Positive Knots And Knots With Braid Index Three Have Property-P

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Abstract: We prove that positive knots and knots with braid index three in the 3-sphere satisfy the Property P conjecture.

1. Introduction

Let K be a knot in the 3-sphere S^3 and $M = M_K$ the complement of an open regular neighborhood of K in S^3. As usual, the set of slopes on the torus ∂M (i.e. the set of isotopy classes of unoriented essential simple loops on ∂M) is parameterized by

$$\{m/n;\ m, n \in \mathbb{Z}, n > 0, (m,n) = 1\} \cup \{1/0\},$$

so that $1/0$ is the meridian slope of K and $0/1$ is the longitude slope of K. The manifold obtained by Dehn surgery on S^3 along the knot K (equivalently, Dehn filling on M along the torus ∂M) with slope m/n, is denoted by $K(m/n)$ or $M(m/n)$. Of course $K(1/0) = S^3$, and thus the surgery with the slope $1/0$ is called the trivial surgery. The celebrated *Property P* conjecture, introduced by Bing and Matin in 1971 [BMa], states that every nontrivial knot K in S^3 has Property P, i.e. every nontrivial surgery on S^3 along K produces a non-simply connected manifold. For convenience we say that a class of knots in S^3 have Property P if every nontrivial knot in this class has Property P. The following classes of knots were known to have Property P: torus knots [H], symmetric knots [CGLS] (the part for strongly invertible knots was proved in [BS]), satellite knots [G1], arborescent knots [W], alternating knots [DR], and small knots with no non-integral boundary slopes [D]. For a simple homological reason, to prove the conjecture for a knot K one only needs to consider the surgeries of K with slopes $1/n$, $n \neq 0$. A remarkable progress on the conjecture was made in [CGLS]; it was proved there that for a nontrivial knot, only one of $K(1)$ or $K(-1)$ could possibly be a simply connected manifold. Another remarkable result was given in [GL], which told us that if the Property P conjecture is false, then the Poincare conjecture is false. For some earlier progresses on the conjecture, see [K, Problem 1.15] for a summary. In this paper we prove

Theorem 1 *Positive (or negative) knots in S^3 satisfy the Property P conjecture.*

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Recall that a knot is positive if it can be represented as the closure of a positive n-braid for some n, i.e. a braid which involves the standard elementary braid generators $\sigma_1, \cdots, \sigma_{n-1}$ (Figure 1 gives σ_1, σ_2 when $n = 3$) but not their inverses. A negative knot is similarly defined, which is actually the mirror image of a positive knot.

Theorem 2 *Knots in S^3 with braid index three satisfy the Property P conjecture.*

Theorem 1 is a quick application of the Casson invariant. We refer to [AM] for the definition and basic properties of the Casson invariant. The Casson invariant is an integer valued topological invariant defined for homology 3-spheres and for knots in homology 3-spheres. If a homology 3-sphere has non-zero Casson invariant, then the manifold has an irreducible representation from its fundamental group to the group $SU(2)$, which implies in particular that the manifold is non-simply connected. For a knot K in S^3, if $\Delta_K(t)$ denotes the normalized Alexander polynomial of K, i.e. satisfying $\Delta_K(1) = 1$ and $\Delta_K(t^{-1}) = \Delta_K(t)$, then the Casson invariant C_K of K is equal to the integer $\frac{1}{2}\Delta''_K(1)$ [AM]. Further the Casson invariant of the manifold $K(1/n)$ is equal to nC_K. So $C_K \neq 0$ implies in particular that the knot K has Property P. Note that the Conway polynomial of a knot in S^3 is a single variable polynomial $\bigtriangledown_K(x) \in \mathbb{Z}[x]$ with only even powers and $\bigtriangledown_K(t^{1/2} - t^{-1/2}) = \Delta_K(t)$, from which one can easily deduce that the coefficient of x^2 in $\bigtriangledown_K(x)$ is equal to the Casson invariant of K. For a nontrivial positive knot K in S^3, it has been proved in [V] that the coefficient of x^2 in $\bigtriangledown_K(x)$ is a positive integer. Hence such knot has Property P. A negative knot is just a mirror image of a positive knot, and obviously a knot has Property P if and only if its mirror image does. Theorem 1 now follows.

The rest of the paper is devoted to the proof of Theorem 2. The main tools we shall use are the Casson invariant and essential laminations. We refer to [GO] for the definition and basic properties of an essential lamination. In section 2 we give an outline of the proof of Theorem 2. Actually the proof of Theorem 2 is reduced there to that of two propositions, Proposition 3 and Proposition 4. These two propositions will then be proven in section 3 and section 4 respectively.

2. Proof of Theorem 2

Recall that the 3-braid group, B_3, has the following well known Artin presentation:

$$B_3 =< \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$$

where σ_1 and σ_2 are elementary 3-braids as shown in Figure 1. If we let $a_1 = \sigma_1, a_2 = \sigma_2, a_3 = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$, then B_3 also has the following presentation in generators a_1, a_2 and a_3 (see [X]):

$$B_3 =< a_1, a_2, a_3 \mid a_2a_1 = a_3a_2 = a_1a_3 > .$$

Figure 1: σ_1 (the left figure) and σ_2 (the right figure)

In this paper we shall always express a 3-braid as a word $w(a_1, a_2, a_3)$ in letters a_1, a_2, a_3. Such a word is called *positive* if the power of every letter in the word is positive. A positive word $w = a_{\tau_1} \cdots \alpha_{\tau_k}$ is said to be in *non-decreasing order* (ND-order) if the array of its subscripts $(\tau_1, ..., \tau_k)$ satisfies

$$\tau_{j+1} = \tau_j \text{ or } \tau_{j+1} = \tau_j + 1 \text{ (mod 3 if } \tau_j + 1 = 4) \text{ for } j = 1, ..., k-1.$$

One can define *negative* word and *non-increasing order* (NI-order) similarly. Let P be the set of positive words in ND-order, let N be the set of negative words in NI-order, and let $\alpha = a_2 a_1$. It is proven in [X] that for any 3-braid, there is a representative in its conjugacy class that is a shortest word in a_1, a_2, a_3 and is of the form

(i) a product of α^k and a word (maybe empty) in P for some non-negative integer k; or

(ii) a product of α^k and a word (maybe empty) in N for some non-positive integer k; or

(iii) a product of a word in P and a word in N.

where the meaning of *the shortest* is that the length, i.e. the number of letters of the representative is minimal among all representatives in the conjugacy class of the braid. Such representative of a 3-braid is said to be in *normal form.* We shall only need to show that if K is a nontrivial knot in S^3 which is the closure of a 3-braid in normal form (i) or (ii) or (iii), then it has Property P.

Recall that a word $w(a_1, a_2, a_3)$ is called *freely reduced* if no adjacent letters are inverse to each other, and is called *cyclically reduced* if it is freely reduced and the first letter and the last letter of the word are not inverse to each other. Given a word $w(a_1, a_2, a_3)$, one can combine all adjacent letters of the same subscript into a single power of the letter, called a *syllabus* of the word in that subscript. A word w is called *syllabus reduced* if it is expressed as a word in terms of syllabuses as

$$w = a_{\tau_1}^{m_1} a_{\tau_2}^{m_2} \cdots a_{\tau_k}^{m_k}$$

such that $a_{\tau_j} \neq a_{\tau_{j+1}}$ for $j = 1, ..., k-1$. A word is called *cyclically syllabus reduced* if it is syllabus reduced and its first and last syllabuses are in different subscripts.

Let P^* denote the set of all positive words in a_1, a_2, a_3 such that between any two syllabuses in a_3 both a_1 and a_2 occurs. Obviously any 3-braid of norm form (i) is contained in P^*. Let β be a 3-braid in P^*. Suppose that a_3^k is a syllabus in β which is proceeded immediately by a_1. Then one can eliminate the syllabus a_3^k with the equality $a_1 a_3^k = a_2^k a_1$

to get an isotopic braid which is still in P^* but with one less number of syllabuses in a_3. Similarly if a syllabus a_3^k is followed immediately by a_2, then one can eliminate the syllabus a_3^k with the equality $a_3^k a_2 = a_2 a_1^k$ to get an isotopic braid which is still in P^* but with one less number of syllabuses in a_3. We shall call this process *index-3 reduction*. So for any given $\beta \in P^*$, we can find, after a finitely many times of index-3 reduction, an equivalent braid representative β' in P^* for β such that every syllabus in a_3 occurring in β' can only possibly be proceeded immediately by a_2 and likewise can only possibly be followed immediately by a_1. We call a word β in P^* *index-3 reduced* if every syllabus in a_3 occurring in β is neither proceeded immediately by a_1 nor followed immediately by a_2.

Let P^a denote the set of 3-braids of the form $\beta = a_i^{-q}\delta$, where $q = 0$ or 1 and $\delta \in P^*$ is a non-empty word, such that β is cyclically reduced and δ is index-3 reduced. Obviously P is contained in P^a.

Proposition 3 *Suppose that K is a knot in S^3 which is the closure of a 3-braid $\beta = a_i^{-q}\delta$ in P^a such that δ contains at least four syllabuses but β is not one of the words in the set $E = \{a_1^{-1}a_2a_3^2a_1a_2,\ a_1^{-1}a_3^2a_1a_2a_3,\ a_1^{-1}a_3a_1a_2^2a_3,\ a_1^{-1}a_2a_3a_1a_2^2,\ a_2^{-1}a_3a_1a_2a_3^2,\ a_2^{-1}a_3a_1^2a_2a_3,\ a_2^{-1}a_1a_2a_3^2a_1,\ a_2^{-1}a_1^2a_2a_3a_1,\ a_3^{-1}a_1a_2a_3a_1^2,\ a_3^{-1}a_1a_2^2a_3a_1,\ a_3^{-1}a_2a_3a_1^2a_2,\ a_3^{-1}a_2^2a_3a_1a_2\}$. Then K has positive Casson invariant and thus has Property P.*

Later on we shall refer the set E given in Proposition 3 as the *excluded* set.

Proposition 4 *Suppose that K is a knot in S^3 which is the closure of a 3-braid $\beta = \delta\eta$ which is in normal form (iii), i.e. $\delta \in P$ and $\eta \in N$. Suppose that either*
(1) each of δ and η has a syllabus of power larger than one, or
(2) each of δ and η contains at least two syllabuses, or
(3) one of δ and η contains at least four syllabuses and the other has length at least two.
Then each of $K(1)$ and $K(-1)$ is a manifold which contains an essential lamination.

