad f(B_t^-) \cap B = \emptyset$$

for any $t \in [0, 1]$ and

$$\psi(B^+ \times [-\epsilon, 0)) \cap B = \emptyset.$$

(The latter condition follows by the definition of proper isolating block.) It follows that W is an isolating block for the flow π_f, $\Sigma_f(K) = \mathrm{Inv}\, W$, and its entrance and exit sets is are given by

$$\begin{aligned} W^+ &= \{[x, t] \in \Sigma_f : x \in B_t^+\} \cup \{[x, 0] \in \Sigma_f : x \in B \setminus f(B_1)\}, \\ W^- &= \{[x, t] \in \Sigma_f : x \in B_t^-\} \cup \{[x, 0] \in \Sigma_f : x \in f(B_1) \setminus B\}, \end{aligned}$$

respectively. For $t \in [0, 1]$ define $f_{t\#} : (B/B^-, [B^-]) \to (B/B^-, [B^-])$ by

$$f_{t\#}([x]) := \begin{cases} [f(\psi_{-t\epsilon}(x))] & \text{if} \quad f(\psi_{-t\epsilon}(x)) \in B, \\ [B^-] & \text{if} \quad f(\psi_{-t\epsilon}(x)) \notin B. \end{cases}$$

It follows that $f_{t\#}$ is a homotopy relative to $[B^-]$ joining $f_\#$ with $f_{1\#}$. Up to an obvious homeomorphism, the adjunction $W \cup_{r_f|W^-} \mathrm{S}^1$ is equal to the suspension $\Sigma_{f_{1\#}}$, hence, by Proposition 9.2, $W \cup_{r_f|W^-} \mathrm{S}^1$ has the same fiberwise deforming homotopy type as $\Sigma_{f_\#}$, hence the result is proved. □

Remark 10.2 The reader familiar with [14] will easily notice that under the assumptions of Theorem 10.1 the pair (B, B^-) is an index pair in the sense of [14] for the discrete dynamical system generated by the homeomorphism f and $f_\#$ is the associated index map (cf also [9]).

The assumption on an isolating block structure of the set B in the Theorem 10.1 is satisfied in reasonable situations as the following result shows:

Theorem 10.3 *Let X be a smooth manifold (without boundary) and B be a C^1-class compact submanifold-with-corners of X,* $\dim B = \dim X$*, such that*

$$\partial B = B_1 \cup B_2,$$

where B_1 and B_2 are codimension-one C^1-class submanifolds-with-boundary of X,

$$\partial B_1 = \partial B_2 = B_1 \cap B_2.$$

Then there exists a flow on X such that B is its proper isolating block, $B^+ = B_1$ and $B^- = B_2$. In particular, if (13) is satisfied then the conclusion of Theorem 10.1 holds.

Proof: Let $\dim X = n$. By definition, if $x \in X$ there exists an open neighborhood V^x of x and a diffeomorphsm $\alpha^x : V^x \to \mathrm{R}^n$, $\alpha^x(x) = 0$, such that:

if $x \in X \setminus \partial B$ then $V^x \subset X \setminus \partial B$,

if $x \in B_i \setminus \partial B_i$ $(i = 1, 2)$ then $V^x \subset X \setminus B_j$, $j \neq i$, and

$$\alpha^x(B \cap V^x) = [0, \infty) \times \mathrm{R}^{n-1}, \qquad \alpha^x(B_i \cap V^x) = \{0\} \times \mathrm{R}^{n-1},$$

if $x \in B_1 \cap B_2$ then

$$\begin{aligned} \alpha^x(B \cap V^x) &= [0, \infty) \times [0, \infty) \times \mathrm{R}^{n-2}, \\ \alpha^x(B_1 \cap V^x) &= \{0\} \times [0, \infty) \times \mathrm{R}^{n-2}, \\ \alpha^x(B_2 \cap V^x) &= [0, \infty) \times \{0\} \times \mathrm{R}^{n-2}. \end{aligned}$$

For each $x \in X$ we define a smooth vector-field $v^x : V^x \to TX$ as follows. Let $c_w : \mathrm{R}^n \to \mathrm{R}^n$ denote the constant vector-field with the value $w \in \mathrm{R}^n$. Define

$$v^x := \begin{cases} 0 & \text{if} \quad x \in X \setminus \partial B, \\ \alpha_*^x(c_{(1,0,\dots,0)}) & \text{if} \quad x \in B_1 \setminus B_2, \\ \alpha_*^x(c_{(-1,0,\dots,0)}) & \text{if} \quad x \in B_2 \setminus B_1, \\ \alpha_*^x(c_{(1,-1,0,\dots,0)}) & \text{if} \quad x \in B_1 \cap B_2. \end{cases}$$

Using a smooth partition of unity subordinate to the covering $\{V^x\}_{x \in X}$ we connect the vector-fields v^x to a vector-field $v : X \to TX$. The flow generated by v has the required properties. $\square$

11 Examples

As examples of the computation of the Conley index over S^1 we study the suspension flows of the G- and U-horseshoes.

