Math 459 Midterm solutions

1. Let $U(x)$ be my utility function normalized such that

$$U(\$100) = 0 \quad \text{and} \quad U(\$2000) = 1.$$

Denote $U(\$1500)$ by z. Consider four assets:

Asset A pays a fixed \$1500.
Asset B pays \$2000 with probability 0.8 or \$100 with probability 0.2.
Asset C pays \$1500 with probability 0.25 or \$100 with probability 0.75.
Asset D pays \$ 2000 with probability 0.2 and \$100 with probability 0.8.

Given a choice between assets A and B, I would prefer A. Given a choice between assets C and D, I would choose D. Explain why this shouldn't happen.

Solution: My choices imply that the expected utility of asset A is greater than that of asset B, and the expected utility of asset D is greater than that of asset C. Computing these gives:

Asset A: $E[U] = z$.
Asset B: $E[U] = 0.8 \cdot U(\$2000) + 0.2 \cdot U(\$100) = 0.8$.
Asset C: $E[U] = 0.25 \cdot U(\$1500) + 0.75 \cdot U(\$100) = \frac{z}{4}$.
Asset D: $E[U] = 0.2 \cdot U(\$2000) + 0.8 \cdot U(\$100) = 0.2$.

Prefering A to B says $z > 0.8$. Prefering D to C says $0.2 > \frac{z}{4}$, i.e., $0.8 > z$. Thus there is no way to define $z = U(\$1500)$ to be consistent with my choices, or more likely, this shows that I am irrational.

This problem is a version of the Allais paradox, named after a prominent critic of subjective expected utility theory.

2. Consider an 8 month forward contract on a stock with a current price of \$50. The stock pays a dividend of \$1 per share after three months and after six months. The risk-free interest rate, with no compounding, is 6% per year. What is the forward price?

Solution: Since the stock *pays* dividends, the dividends act as a negative carrying cost. So we have

$$F = \frac{\$50}{d(0,8)} - \frac{\$1}{d(3,8)} - \frac{\$1}{d(6,8)} = \$49.965.$$

3. A hamburger chain plans a purchase at the end of the year of 6,000,000 pounds of soybean oil, hedging this with a position in January 2010 soybean oil futures, now priced at 32.50 cents per pound with a standard deviation of 24%. The spot price is 30.85 cents per pound with a standard deviation of 28%, and the correlation between the two prices is 0.72. One futures contract is for 60,000 pounds. What futures position should the hamburger chain take for a minimum variance hedge, i.e., how many contracts, and long or short?

Solution: Since the chain is buying at the end of the year, they should go long in futures. The hedge ratio is

$$\rho\frac{\sigma_S\sigma_F}{\sigma_F^2} = \rho\frac{\sigma_S}{\sigma_F} = 0.72\frac{0.28 \cdot 30.85}{0.24 \cdot 32.50} = 0.80.$$

So the chain should buy $\frac{0.80 \cdot 6{,}000{,}000}{60{,}000}$ or 80 contracts.

4. Suppose you want to approximate $\sqrt{3}$. You can use Newton's method with initial guess 2, or the midpoint method with initial interval $[1.5, 2]$. Which do you think will be better after three iterations? Do the interations and verify your intuition.

Solution: I think Newton will be better. Let $F(x) = x^2 - 3$.

Bisector Method: $F(1.5) = -0.75 < 0$, $F(2) = 1 > 0$. $F(1.75) = 0.0625 > 0$, so $1.5 < \sqrt{3} < 1.75$. $F(1.625) = -0.359 < 0$, so $1.625 < \sqrt{3} < 1.75$. Finally, $F(1.6875) = -0.152$, so $1.6875 < \sqrt{3} < 1.75$.

Newton's Method: $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$. $x_0 = 2$. Doing the computations to three decimal places, $x_1 = \frac{7}{4}$, $x_2 = \frac{97}{56} = 1.732$, and $x_3 = x_2$. So two interations of Newton's method gives $\sqrt{3}$ to three decimal places, while three interations of the bisector method gives a little better than one decimal place.

5. (Essay Question) Discuss the notion of implied volatility, including the following: a definition; how you know there is a unique solution; and how to find the solution.

Solution: In the Black-Scholes formula for a European call we can obtain current exact values for S, r, K, T, t, and the option price C, so the only quantity in the Black-Scholes formula which we can't measure exactly is the volatility σ. If we fix the values of S, r, K, T and t, then we can consider C as a function of σ. Given a price C^* for the option, the implied volatility is the value σ^* such that $C(\sigma^*) = C^*$. Since $\partial C/\partial\sigma > 0$, the value of σ^* is unique. One cannot solve for σ^* analytically. Newton's method used to find a root of $F(\sigma) = C(\sigma) - C^*$ is an efficient numerical way of finding the implied volatility. If one uses the inflection point as the starting point, then Newton's method will converge.