If a closed 3-manifold has an essential lamination, then its universal cover is $\mathbb{R}^3$ [GO] and thus in particular the manifold cannot be simply connected. Hence any knot as given in Proposition 4 has Property P by [CGLS].

Given Propositions 3 and 4, we can finish the proof of Theorem 2 as follows. For a braid β, we use $\hat{\beta}$ to denote the closure of β. Let $K \subset S^3$ be a nontrivial knot with index 3. Let β be a 3-braid in normal form such that $\hat{\beta} = K$.

First we consider the case that β is in normal form (i), i.e. $\beta = \alpha^k\delta$ with $\delta \in P$ and $k \geq 0$. If β contains at least four syllabuses and belongs to P^a, then $K = \hat{\beta}$ has positive Casson invariant by Proposition 3. So the knot K has Property P in this case. If β has less than four syllabuses, then up to conjugation in B_3, $\beta = a_2a_1a_3^i$ or $\beta = a_1^ia_2^ja_3^m$ for

$i, j, m \geq 0$. It is easy to see that in this case $\hat{\beta}$ is an arborescent knot and thus by [W], $K = \hat{\beta}$ has Property P. For instance if $\beta = a_1^i a_2^j a_3^m$, then $\hat{\beta}$ is as shown in Figure 2 which shows in fact that $\hat{\beta}$ is a Montesinos knot. So we may assume that β has at least four

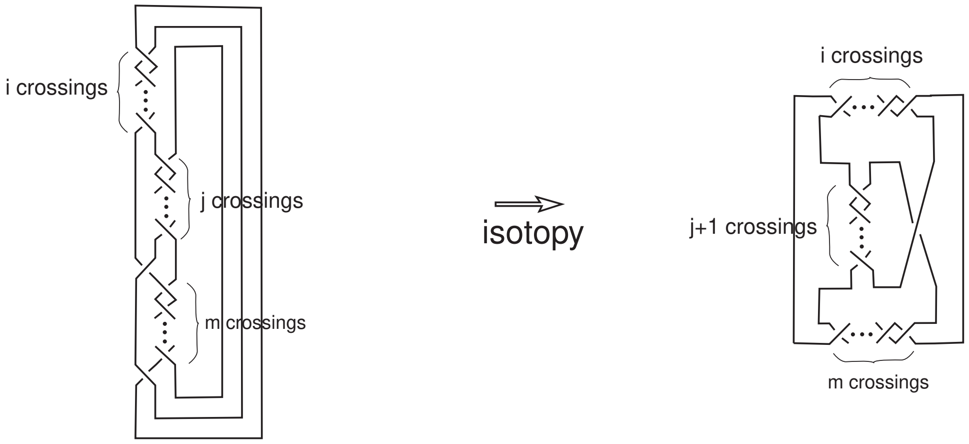

Figure 2: the closure of $a_1^i a_2^j a_3^m$ is a Montesinos link

syllabuses but is not in P^a. This implies that in $\beta = \alpha^k \delta$, we have $k > 0$ and δ starts with a syllabus in a_3. Performing index-3 reduction on β once, we get an equivalent 3-braid $\beta' \in P^*$ which is index-3 reduced, i.e. $\beta' \in P^a$. If β' has less than four syllabuses, then again $K = \hat{\beta} = \hat{\beta}'$ is an arborescent knot and thus has Property P. If β' contains at least four syllabuses, we may apply Proposition 3 to get Property P for the knot.

If β is of normal form (ii), then the mirror image of β is a braid of normal form (i) and thus the knot $K = \hat{\beta}$ has Property P.

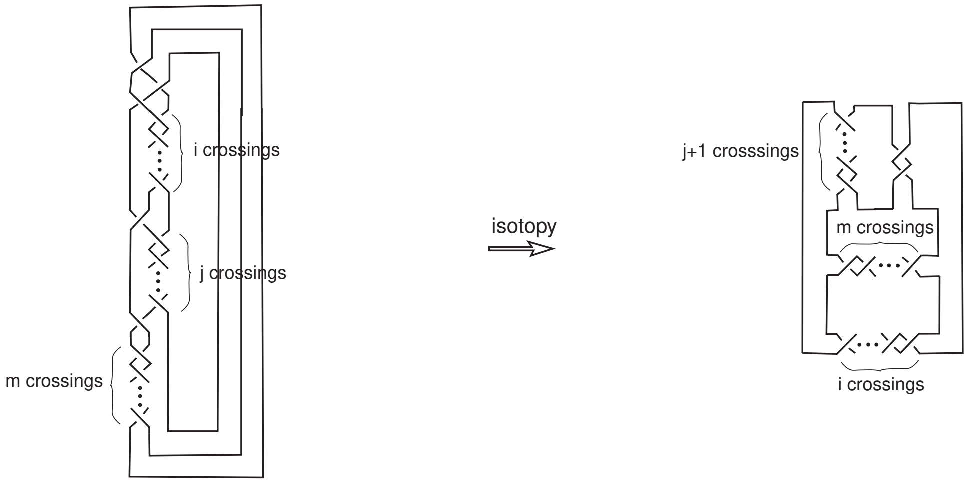

Figure 3: the closure of $a_3^{-1} a_2^i a_3^j a_1^m$ is an arborescent link

Suppose finally that $\beta = \delta\eta$ is a braid of normal form (iii) (recall that $\delta \in P$ and $\eta \in N$). By Proposition 4, we may assume that each of the conditions (1)-(3) in Proposition 4 does not hold for $\beta = \delta\eta$. Then β is a word of the form δa_i^{-1} or $a_i\eta$, or $\beta = \delta a_i^{-k}$ where $k > 1$ and δ contains at most three syllabuses each having power 1, or $\beta = a_1^k\eta$ where $k > 1$ and η contains at most three syllabuses each having power -1. If β contains at most four syllabuses, then $K = \hat{\beta}$ is an arborescent knot and thus has Property P. (Figure 3 shows this for the case that $\beta = a_3^{-1}a_2^i a_3^j a_1^m$. Other cases can be treated similarly). So we may only consider the cases when $\beta = \delta a_i^{-1}$ or $\beta = a_i\eta$, each containing at least five syllabuses. If $\beta = \delta a_i^{-1}$, then β is conjugate to $\beta' = a_i^{-1}\delta$ which is in P^a. Hence if β' does not belong to the excluded set $E =\{ a_1^{-1}a_2a_3^2a_1a_2,\ a_1^{-1}a_3^2a_1a_2a_3,\ a_1^{-1}a_3a_1a_2^2a_3, a_1^{-1}a_2a_3a_1a_2^2,\ a_2^{-1}a_3a_1a_2a_3^2,\ a_2^{-1}a_3a_1^2a_2a_3,\ a_2^{-1}a_1a_2a_3^2a_1,\ a_2^{-1}a_1^2a_2a_3a_1,\ a_3^{-1}a_1a_2a_3a_1^2, a_3^{-1}a_1a_2^2a_3a_1,\ a_3^{-1}a_2a_3a_1^2a_2,\ a_3^{-1}a_2^2a_3a_1a_2 \}$, then $K = \hat{\beta}'$ has positive Casson invariant by Proposition 3 and thus has Property P. If β' is in the excluded set E, then $K = \hat{\beta}'$ is an arborescent knot and thus has Property P [W]. (Figure 4 illustrates this for the case $\beta = a_1^{-1}a_3^2a_1a_2a_3$. Other cases can be checked similarly). Finally, if $\beta = a_i\eta$, then its mirror image is a braid in P^a, which is a case we have just discussed. This completes the proof of Theorem 2.

Propositions 3 and 4 will be proved in subsequent two sections which constitutes the rest of the paper.

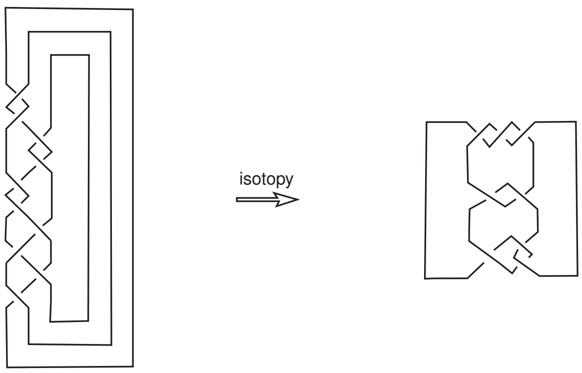

Figure 4: the closure of the braid $a_1^{-1}a_3^2a_1a_2a_3$ is a Montesinos knot

3. Proof of Proposition 3

We retain all definitions and notations established earlier. For a 3-braid β, we use n_β to denote the number of syllabuses in a_3 occurring in β and use s_β to denote the number of syllabuses of β. Obviously if β' is the braid obtained from $\beta \in P^*$ after some non-trivial index-3 reduction, then $\hat{\beta}' = \hat{\beta}$ but $n_{\beta'} < n_\beta$. For a syllabus $a_3^k = a_1^{-1}a_2^k a_1$, we shall always assume its plane projection corresponds naturally to $a_1^{-1}a_2^k a_1$ as shown in Figure 6 (a). Hence, every 3-braid in letters a_1, a_2, a_3 has its *canonical* plane projection; namely in the projection plane we place vertically (from top to bottom) and successively the projections of letters occurring in the braid, corresponding to their natural order from left to right.

Whenever we need consider a plane projection of a 3-braid, the canonical one is always assumed unless specifically indicated otherwise.

The basic tool we are going to use to prove the proposition is the following crossing change formula $(*)$ of the Casson invariant. If K_+, K_- are oriented knots and L_0 an oriented link with two components in S^3 such that they have identical plane projection except at one crossing they differ as shown in Figure 5, then the Casson invariant of K_+ and K_- satisfy the following relation:

$$C_{K_+} - C_{K_-} = lk(L_0) \qquad (*)$$

where $lk(L_0)$ is the linking number of L_0. This formula can be found on page 141 of [AM] (note that there was a print error there, K_+ and K_- in Figure 36 and Figure 37 of [AM] should be exchanged). The idea of proof of the proposition is repeatedly applying the

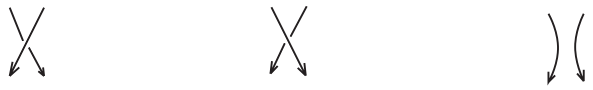

Figure 5: K_+ (the left figure), K_- (the middle figure) and L_0 (the right figure)

formula $(*)$ to a 3-braid of the type as given in the proposition to reduce the complexity of the braid and inductively prove its positivity. To do so, we first need to estimate the linking number of a two-component link which is the closure of a 3-braid of relevant type. Given an oriented link L of two components L_1 and L_2, we shall calculate the linking number of L as follows (cf. [Rn]). Take a plane projection of L. A crossing of L as shown on the left of Figure 5 has positive sign 1 and the crossing in the middle of the figure has negative sign -1. The linking number of L is the algebraic sum of the crossing signs at those crossings of L where L_1 goes under L_2.