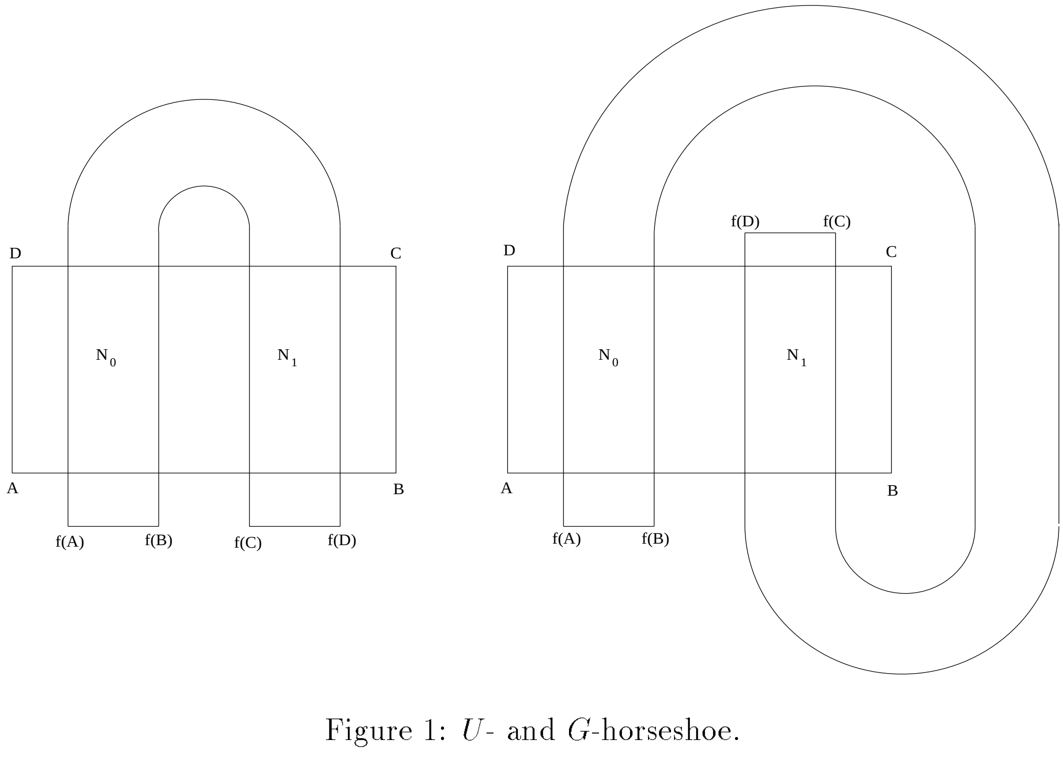

Figure 1: U- and G-horseshoe.

Let us recall that the G- and U-horseshoes are homeomorphisms of the plane which transform a rectangle to the shape of U and G, respectively, as depicted in Figure 1. The rectangle $B := \mathbf{ABCD}$ is a submanifold-with-corners, its boundary is the union of B_1 and B_2 where $B_1 := \mathbf{AD} \cup \mathbf{BC}$, the union of the sides, and $B_2 := \mathbf{AB} \cup \mathbf{CD}$, the union of the bases. By Theorem 10.3, B is a proper isolating block for some planar flow, $B^+ = B_1$ and $B^- = B_2$. It is clear from the picture that the condition (13) is satisfied, hence, by Theorem 10.1, in both cases the index $h_{r_f}(\Sigma_f(K), \pi_f)$ of the suspension of the isolated invariant set K determined by a horseshoe f is equal to the fiberwise deforming homotopy type of $\Sigma_{f_\#}$. After collapsing of B/B_1 to a circle, by Proposition 9.4, the latter type can be expressed as $[\Sigma_D]_{\mathrm{S}^1}$ in the case of U-horseshoe and as $[\Sigma_E]_{\mathrm{S}^1}$ in the case of G-horseshoe, where $D, E : (\mathrm{S}^1, 1) \to (\mathrm{S}^1, 1)$ are defined by

$$D(e^{2\pi it}) := \begin{cases} e^{4\pi it} & \text{if} \quad t \in [0, \frac{1}{2}], \\ e^{4\pi i(1-t)} & \text{if} \quad t \in [\frac{1}{2}, 1] \end{cases}$$

and by $E(z) := z^2$. Since $(\mathrm{S}^1, 1)$ is a well-pointed space and D is homotopic to the constant map c_1, one easily concludes from Propositions 9.2 and 9.3 that the Conley index over S^1 of the suspension of the isolated invariant set determined by the U-horseshoe is trivial.

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