Chapters 7 Solutions

Note: These solutions are very brief and I have left out the justificaitons for some of the steps. It should be easy to figure out what the justifications are, and you should include these in your homework and exam solutions.

7.1. Fix $a \in \mathbf{N}$. Recall that $\mathbf{N}$ is closed under addition, i.e., if $m, n \in \mathbf{N}$, then $m + n \in \mathbf{N}$. We will show by induction that for all $b \in \mathbf{N}, a \cdot b \in \mathbf{N}$. Let

$$M = \{\, b \in \mathbf{N} \mid a \cdot b \in \mathbf{N} \,\}..$$

We have $1 \in M$ since by definition of multiplication, $a \cdot 1 = a$. Now we assume $b \in M$ and show that $b + 1 \in M$. We have $a \cdot (b + 1) = a \cdot b + a \cdot 1$ by the distributive law. Since $b \in M$, $a \cdot b \in \mathbf{N}$, and by definition of multiplication, $a \cdot 1 = a \in \mathbf{N}$. Since $\mathbf{N}$ is closed under addion, we have $a \cdot (b + 1) \in \mathbf{N}$, so $b + 1 \in M$. By induction, $M = \mathbf{N}$, so $a \cdot b \in \mathbf{N}$ for all $a, b \in \mathbf{N}$, so $a \cdot b \neq 0$.

7.2.

1) $0 = 0 + 0$, so $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. By cancellation for addition, $0 = a \cdot 0$.
2) To show $a \cdot (-1) = -a$, we need to show $a + a \cdot (-1) = 0$. We have $a + a \cdot (-1) = a \cdot 1 + a \cdot (-1) = a \cdot (1 + (-1)) = a \cdot 0 = 0$.
3) This is done as part of the solution to Problem 7.1.

7.3. Suppose $a \neq 0$, and $a \cdot b = a \cdot c$. Then $a \cdot b - a \cdot c = 0$, so $a \cdot (b - c) = 0$. By Theorem 7.11, since $a \neq 0$, we must have $b - c = 0$, so $b = c$.

7.4.

1) $a < b$ means $b - a = m \in \mathbf{N}$, and $b < c$ means $c - b = n \in \mathbf{N}$. We have $c - a = c - b + b - a = n + m \in \mathbf{N}$, so $a < c$.
2) We have $b - a = m \in \mathbf{N}$. $b + c - (a + c) = b - a + c - c = b - a \in \mathbf{N}$, so $a + c < b + c$.
3) Again, $b - a = m \in \mathbf{N}$, and $c \in \mathbf{N}$. We have $b \cdot c - a \cdot c = (b - a) \cdot c = m \cdot c \in \mathbf{N}$ by Problem 7.2 part 3), so $ac < bc$.