For a 3-braid β, we shall always orient each component of $\hat{\beta}$ in such a way that the induced orientation on each strand of β in its canonical projection is pointing downward in the projection plane.

Suppose that a two component link $L = L_1 \cup L_2$ is the closure of a 3-braid β. At a crossing corresponding to a letter a_1 or a_2 or a_1^{-1} or a_2^{-1} appeared in β, if the under strand is from L_1 and the upper strand is from L_2, then this crossing contributes negative one to the linking number of $L = \hat{\beta}$ when the crossing is corresponding to a_1 or a_2 and positive one when the crossing is corresponding to a_1^{-1} and a_2^{-1}; and if the under strand is from L_2 or the upper strand is from L_1, then this crossing contributes zero to the linking number of $L = \hat{\beta}$. The linking number of $L = \hat{\beta}$ is the sum of the contributions from all the crossings of β. If β' is a portion of β, we use $l(\beta')$ to denote the total contribution to the linking number of L coming from all the crossings of β'. In particular $l(\beta) = lk(L)$. Also if we decompose β into portions $\beta = \beta_1\beta_2\cdots\beta_k$, then $l(\beta) = l(\beta_1) + l(\beta_2) + \cdots + l(\beta_k)$.

Lemma 5 *Let $\delta \in P^*$ be a 3-braid satisfying: (1) $n_\delta = 0$; (2) $\hat{\delta}$ is a link of two components; and (3) both a_1 and a_2 appear in δ. Then $lk(\hat{\delta}) < 0$.*

Proof. Recall that $n_\delta = 0$ means that δ is a word in letters a_1 and a_2 only. Also, $\delta \in P^*$ is a positive word. The conclusion of the lemma is now obvious. $\diamondsuit$

Lemma 6 *Let $\beta = a_i^{-1}\delta$ be an element in P^a satisfying: (1) $n_\delta = 0$; (2) $\hat{\beta}$ is a link of two components; and (3) δ contains at least four syllabuses. Then $lk(\hat{\beta}) < 0$.*

Proof. Again δ is a positive word without letter a_3. If $i = 1$, then δ must start with a syllabus in a_2 and also ends with a syllabus in a_2 since β is cyclically reduced. Since δ contains at least four syllabuses, it follows that $\beta = a_1^{-1}a_2^j a_1^k a_2^m a_1^n a_2^p \cdots$ for some $j, k, m, n, p > 0$. One can easily verify that $lk(\hat{\beta}) \leq l(a_1^{-1}a_2^j a_1^k a_2^m a_1^n a_2^p) < 0$. Similarly one can treat the $i = 2$ case. Consider now the case $i = 3$. If δ starts with a_2, then $\beta = a_3^{-1}a_2^j a_1^k a_2^m a_1^n \cdots$ for some $j, k, m, n > 0$, which is conjugate to $a_1^{-1}a_2^{j-1}a_1^k a_2^m a_1^n \cdots a_2$. So if $j > 1$, then we are back to the case $i = 1$ (all required conditions remain valid). If $j = 1$, then β is conjugate to $\beta' = a_1^{k-1}a_2^m a_1^n \cdots a_2$. Hence by Lemma 5 we see that $lk(\hat{\beta}) = lk(\hat{\beta}') < 0$. Similarly one can treat the case when δ starts with a_1. $\diamondsuit$

Lemma 7 *Let $L = L_1 \cup L_2$ be a link of two components in S^3 which is the closure of a braid $\beta = a_i^{-q}\delta$ in P^a such that δ contains at least four syllabuses. Then the linking number $lk(L)$ of L is non-positive.*

Proof. We may assume that β has been chosen in its conjugacy class, subject to satisfying all conditions of the lemma, to have the minimal number n_δ. By Lemmas 5 and 6, we may assume that $n_\delta > 0$.

Given the canonical plane projection of a 3-braid η, we shall always call the strand of η which starts at the top left corner *the strand 1* of η, call the strand which starts at the top middle place *the strand 2* of η, and call the strand which starts at the top right corner *the strand 3* of η.

Let $a_3^k = a_1^{-1}a_2^k a_1$ be a syllabus in a_3 occurring in δ. Its crossings are shown in Figure 6 (a). Obviously the total contribution of the syllabus to the linking number is at most one; i.e. $l(a_3^k) \leq 1$. More precisely, one can easily verify that the following claim holds.

Claim 1. If $l(a_3^k) = 1$, then the strand 1 of a_3^k (Figure 6 (a)) is from L_1, the strand 2 of a_3^k is from L_2, $k = 1$, and the strand 3 is from L_2.

Claim 2. If $\delta_0 = a_3^k a_1^m a_2^j$ is a portion of δ for some $k, m, j > 0$, then $l(\delta_0) \leq 0$.

Since $l(a_3^k a_1^m a_2^j) = l(a_3^k) + l(a_1^m a_2^j)$, the claim is obviously true if $l(a_3) \leq 0$. So suppose that $l(a_3^k) = 1$. Then by Claim 1, we have that the strand 1 of δ_0 (Figure 6 (b)) is from L_1,

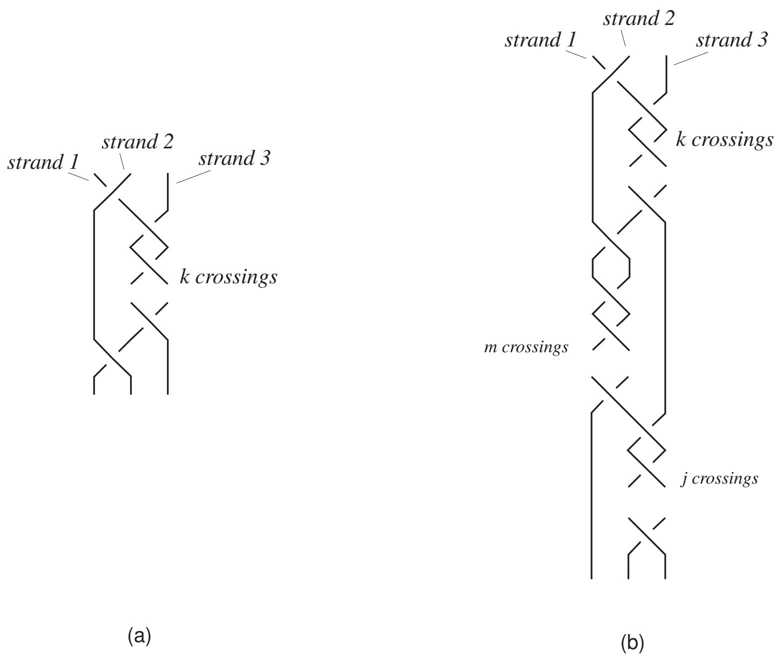

Figure 6: (a) a_3^k (b) $\delta_0 = a_3^k a_1^m a_2^j$

$k = 1$ and the strands 2 and 3 of δ_0 are from L_2. So there will be a negative one contribution to the linking number at the first crossing of the syllabus in a_2. This proves the claim.

We also need to consider under what conditions we have $l(\delta_0) = 0$. Consider Figure 6 (b).

Case (A1). If the strand 1 of δ_0 is from L_1 and the strands 2 and 3 of δ_0 are from L_2, then $l(\delta_0) = 0$ happens exactly in the following conditions: $k = 1$ and $j \leq 2$;

Case (A2). If the strand 2 of δ_0 is from L_1 and the strands 1 and 3 of δ_0 are from L_2, then $l(\delta_0) = 0$ never happen.

Case (A3). If the strand 3 of δ_0 is from L_1 and the strands 1 and 2 of δ_0 are from L_2, then $l(\delta_0) = 0$ never happen.

Case (A4). If the strand 1 of δ_0 is from L_2 and the strands 2 and 3 of δ_0 are from L_1, then $l(\delta_0) = 0$ never happen.

Case (A5). If the strand 2 of δ_0 is from L_2 and the strands 1 and 3 of δ_0 are from L_1, then $l(\delta_0) = 0$ happens exactly in the following condition: $m = 1$.

Case (A6). If the strand 3 of δ_0 is from L_2 and the strands 1 and 2 of δ_0 are from L_1, then $l(\delta_0) = 0$ never happen.

Suppose that δ_1 is a portion of the braid δ which starts with a syllabus in a_3 and is followed by a positive word which starts with a_1 and contains both a_1 and a_2, but not a_3.

It follows from Claim 2 that

Claim 3. $l(\delta_1) \leq 0$.

We also need to know exactly when $l(\delta_1) = 0$. By the discussion following Claim 2, we have that the portion of δ_1 which consists of the first three syllabuses of δ_1 must be as in Cases A1 or A5. One can then verify that $l(\delta_1) = 0$ happens exactly in one of the following situations:

(B1) $\delta_1 = a_3 a_1^m a_2^j$, for some $m > 0, 0 < j \leq 2$, and the strand 1 of δ_1 is from L_1 and the strands 2 and 3 from L_2; or

(B2) $\delta_1 = a_3 a_1^m a_2^2 a_1^p$, for some $m, p > 0$,and the strand 1 of δ_1 is from L_1 and the strands 2 and 3 from L_2; or

(B3) $\delta_1 = a_3^k a_1 a_2^j$, for some $k, j > 0$, and the strands 1 and 3 of δ_1 are from L_1 and the strand 2 from L_2.

Suppose that δ_2 is a portion of the braid δ between and including two syllabuses in a_3.

Claim 4. $l(\delta_2) \leq 0$.

We have $\delta_2 = \delta_1 a_3^r$ for some $r > 0$ where δ_1 is a braid as in Claim 3 and δ_1 must also end with a syllabus in a_2 since δ is index-3 reduced. Suppose otherwise, that is $l(\delta_2) > 0$. We will get a contradiction. By Claim 3 and Claim 1, we may assume that $l(\delta_1) = 0$ and $l(a_3^r) = 1$. So δ_1 is as in case (B1) or (B3) and a_3^r is as in Claim 1. One can easily check that the link component assignment to the strands of δ_1 and the link component assignment to the strands of a_3^r never match.

Similarly one can show

Claim 5. Suppose that δ_3 is a portion of δ satisfying (1) it ends with a syllabus in a_3; (2) this ending syllabus is proceeded by a positive word in a_1 and a_2 and the word contains at least two syllabuses. Then $l(\delta_3) \leq 0$ and $l(\delta_3) = 0$ happens exactly in one of the following situations:

(C1) $\delta_3 = a_1 a_2^m a_3$, for some $m > 0$, and the strand 2 of δ_3 is from L_1 and the strands 1 and 3 from L_2; or

(C2) $\delta_3 = a_1^2 a_2^m a_3$, for some $m > 0$, and the strand 1 of δ_3 is from L_1 and the strands 2 and 3 from L_2; or

(C3) $\delta_3 = a_2^p a_1^2 a_2^m a_3$, for some $m, p > 0$, and the strand 1 of δ_3 is from L_1 and the strands 2 and 3 from L_2; or

(C4) $\delta_3 = a_1^j a_2 a_3^k$, for some $k, j > 0$, and the strands 1 and 2 of δ_3 are from L_1 and the strands 3 from L_2.

Claim 6. Suppose that $\delta_4 = \omega\delta_1$ is a portion of δ such that δ_1 is a braid as in Claim 3 and ω is a positive word in a_1 and a_2 and the word contains at least three syllabuses (and ends with a_2). Then $l(\delta_4) < 0$.

Suppose that the claim is not true. Then we have $l(\omega) = 0$, $l(\delta_1) = 0$ and δ_1 is a word as in Case (B1) or (B2) or (B3). But in each of such cases for δ_1, one can easily check that $l(\omega) = 0$ cannot not hold. This contradiction proves Claim 6.

We are now ready to finish the proof of Lemma 7. We first consider the case when $q = 0$, i.e. $\beta = \delta \in P^a$ is an index-3 reduced positive word containing at least four syllabuses and $n_\delta > 0$. If $n_\delta = 1$, then the only syllabus in a_3 occurring in δ must belong to a portion of δ that looks like either in Claim 3 or in Claim 5, since δ contains at least four syllabuses. Hence, Lemma 7 holds in this case by Claim 3 and Claim 5. If $n_\delta \geq 2$, then Lemma 7 follows from Claims 3 and 4.

We now consider the case when $q = 1$, i.e. $\beta = a_i^{-1}\delta \in P^a$ is cyclically reduced, where $\delta \in P^*$ is index-3 reduced with $n_\delta > 0$ and contains at least four syllabuses. Suppose otherwise that $lk(L) > 0$. We will get a contradiction. We have three subcases to consider, corresponding to $i = 1, 2, 3$. Note that by Claims 3-5, we have $l(\delta) \leq 0$.

If $i = 1$ (i.e. $\beta = a_1^{-1}\delta$), then we must have $l(\delta) = 0$ and $l(a_1^{-1}) = 1$, and the first strand of β is from L_1 and the second strand of β from L_2. Also, δ must start with a syllabus in a_2 or a_3, and end with a syllabus in a_2 or a_3.

Suppose that δ starts with syllabus a_2^k and ends with syllabus a_3^j. Then $\beta = a_1^{-1}a_2^k \cdots a_1^p a_2^m a_3^j$, which is conjugate to $\beta' = a_1^{-1}a_2^{k+j} \cdots a_1^p a_2^m$. The braid β' is still in P^a but has one less number of syllabuses in a_3. So if β' contains at least five syllabuses, we may use induction on n_δ to conclude that Lemma 7 holds in this case. We may then assume that β' contains four syllabuses and, thus, $\beta = a_1^{-1}a_2^k a_1^p a_2^m a_3^j$. But by Claim 5, δ is as in case (C3); the strand 1 of δ is from L_1 and the strands 2 and 3 of δ are from L_2. But this does not match with the link component assignment already given to the strands of a_1^{-1}.

Suppose that δ starts with syllabus a_2^k and ends with syllabus a_2^j. Then each syllabus in a_3 is followed by a word without a_3 in which both a_1 and a_2 occur. So we may decompose β into portions as $\beta = a_1^{-1}\eta_0\eta_1 \cdots \eta_n$ such that $\eta_0 \in P^a$ is a word without a_3 and each of $\eta_1, ..., \eta_n \in P^a$ is a word that starts with a syllabus in a_3 which is followed by a word without a_3 but containing both a_1 and a_2. It follows from Claim 3 we must have $l(\eta_0) = l(\eta_1) = \cdots = l(\eta_n) = 0$ and each of η_1,..., η_n is as in one of the situations described in (B1)-(B3). But any two cases of (B1)-(B3) will not match with their link component assignments if they are adjacent portions of δ. Hence $n = 1$. By Claim 6, we have $\eta_0 = a_2^k$. One can now easily check for $\beta = a_1^{-1}a_2^k\eta_1$ that the conditions $l(a_1^{-1}) = 1$, $l(a_2^k) = 0$ and $l(\eta_1) = 0$ (so η_1 is as in one of (B1)-(B3)) imply that there is no consistent link component assignment to the strands of a_1^{-1}, a_2^k and η_1.

Similarly as in the previous case, we can get a contradiction for the case when δ starts with syllabus a_3^k and ends with syllabus a_2^j.

Suppose that δ starts with syllabus a_3^k and ends with syllabus a_3^j. Then $\beta = a_1^{-1}a_3^k \cdots a_1^p a_2^m a_3^j$, which is conjugate to $\beta' = a_1^{-1}a_2^j a_3^k \cdots a_1^p a_2^m$. The braid β' is still in P^a but has one less number of syllabuses in a_3. Also β' contains at least five syllabuses, so we may use induction to see that Lemma 7 holds in this case.

Similarly one can deal with the case that $i = 2$.

Finally, we consider the case that $i = 3$, i.e. $\beta = a_3^{-1}\delta$. Note that δ cannot start or end with a_3.

Suppose that δ starts with a_1 and ends with a_2. Then $\beta = a_3^{-1}a_1^j a_2^k \cdots a_1^m a_2^n$ for some $j, k, m, n > 0$. Using the identity $a_3^{-1}a_1 = a_2 a_3^{-1}$, we get $\beta = a_2^j a_3^{-1} a_2^k \cdots a_1^m a_2^n = a_2^j a_2 a_1^{-1} a_2^{k-1} \cdots a_1^m a_2^n$. So β is conjugate to $\beta' = a_1^{-1}a_2^{k-1} \cdots a_1^m a_2^{n+j+1}$. Obviously $\beta' \in P^a$ unless $\beta = a_3^{-1}a_1^j a_2 a_1^p a_2^q \cdots a_1^m a_2^n$. If $\beta = a_3^{-1}a_1^j a_2 a_1^p a_2^q \cdots a_1^m a_2^n$, then it conjugates to $\beta'' = a_1^p a_2^q \cdots a_1^m a_2^n$ and thus $l(\beta) \le 0$. Hence we may assume that $\beta \in P^a$. So if β' contains at least five syllabuses, then we are back to the treated case $i = 1$. Note that if $k > 1$, then $s_{\beta'} > 4$. So we may assume that $k = 1$ and that $\beta' = a_1^{-1}a_3^p a_1^m a_2^{n+j+1}$. Applying Claim 3, we see that $l(\beta) = l(\beta') \le 0$.

Suppose that δ starts with a_2 and ends with a_2. Then $\beta = a_3^{-1}a_2^k \cdots a_1^m a_2^n = a_2 a_1^{-1} a_2^{k-1} \cdots a_1^m a_2^n$ for some $k, m, n > 0$. So β is conjugate to $\beta' = a_1^{-1}a_2^{k-1} \cdots a_1^m a_2^{n+1}$. Again using Claim 3, we see that $l(\beta) = l(\beta') \le 0$.

Similarly one can deal with the case that δ starts with a_1 and ends with a_1, and the case that δ starts with a_2 and ends with a_1. The proof of Lemma 7 is now finished. $\diamondsuit$

We are now ready to prove Proposition 3, which is the content of the rest of this section. Let $\beta = a_i^{-q}\delta$ be a 3-braid in P^a as given in the proposition, whose closure is the given knot K. If some syllabus in a_3 occurring in δ has power $k > 2$, we apply the crossing change formula $(*)$ for Casson invariant to $K = \hat{\beta}$ at the second crossing of the syllabus. The link L (of two components) obtained by smoothing the crossing is the closure of a braid of the type as described in Lemma 7 and thus has non-positive linking number. So we get a new braid β' which is identical with β except with two less in power at the syllabus in a_3 and the Casson invariant of $\hat{\beta}'$ is less than or equal to that of $\hat{\beta}$. Obviously β' is still in P^a. So it suffices to show that $\hat{\beta}'$ has positive Casson invariant if it is not in the excluded set E given in Proposition 3. If β' is in the set E, then one can verify directly that the original braid β has positive Casson invariant. Similarly we may reduce the power of a syllabus in a_2 or in a_1 to one or two, using the formula $(*)$, so that the resulting new braid is still in P^a, without increasing the Casson invariant, and that if the new braid is in the set E, then the old braid has positive Casson invariant. Therefore we only need to show the proposition under the extra condition that every syllabus of δ has power one or two. We shall prove this by induction on the number n_δ.

We first consider the initial step of the induction, i.e. the case when $n_\delta = 0$. If $q = 0$, then $K = \hat{\beta} = \hat{\delta}$ is a positive knot. So by the proof of Theorem 1, Proposition 3 holds in this case. Actually in this case one can easily give a quick self-contained proof as follows. First apply the formula $(*)$ to any syllabus of β which has power larger than one so that β becomes a new braid with two less crossings and the Casson invariant of K is equal to the Casson invariant of the closure of the new braid minus some negative integer (by Lemma 5). So after several such steps, the braid β is simplified to a braid β_0 in which every syllabus has power one such that the Casson invariant of $K = \hat{\beta}$ is larger than that of $\hat{\beta}_0$ unless $\beta = \beta_0$. But $\hat{\beta}_0$ is a $(3, n)$ torus knot. The normalized Alexander polynomial of a $(3, n)$-torus knot $T(3, n)$ is

$$\Delta_{T(3,n)}(t) = \frac{(t^{3n} - 1)(t - 1)}{(t^3 - 1)(t^n - 1)t^{n-1}}.$$

Pure calculation of the second derivative of $\Delta_{T(3,n)}(t)$ valued at $t = 1$ gives

$$\frac{1}{2}\Delta''_{T(3,n)}(1) = \frac{n^2 - 1}{3}.$$

The conclusion of Proposition 3 follows in this case.

Suppose then that $q = 1$ (and $n_\delta = 0$). If β contains exactly five or six syllabuses and at least one of them has power 2 then one can verify directly that $\hat{\beta}$ has positive Casson invariant (the checking is pretty quick since every syllabus of β has power at most two and also note that β is cyclically reduced and $\hat{\beta}$ is a knot). If β has more than six syllabuses and one of them has power 2, then we apply the formula $(*)$ at such syllabus to reduce the number of syllabuses of β. If the new braid β' is not in P^a, then a cancellation must occur between a_i^{-1} and an a_i in β'. After the cancellation, we get a braid which is a positive word in a_1 and a_2 and thus its closure has positive Casson invariant unless it is a trivial knot. So Lemma 6 implies that the Casson invariant of the old knot $\hat{\beta}$ is positive. If the new braid β' contains at least seven syllabuses, then we may continue to do such simplification. Note that $s_{\beta'}$ is one or two less than s_β when β' is still in P^a. So we may assume that every syllabus of δ has power one. Hence δ looks like $\delta = (a_1a_2)^p$ or $\delta = (a_1a_2)^pa_1$ or $\delta = (a_2a_1)^p$ or $\delta = (a_2a_1)^pa_2$ for some $p \geq 2$. The case $\delta = (a_1a_2)^p$ or the case $\delta = (a_2a_1)^p$ cannot happen since in such case we must have $i = 3$ and the closure of β is then not a knot. Consider the case $\delta = (a_1a_2)^pa_1$. We have $i = 2$ or 3. If $i = 2$, then to be a knot, $\beta = a_2^{-1}(a_1a_2)^pa_1$ for $p > 2$ and $p \neq 2 \pmod 3$. Now one can verify directly that $C_{\hat{\beta}} > 0$. If $i = 3$, then $\beta = a_3^{-1}(a_1a_2)^pa_1$ is conjugate to $a_1(a_1a_2)^{p-1}a_1$ which obviously has positive Casson invariant. The case $\delta = (a_2a_1)^pa_2$ can be treated similarly. The proof of Proposition 3 for the initial step $n_\delta = 0$ is complete.

Now we may assume that $n_\delta > 0$. We warn the reader that the rest of the proof will involve more patient case counting. Nevertheless the guiding idea will still be more or less as follows: apply the crossing change formula $(*)$ for the Casson invariant and Lemma 7 at a suitable chosen crossing of the given braid $\beta = a_i^{-q}\delta$ to reduce its complexity, simplify the new braid (if necessary) to get a braid $\beta' = a_i^{-q}\delta'$ in P^a, use the induction if $s_{\delta'} \geq 4$ and β' is not in the excluded set

$$E = \left\{ \begin{array}{llll} a_1^{-1}a_2a_3^2a_1a_2, & a_1^{-1}a_3^2a_1a_2a_3, & a_1^{-1}a_3a_1a_2^2a_3, & a_1^{-1}a_2a_3a_1a_2^2, \\ a_2^{-1}a_3a_1a_2a_3^2, & a_2^{-1}a_3a_1^2a_2a_3, & a_2^{-1}a_1a_2a_3^2a_1, & a_2^{-1}a_1^2a_2a_3a_1, \\ a_3^{-1}a_1a_2a_3a_1^2, & a_3^{-1}a_1a_2^2a_3a_1, & a_3^{-1}a_2a_3a_1^2a_2, & a_3^{-1}a_2^2a_3a_1a_2 \end{array} \right\},$$

otherwise verify directly that the original braid β has positive Casson invariant. Besides we shall often make use of the conditions such as $\hat{\beta}$ is a knot, β is cyclically reduced and δ is index-3 reduced.

Claim D1. If a_3^2 is a syllabus of δ, then the Proposition holds.

Consider the first such syllabus occurring in δ. Applying the formula $(*)$ to $\hat{\beta}$ at the second crossing of the syllabus a_3^2, we get a new braid $\beta' = a_i^{-q}\delta'$, where δ' is the braid obtained from δ by deleting the syllabus a_3^2, such that $\delta' \in P^*$, $n_{\delta'} = n_\delta - 1$, and $C_{\hat{\beta}} \geq C_{\hat{\beta}'}$ (by Lemma 7). If β' is still in P^a and δ' contains at least four syllabuses but β' is not a word in the excluded set E, then we may use induction.

Suppose that $\beta' \in E$. Since the syllabus we are considering is the first such occurring in β and since δ is index-3 reduced, β can only be the word $a_2^{-1}a_3^2a_1a_2a_3^2a_1$ or $a_2^{-1}a_3^2a_1^2a_2a_3a_1$ or $a_2^{-1}a_3^2a_1a_2a_3a_1^2$ or $a_1^{-1}a_2a_3a_1a_2^2a_3^2$. In such a case one can verify directly that $C_{\hat{\beta}}$ is positive.

Hence, we may assume that either β' is still in P^a with δ' containing exactly three syllabuses, or β' is no longer in P^a. In the former case, β is a word in the set

$$\left\{ \begin{array}{lllll} a_1^ja_2^ka_3^2a_1^m, & a_2^ja_1^ka_2^ma_3^2, & a_2^ja_3^2a_1^ka_2^m, & a_3^2a_1^ka_2^ja_3^m, & a_3^2a_1^ka_2^ja_1^m, \\ a_3a_1^ka_2^ma_3^2, & a_1^{-1}a_2^ja_1^ka_2^ma_3^2, & a_1^{-1}a_2^ja_3^2a_1^ka_2^m, & a_1^{-1}a_3a_1^ka_2^ma_3^2, & a_2^{-1}a_1^ja_2^ka_3^2a_1^m, \\ a_2^{-1}a_3^2a_1^ja_2^ka_3^m, & a_2^{-1}a_3^2a_1^ka_2^ja_1^m, & a_3^{-1}a_1^ja_2^ka_3^2a_1^m, & a_3^{-1}a_2^ja_3^2a_1^ka_2^m & \end{array} \right\}$$

for some $j, k, m \in \{1, 2\}$. When β is one of words in this set but is not in the excluded set E, one can verify directly using the formula $(*)$ that $\hat{\beta}$ has positive Casson invariant. For instance, when $\beta = a_1^{-1}a_2^ja_1^ka_2^ma_3^2$, it is conjugate to $a_2^ja_1^ka_2^ma_3^2a_1^{-1} = a_2^ja_1^ka_2^ma_1^{-1}a_2^2$ which in turn is conjugate to $a_1^{-1}a_2^{j+2}a_1^ka_2^m$. So we need to show that the Casson invariant of the closure of the braid $\eta = a_1^{-1}a_2^{j+2}a_1^ka_2^m$ is positive. We have eight possible cases for η corresponding to various possible values of j, k and m. But only in case $j = k = m = 1$ or case $j = 1, k = m = 2$ or case $j = k = 2, m = 1$, the closure of the involved braid is a knot, and in such a case one can verify directly using formula $(*)$ that $\hat{\beta} = \hat{\eta}$ has positive Casson invariant. In a similar way one can verify the proposition for each of the other words in the above set. (We did the checking!)

We now consider the latter case when β' is not in P^a. It follows that the syllabus a_3^2 is either the first or the last syllabus of δ and $q = 1$. We consider the case when the syllabus a_3^2 is the first syllabus of δ. The case when the syllabus a_3^2 is the last syllabus of δ can be treated similarly. It follows that $\beta = a_1^{-1}a_3^2a_1^ja_2^k\cdots$, for some $j,k \in \{1,2\}$, and β does not end with a_1. If $j = 2$, then $\beta' = a_1^{-1}a_1^2a_2^k\cdots$ which is isotopic to $\beta'' = a_1a_2^k\cdots$ which is in P^a. So if β'' contains at least four syllabuses, we may use the induction. We may then assume that β'' has less than four syllabuses. It follows that $\beta'' = a_1a_2^ka_3^m$ which is a knot only when $k = 2, m = 1$ or $k = 1, m = 2$ and in these two cases $C_{\hat\beta} \geq C_{\hat\beta''} > 0$. So we may assume that $j = 1$. In this case β' is isotopic to $\beta'' = a_2^k\cdots$ which is in P^a. Hence if β'' contains at least four syllabuses, we may use the induction. If β'' has less than four syllabuses, then we must either have $\beta = a_1^{-1}a_3^2a_1a_2^ka_1^na_2^m$ or $\beta = a_1^{-1}a_3^2a_1a_2^ma_3^j$. In the former case, β'' has positive Casson invariant (a nontrivial positive knot). In the latter case, only when $m = 1, j = 1$, $\hat\beta$ is a knot. But this braid is in the set E. The proof of Claim D1 is now complete.

By Claim D1, we may now assume that every syllabus in a_3 occurring in δ has power equal to one.

Claim D2. We may assume that every syllabus in a_1 occurring in δ has power equal to one.

Suppose that δ contains syllabuses in a_1 of power two. Consider the first such syllabus occurring in δ. Applying the formula $(*)$ to $\hat\beta$ at the first crossing of the syllabus a_1^2, we get a new braid $\beta' = a_i^{-q}\delta'$ such that the Casson invariant of $\hat\beta'$ is less than or equal to that of $\hat\beta$ by Lemma 7. Obviously δ' is still a positive word in a_1, a_2, a_3 but may not be in P^* or P^a. We have several possibilities for δ around the given syllabus a_1^2: $\delta = \cdots a_2^ja_1^2a_2^k\cdots$ or $\delta = \cdots a_3a_1^2a_2^k\cdots$ or $\delta = a_1^2a_2^k\cdots$ or $\delta = \cdots a_1^2$, for some $j,k \in \{1,2\}$.

Case (D2.1). $\delta = \cdots a_2^ja_1^2a_2^k\cdots$.

Then $\delta' = \cdots a_2^{j+k}\cdots$ and $\beta' = a_i^{-q}\delta'$ is still in P^a. Also if $j+k > 2$, we may apply the formula $(*)$ one more time to bring it down to one or two. Let $\beta'' = a_i^{-q}\delta''$ be the resulting braid. Then if δ'' contains at least four syllabuses and β'' is not in E, then we have eliminated one a_1^2. If β'' is in E, then one can verify directly that the original braid β has positive Casson invariant. Note that $s_{\delta'} = s_\delta - 2$. Suppose that $s_{\delta'} < 4$. Then s_δ is four or five and $\beta = a_i^{-q}a_2^ja_1^2a_2^ka_3$ or $\beta = a_i^{-q}a_1a_2^ja_1^2a_2^ka_3$ or $\beta = a_i^{-q}a_3a_1a_2^ja_1^2a_2^k$. Easy to check that when $q = 0$, any knot from these cases has positive Casson invariant. So assume that $q = 1$ in these cases.

If $\beta = a_i^{-1}a_2^ja_1^2a_2^ka_3$, then $i = 1$ since β is cyclically reduced. So $\beta = a_1^{-1}a_2^ja_1^2a_2^ka_3$. To be a knot, we have $k = j = 1$ or $k = j = 2$. In each of the two cases, one can verify directly that the Casson invariant of the knot is positive.

If $\beta = a_i^{-q}a_1a_2^ja_1^2a_2^ka_3$, then $i = 2$. That is $\beta = a_2^{-1}a_1a_2^ja_1^2a_2^ka_3$. One can also directly verify that $C_{\hat{\beta}} > 0$ (for those values of $k, j \in \{1,2\}$ which make $\hat{\beta}$ a knot).

The case that $\beta = a_i^{-q} a_3 a_1 a_2^j a_1^2 a_2^k$ can be treated similarly.

Case (D2.2). $\delta = \cdots a_3 a_1^2 a_2^k \cdots$.

Then $\delta' = \cdots a_3 a_2^k \cdots = \cdots a_2 a_1 a_2^{k-1} \cdots$.

Case (D2.2.1.) $k = 2$.

If δ does not start with a_3, then $\beta' = a_i^{-q} \delta' = a_i^{-q} \cdots a_2^{j+1} a_1 a_2 \cdots$ is still in P^a and is not in the set E. In such case if δ' contains at least four syllabuses, we may apply induction since $n_{\delta'} = n_\delta - 1$. If $s_{\delta'} < 4$, then $\beta = a_i^{-q} a_2^j a_3 a_1^2 a_2^2$. To be a knot, we have $q = 0, j = 1$ or $q = 1, i = 1, j = 2$. In each of the two cases we have $C_{\hat{\beta}} > 0$ by direct calculation.

If δ starts with a_3 but $q = 0$, then $\beta' = \delta'$ is still in P^a but not in E. Also $s_\beta = s_\delta = s_{\beta'} = s_{\delta'}$ and $n_{\delta'} = s_\delta - 1$. So we may apply induction.

If δ starts with a_3 and $q = 1$, then $i = 1$ or 2. If $i = 1$, then $\beta' = a_1^{-1} a_2 a_1 a_2 \cdots$ is still in P^a but not in E and contains at least five syllabuses. So we may use induction since $s_{\delta'} = s_\delta$ but $n_{\delta'} = n_\delta - 1$. If $i = 2$, then $\beta' = a_1 a_2 \cdots$ is in P^a and is not in E. Also $s_{\beta'} = s_\delta - 1$. Hence if $s_\delta > 4$, we may use induction. So suppose that $s_\delta = 4$. Then $\beta = a_2^{-1} a_3 a_1^2 a_2^2 a_3$ or $\beta = a_2^{-1} a_3 a_1^2 a_2^2 a_1^j$. But the former is not a knot. The latter is a knot when $j = 2$, in which case $C_{\hat{\beta}} > 0$.

Case (D2.2.2). $k = 1$.

Then $\delta = \cdots a_3 a_1^2 a_2 a_3 \cdots$ or $\delta = \cdots a_3 a_1^2 a_2 a_1^j \cdots$ or $\delta = \cdots a_3 a_1^2 a_2$. Correspondingly, we have $\delta' = \cdots a_2 a_1 a_3 \cdots = \cdots a_2^2 a_1 \cdots$ or $\delta = \cdots a_2 a_1^{j+1} \cdots$ or $\delta = \cdots a_2 a_1$. If β' is in $P^a - E$ and $s_{\delta'} \geq 4$, we may use induction since $n_{\delta'} < n_\delta$. If $\beta' \in E$, then one can check that the old braid β has positive Casson invariant. If $\beta' \in P^a$ but $s_{\delta'} < 4$, then one can also verify that β always has positive Casson invariant, applying the conditions that (1) $\beta \in P^a - E$, (2) $s_\delta \geq 4$ and (3) $\hat{\beta}$ is a knot.

Case (D2.3). $\delta = a_1^2 a_2^k \cdots$.

Then $\delta = a_1^2 a_2^k a_1^j a_2^m \cdots$ or $\delta = a_1^2 a_2^k a_3 a_1^j \cdots$. And $\delta' = a_2^k a_1^j a_2^m \cdots$ or $\delta' = a_2^k a_3 a_1^j \cdots$. So β' is in P^a unless β starts with a_2^{-1}. If β starts with a_2^{-1}, then $\beta' = a_2^{k-1} a_1^j a_2^m \cdots$ or $\beta' = a_2^{k-1} a_3 a_1^j \cdots$ which is in P^a. Again in each of these cases, if β' is the set E or $s_{\delta'} < 4$, one can calculate directly that the old braid β has positive Casson invariant. Otherwise one can use the induction.

Case (D2.4). $\delta = \cdots a_1^2$.

This case can be treated similarly as in the previous case.

So by Claims D1 and D2, we now assume that every syllabus in a_3 and in a_1 occurring in δ have power equal to one.

Claim D3. We may assume that every syllabus in a_2 occurring in δ has power equal to one.

This can be proved similarly as Claim D2.

So we now assume that every syllabus occurring in δ has power one.

If $a_1a_2a_1$ appears immediately after an a_3, then $\beta = \cdots a_3a_1a_2a_1 \cdots = \cdots a_3a_1a_3a_2 \cdots = \cdots a_3a_2a_1a_2 \cdots = \cdots a_2a_1^2a_2 \cdots$ and so we get an isotopic braid in P^a with less number of syllabuses in a_3. Hence we may apply induction unless β is in the set $\{a_3a_1a_2a_1,\ a_2^{-1}a_3a_1a_2a_1,\ a_3^{-1}a_2a_3a_1a_2a_1,\ a_1^{-1}a_3a_1a_2a_1a_2,\ a_1^{-1}a_2a_3a_1a_2a_1a_2,\ a_3^{-1}a_2a_3a_1a_2a_1a_2\}$. If β is a word in this set, then either $\hat{\beta}$ is not a knot or $\hat{\beta}$ has positive Casson invariant. So we may assume that no $a_3a_1a_2a_1$ occurs in β. A similar argument shows that we may assume that no $a_2a_1a_2a_3$ occurs in β. Hence we may assume that β is one of the words in the set

$$\left\{\begin{array}{llll}
a_1a_2a_3a_1, & (a_3a_1a_2)^m, & (a_3a_1a_2)^m a_3, & (a_3a_1a_2)^m a_3a_1, \\
a_2(a_3a_1a_2)^m, & a_1a_2(a_3a_1a_2)^m, & a_2(a_3a_1a_2)^m a_3, & a_2(a_3a_1a_2)^m a_3a_1, \\
a_1a_2(a_3a_1a_2)^m a_3, & a_1a_2(a_3a_1a_2)^m a_3a_1, & a_2^{-1}a_1a_2a_3a_1, & a_3^{-1}a_1a_2a_3a_1, \\
a_1^{-1}(a_3a_1a_2)^m, & a_1^{-1}(a_3a_1a_2)^m a_3, & a_2^{-1}(a_3a_1a_2)^m a_3, & a_2^{-1}(a_3a_1a_2)^m a_3a_1 \\
a_1^{-1}a_2(a_3a_1a_2)^m, & a_3^{-1}a_2(a_3a_1a_2)^m, & a_3^{-1}a_1a_2(a_3a_1a_2)^m, & a_1^{-1}a_2(a_3a_1a_2)^m a_3, \\
a_3^{-1}a_2(a_3a_1a_2)^m a_3a_1 & a_2^{-1}a_1a_2(a_3a_1a_2)^m a_3, & a_2^{-1}a_1a_2(a_3a_1a_2)^m a_3a_1, & a_3^{-1}a_1a_2(a_3a_1a_2)^m a_3a_1
\end{array}\right\}$$

where $m > 0$.

If $\beta = a_1a_2a_3a_1$, then it has positive Casson invariant. The case $\beta = (a_3a_1a_2)^m$ cannot occur since its closure is not a knot for all $m > 0$. Similarly each of the cases $a_2(a_3a_1a_2)^m a_3a_1$, $a_1a_2(a_3a_1a_2)^m a_3$, $a_2^{-1}a_1a_2a_3a_1$, $a_3^{-1}a_1a_2a_3a_1$, $a_1^{-1}(a_3a_1a_2)^m$, $a_2^{-1}(a_3a_1a_2)^m a_3a_1$ and $a_1^{-1}a_2(a_3a_1a_2)^m a_3$ cannot occur as β. If $\beta = (a_3a_1a_2)^m a_3$, then it is conjugate to $\beta' = a_3^2a_1a_2(a_3a_1a_2)^{m-1}$. So we may apply Claim D1 (note that $n_{\beta'} < n_\beta$) unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_3a_1a_2a_3$ has positive Casson invariant. If $\beta = (a_3a_1a_2)^m a_3a_1$, then it is conjugate to $\beta' = a_2a_1^3a_2(a_3a_1a_2)^{m-1}$. So we are back to a previous case (note that $n_{\beta'} < n_\beta$) unless $m = 1$. But when $m = 1$, $\hat{\beta}$ is not a knot. If $\beta = a_2(a_3a_1a_2)^m$, then it is conjugate to $\beta' = (a_3a_1a_2)^{m-1}a_3a_1a_2^2$. So we may apply Claim D2 (note that $n_{\beta'} = n_\beta$) unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_2a_3a_1a_2$ has positive Casson invariant. Similarly one can deal with the cases $a_1a_2(a_3a_1a_2)^m$, $a_2(a_3a_1a_2)^m a_3$ and $a_1a_2(a_3a_1a_2)^m a_3a_1$.

Suppose that $\beta = a_1^{-1}(a_3a_1a_2)^m a_3$. To be a knot, we have $m \geq 3$ and $m = 3(mod 2)$. Also β is conjugate to $\beta' = a_1^{-1}a_2(a_3a_1a_2)^m$ which has less number of syllabuses in a_3 and thus we may apply the induction. Similarly one deal with the case when $\beta = a_2^{-1}(a_3a_1a_2)^m a_3$.

If $\beta = a_1^{-1}a_2(a_3a_1a_2)^m$, then to be a knot m must be even. Applying the formula $(*)$ at the first crossing of β, we get $C_{\hat{\beta}} = C_{\hat{\beta}_1} + lk(\hat{\lambda}_1)$ where $\beta_1 = a_1a_2(a_3a_1a_2)^m$ and $\lambda_1 = a_2(a_3a_1a_2)^m$. One can easily deduce from the projection of λ_1 that $lk(\hat{\lambda}_1) = -m/2$.

Hence, we have $C_{\hat\beta} = C_{\hat\beta_1} - m/2$. In such case it suffices to show the following.

Claim D4. $C_{\hat\beta_1} > m/2$.

We knew that $m = 2p$ with $p > 0$. We shall prove the claim by induction on the number p. Write β_1 as $\beta_1 = a_1a_2a_1^{-1}a_2a_1^2a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. Applying formula $(*)$ to β_1 at the first crossing of the first syllabus a_1^2, we get $C_{\hat\beta_1} = C_{\hat\beta_2} - lk(\hat\lambda_2)$ where $\beta_2 = a_1a_2a_1^{-1}a_2^2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_2 = a_1a_2a_1^{-1}a_2a_1a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. One can easily deduce from the projection of λ_2 that $lk(\hat\lambda_2) = -2(m-2) - 3 = -4p + 1$. Hence, we have $C_{\hat\beta_1} = C_{\hat\beta_2} + 4p - 1$. We then apply formula $(*)$ to β_2 at the first crossing of the first syllabus a_2^2, we get $C_{\hat\beta_2} = C_{\hat\beta_3} - lk(\hat\lambda_3)$ where $\beta_3 = a_1a_2a_1^{-2}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_3 = a_1a_2a_1^{-1}a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. One can calculate to see that $lk(\hat\lambda_3) = -p - 1$. Hence, we have $C_{\hat\beta_2} = C_{\hat\beta_3} + p + 1$. We then apply formula $(*)$ to β_3 at the first crossing of the syllabus a_1^{-2}, we get $C_{\hat\beta_3} = C_{\hat\beta_4} + lk(\hat\lambda_4)$ where $\beta_4 = a_1a_2^2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_4 = a_1a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. Also one can calculate to see that $lk(\hat\lambda_4) = -p$. Hence, we have $C_{\hat\beta_3} = C_{\hat\beta_4} - p$. We then apply formula $(*)$ to β_4 at the first crossing of the syllabus a_2^2, we get $C_{\hat\beta_4} = C_{\hat\beta_5} - lk(\hat\lambda_5)$ where $\beta_5 = a_1^3a_2(a_3a_1a_2)^{m-2}$ and $\lambda_5 = a_1a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. We also have $lk(\hat\lambda_5) = -4(p-1) - 2$. Hence we have $C_{\hat\beta_4} = C_{\hat\beta_5} + 4(p-1) + 2$. Applying formula $(*)$ to β_5 at the first crossing of the syllabus a_1^2, we get $C_{\hat\beta_5} = C_{\hat\beta_6} - lk(\hat\lambda_6)$ where $\beta_6 = a_1a_2(a_3a_1a_2)^{m-2}$ and $\lambda_6 = a_1^2a_2(a_3a_1a_2)^{m-2}$. We also have $lk(\hat\lambda_6) = -(p-1)-1 = -p$. Hence we have $C_{\hat\beta_5} = C_{\hat\beta_6} + p$. In summary, we get

$$C_{\hat\beta_1} = C_{\hat\beta_6} + 9p - 2.$$

Now one can easily see that the claim follows.

Similarly we can treat the rest of cases. The proof of Proposition 3 is now complete.

4. Proof of Proposition 4

Given a 3-braid β in letters a_1, a_2, a_3, whose closure is a knot, there is a canonical way to construct a Seifert surface for $\hat\beta$ as follows: in the projection plane we have the braid diagram in its canonical form, place three rectangular disks in the space so that disk 1 lies in the projection plane and is on the left hand side of the braid, disk 3 also lies in the projection plane but on the right hand side of the braid, disk 2 lies perpendicularly above the projection plane, each disk having one side running parallel to the braid from top to the bottom, then to each letter a_1 (a_1^{-1}) occurring in β use a half negatively (positively) twisted band connecting disks 1 and 2, to each letter a_2 (a_2^{-1}) use a half negatively (positively) twisted band connecting disks 2 and 3, and to each letter a_3 (a_3^{-1}) use a half negatively (positively) twisted band connecting disks 1 and 3 (behind disk 2). Figure 7 illustrates such construction for $\beta = a_1a_2a_3a_1^{-1}a_3^{-1}a_2^{-1}$. We call the Seifert surface of $\hat\beta$ so constructed *canonical* Seifert surface of $\hat\beta$. In [X], it was proved that if β is a 3-braid of norm form (whose

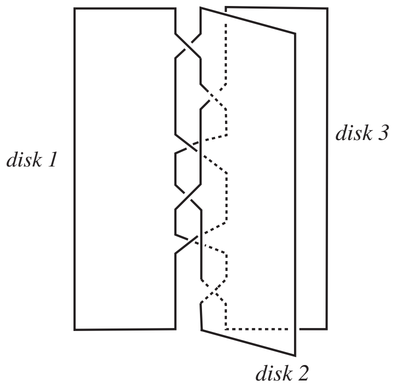

Figure 7: construction of the canonical Seifert surface

definition we recalled in Section 2), then its canonical Seifert surface has the minimal genus (thus is an essential and Thurston norm minimizing surface in the exterior of $\hat{\beta}$).

Lemma 8 *Let β be a 3-braid such that $\hat{\beta}$ is a knot. Let S be the canonical Seifert surface of β and suppose that it has minimal genus.*
(1) If S contains two half twisted bands corresponding to the same letter a_1 or a_2 or a_3, then the -1-surgery on $\hat{\beta}$ is a manifold with essential lamination.
(2) If S contains two half twisted bands corresponding to the same letter a_1^{-1} or a_2^{-1} or a_3^{-1}, then the 1-surgery on $\hat{\beta}$ is a manifold with essential lamination.

Proof. We shall only prove part (1) when S contains two half twisted bands corresponding to the same letter a_1. All other cases can be proved similarly.

Let M be the knot exterior of $\hat{\beta}$ in S^3. We shall also use $M(-1)$ to denote the manifold obtained by Dehn surgery on the knot $\hat{\beta}$ with the slope -1. Let V be the solid torus filled in M to obtain the manifold $M(-1)$. We first construct an essential branched surface B in the exterior M and then prove that B (which has boundary on ∂M) can be capped off by a branched surface in V to yield an essential branched surface $\hat{B}$ in $M(-1)$. The construction of B is similar to that given in [Rs].

Since S contains two half negatively twisted bands corresponding to a_1, there is a disk D in M as shown in Figure 8 (1) whose boundary lies in $S \cup \partial M$ and whose interior is disjoint from $S \cup \partial M$. With more detail, the boundary of ∂D intersects S in two disjoint arcs and intersects ∂M in two disjoint arcs with the latter happening around the places corresponding to the two bands of a_1. (Similar disks were used in [BMe] for a different purpose). The branched surface B is the union of the Seifert surface S and the disk D with their intersection locus smoothed as shown in Figure 8 (2). The arrows in the figure indicate the cusp direction of branched locus. An argument as in [Rs] shows that the branched surface fully carries a lamanition with no compact leaves and each negative slope

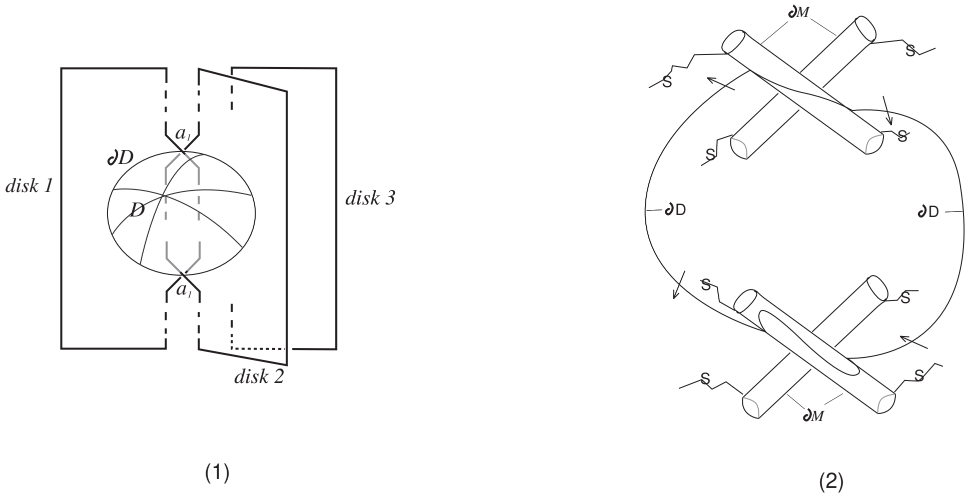

Figure 8: construction of B

can be realized as the boundary slope of a lamination fully carried by B. Since the disk D intersects the knot exactly twice, the branched surface B is essential in M by [G2, 3.12]. The branched locus of B is a set of two disjoint arcs properly embedded in S, each being non-separating. The branched surface B meets ∂M yielding a train track in ∂M as show in Figure 9 (1). Let $\mathcal{L}$ be a lamination fully carried by B whose boundary slope is -1. Then $\partial\mathcal{L}$ must look like as shown in Figure 9 (2).

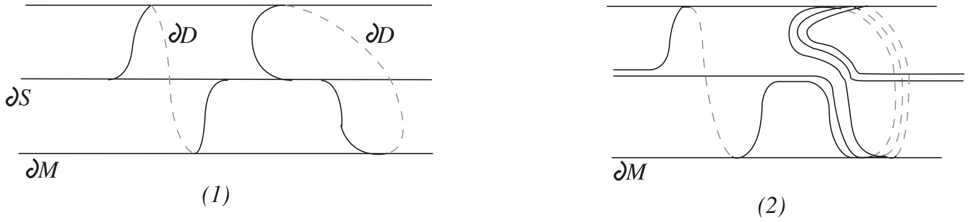

Figure 9: (1) ∂B on ∂M (2) the curve of slope -1 fully carried by ∂B

We now construct a branched surface B_V in the sewn solid torus V such that the train track $B_V \cap \partial V$ is equivalent to the train track $B \cap \partial M$ and B_V fully carries a lamination which is a set of meridian disks of V. Hence, B and B_V match together and form a branched surface $\hat{B}$ in $M(-1)$. Take a meridian disk D_0 of V and push part of it near and around ∂V as shown in Figure 10 (1) and then identify two disjoint sub-disks of D_0 as shown in Figure 10 (2). This gives a branched surface B_1 with the cusp direction along its singular locus (an arc) as shown in Figure 10 (2). Then we split B_1 locally at a place around a point of ∂B_1 as shown in Figure 10 (3) and then we start pinch the resulting branched surface along part of its boundary as shown in Figure 10 (4). The pinching continues as shown in Figure 10 (5) until we get the branch surface whose boundary is as shown in Figure 10 (6). The resulting surface is the branched surface B_v. Obviously the train track ∂B_v on ∂V is equivalent to the train track ∂B in ∂M and they can be matched in $\partial V = \partial M$.

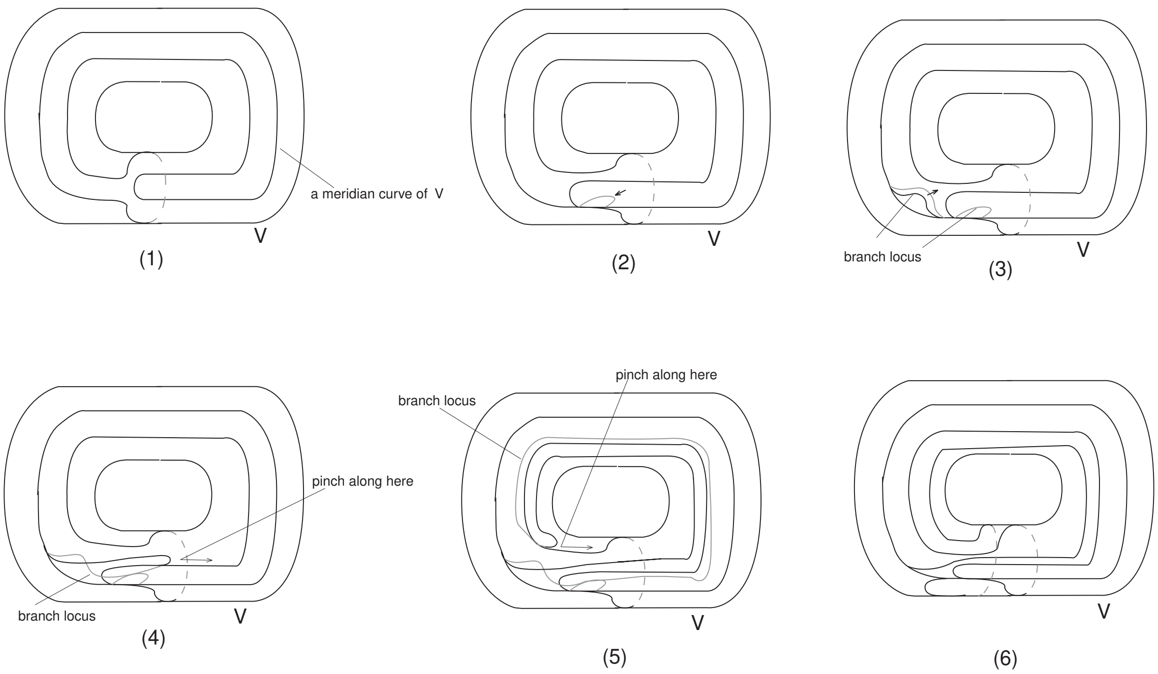

Figure 10: construction of B_V

To see that $\hat{B}$ is essential we have five things to check by [GO, Definition 2.1]:
(i) $\hat{B}$ has no discs of contact;
(ii) The horizontal surface $\partial_h N(\hat{B})$ is incompressible and ∂-incompressible in $M(-1)-\mathring{N}(\hat{B})$, there are no monogons in $M(-1)-\mathring{N}(\hat{B})$ and no component of $\partial_h N(\hat{B})$ is a 2-sphere;
(iii) $M(-1)-\mathring{N}(\hat{B})$ is irreducible;
(iv) $\hat{B}$ contains no Reeb branched surface;
(v) $\hat{B}$ fully carries a lamination.

Condition (v) follows automatically by the construction since leaves of a lamination fully carried by B with boundary slope -1 match on $\partial M = \partial V$ with (disk) leaves of a lamination fully carried by B_V. It also follows that $\hat{B}$ does not carry any compact surface since B does not. Hence in particular condition (iv) holds also for $\hat{B}$. By the construction, one can easily see that $V-\mathring{N}(B_v)$ has two components, each of which topologically looks like as shown in Figure 11. It follows that $M(-1)-\mathring{N}(\hat{B})$ is topologically the same as $M-\mathring{N}(B)$, with the same horizontal surface. From this we get conditions (ii) and (iii) for $\hat{B}$.

We now show that $\hat{B}$ has no disk of contact. Note that $\partial_v(N(\hat{B}))$ is a set of two annuli and each of the annuli is obtained from matching a component of $\partial_v(N(B))$ (a vertical disk) and a component of $\partial_v(N(B_V))$ (a vertical disk). Hence, if D_c were a contact disk in $N(\hat{B})$, then its boundary would have to intersect a component of $\partial_v(N(B))$. It follows then that the interior of D_c must enter into the region of $N(B)$ correspond to a branch of $(S-$

the singular locus of B). But one can easily see from Figure 8 that the complement of the singular locus of B in S is a connected surface. It follows that D_c has to intersect every I-fiber of B since D_c is transverse to I-fibers of B. In particular ∂D_c has to intersect both of the vertical annuli of $\partial_v(\hat{B}_v)$, which gives a contradiction. $\diamondsuit$

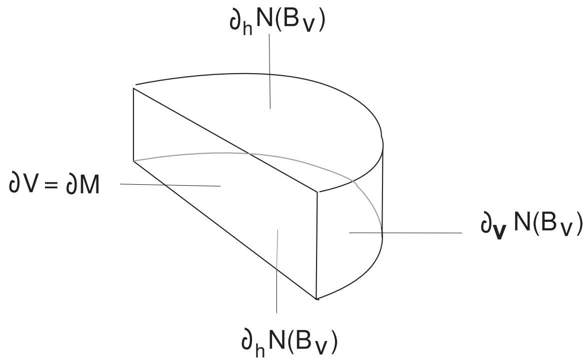

Figure 11: a component of $V - \overset{\circ}{N}(B_v)$

We now prove Proposition 4. By [X], the canonical Seifert surface of $\hat{\beta}$ has minimal genus. If the condition (1) of Proposition 4 holds, then the conclusion of Proposition 4 follows obviously from Lemma 8. Suppose that the condition (2) of Proposition 4 holds. To show that the 1-surgery on $\hat{\beta}$ gives a manifold with an essential lamination, we may assume, by Lemma 8, that η contains at most three syllabus, and they have different subscripts and all have power -1. But η contains at least two syllabuses. Suppose that the first and the second syllabuses of η are a_3^{-1} and a_2^{-1}, i.e. $\eta = a_3^{-1}a_2^{-1}\cdots$. Then since β is a shortest word, the word δ does not end with a syllabus in a_3. Suppose that δ ends with a syllabus in a_1. Then we have $\beta = \cdots a_1 a_3^{-1} a_2^{-1} \cdots$. By a band move isotopy of the Seifert surface as shown in Figure 12 (1) we get an isotopic 3-braid β' which contains two a_2^{-1}. (Algebraically, $\beta = \cdots a_1 a_3^{-1} a_2^{-1} \cdots = \cdots a_2^{-1} a_1 a_2^{-1} \cdots = \beta'$). Further the canonical Seifert surface of $\hat{\beta}'$ is isotopic to that of $\hat{\beta}$ and thus has minimal genus. So we may apply Lemma 8 to see that for the knot $\hat{\beta}' = \hat{\beta}$, the 1-surgery gives a manifold with essential lamination. Suppose then that δ ends with a syllabus in a_2. Since δ is assumed to contain at least two syllabuses, $\beta = \cdots a_1 a_2^k a_3^{-1} a_2^{-1} \cdots$. Again we may first use the band-isotopy as shown in Figure 12 (2) and then use the band isotopy of Figure 12 (1) to get an isotopic 3-braid whose canonical Seifert surface contains two bands corresponding to a_2^{-1}. Hence Proposition 4 follows from Lemma 8 in this case as well. Similarly one can treat the cases when η starts with $a_2^{-1}a_1^{-1}$ or with $a_1^{-1}a_3^{-1}$. The case when δ contains at most three syllabuses, each having power at most one, can be proved similarly. This proves Proposition 4 under its condition (2). Finally if the condition (3) of Proposition 4 holds then either condition (2) of Proposition 4 holds or one can get directly two letters a_i of the same subscript in δ and two letters a_j^{-1} of the same subscript in η. So again the Proposition follows from Lemma 8. $\diamondsuit$

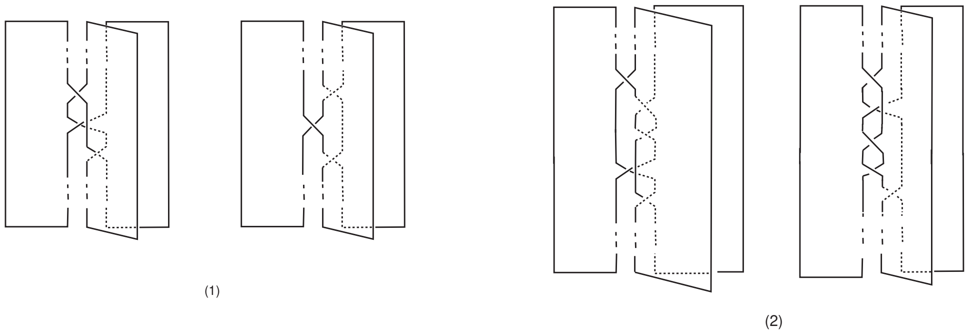

Figure 12: the band move isotopies

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