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property such as the existence of the steady-state solution or symmetry. Numerical examples for elliptic and hyperbolic equations are provided.
On the spectral collocation approximation of the discontinuous solution of singularly perturbed differential equations in one dimension

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Abstract

The solution of singularly perturbed differential equations contains the local jump discontinuity and its spectral approximation is oscillatory due to the Gibbs phenomenon in general. To minimize the Gibbs oscillations near the local jump discontinuity and improve convergence, the regularization of the approximation is needed. In this brief note, a simple derivative of the discrete Heaviside function $H_c(x)$ on the collocation points is used for the approximation of singular source terms $\delta(x - c)$ or $\delta^{(n)}(x - c)$ without any regularizations. The direct projection of $H_c(x)$ yields highly oscillatory approximations of $\delta(x - c)$ and $\delta^{(n)}(x - c)$. In this note, however, it is shown that the direct projection approach can yield a non-oscillatory approximation and the error can also decay uniformly for certain types of differential equations. For some differential equations, spectral accuracy is also recovered. This method is limited to some types of equations but can be applied when the given equation has a nice property such as the existence of the steady-state solution or symmetry. Numerical examples for elliptic and hyperbolic equations are provided.

Key words: Singularly perturbed differential equations, Spectral collocation method, $\delta$-distribution, Gibbs phenomenon, Convergence.

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1 Introduction

Differential equations with singular sources are found in various physical and engineering applications. Singular source terms are usually described by the Dirac delta function, $\delta(x-c)$ and its derivatives $\delta^{(n)}(x-c)$ where the singularity exists at $x=c$. For example, in acoustics the sound source in a distance is usually approximated as a $\delta$-function. For relativistic problems, a single star moving around the massive black hole can be treated as a point source. The wave equations of such system have the $\delta$-function and its derivative in the source terms [5]. Solutions of these singular differential equations are, in general, discontinuous. High order approximations of such discontinuous functions suffer from the Gibbs phenomenon and they are highly oscillatory near the jump discontinuity. The jump discontinuity can also deteriorate the stability when time-dependent problems are considered. Convergence is also degraded accordingly [2,3]. To minimize the Gibbs oscillations near the jump discontinuity in high order approximations, the regularization of singular source terms is necessary. In [10], it is shown that the right regularization of the singular source term can improve convergence of the solution for some elliptic equations with the finite difference methods. In [1,8], more general framework has been provided with the finite difference methods also in the context of the level set method (see these references for the brief review of the previous research on the approximation of singular source terms).

For spectral methods, filtering or post-processing techniques can be recommended for the regularization of the approximation and singular source terms. Such regularization of the spectral data yields, however, only $O(1)$ convergence due to the local jump discontinuity. In this note, we consider the spectral collocation approximation of some singularly perturbed differential equations and show that the direct projection method explained below can yield an accurate result. The $\delta$ distribution and its derivatives are used to represent the singular source terms.

The spectral collocation method seeks an approximation by requiring the discrete residue to be orthogonal to the polynomial subspace on the collocation points. The approximation of $\delta$ and its derivatives are then also defined on the collocation points. We consider the several candidates of the spectral approximation of the $\delta$-function before we consider the approximation of the $\delta$-function based on the direct projection of the Heaviside function.

Let $\{x_j\}_{j=0}^N$, and $\{u_j\}_{j=0}^N$ be the collocation points and the corresponding spectral approximation, respectively on the polynomial space $\mathcal{B}_N$, for example, on the polynomial space expanded by the Chebyshev polynomials $T_l(x)$, $l =$
0, \cdots, N. The interpolation \( u_N(x) \) in \( \mathcal{B}_N \) based on \( \{u_j\}_{j=0}^N \) is then given by

\[
u_N(x) = \sum_{j=0}^N l_j(x)u_j,
\]

where \( l_j(x) \) is known as the cardinal function which has the property \( l_j(x_i) = \delta_{ji} \) by definition. Here \( \delta_{ji} \) is the Kronecker delta in the usual sense. The derivative of \( u(x) \) on the collocation points is then given by

\[
u_N(x_i) = \sum_{j=0}^N l_j'(x_i)u_j = \sum_{j=0}^N D_{ij}u_j,
\]

where the superscript \( ' \) denotes the first derivative of \( l_j(x) \) with respect to \( x \) and \( D \) denotes the differential operator. It can be easily shown that the \( j \)th column of \( D \) is the derivative of \( l_j(x) \) on the collocation points \([2,3,9]\). As they are obtained from the spectral approximation of the \( \delta \)-function, the cardinal functions \( l_j(x) \) can be considered as the approximation of \( \delta(x), \delta_N^c(x) \). For the Chebyshev method on the Gauss-Lobatto collocation points,

\[
\delta_N^c(x) = [(-1)^{N+j+1}(1 - x^2)T_j'(x)]/[\bar{c}_jN^2(x - x_j)]
\]

where \( \bar{c}_j = 2 \) if \( j = 0, N \) and \( \bar{c}_j = 1 \) if \( j = 1, \cdots, N - 1 \) \([3]\). For the Fourier case, the Dirichlet kernel is also known as \( \delta_N^c(x) = \frac{1}{2\pi} \frac{\sin[(N+1/2)x]}{\sin(x/2)} \). Let the discretized \( \delta \)-function be \( 1_j^N = (0, \cdots, 0, N, 0, \cdots, 0)^T \) where the \( j \)th element of \( 1_j^N \) is \( N \) and other elements all vanish. Although this approach can provide a nice approximation of the \( \delta \)-function on the collocation points, it can be defined only at the collocation points, that is, \( l_j \) can not be used when \( c \) is not one of the collocation points for \( \delta(x - c) \). Furthermore, this approach results in highly oscillatory solution when applied to the differential equation and overall convergence is also poor as shown in the following section.

The second approach is to use more regularized \( \delta \)-function. For example, one can use the Gaussian type distribution

\[
\delta_N^c(x) = \frac{1}{\epsilon\sqrt{2\pi}} \exp(-\frac{x^2}{2\epsilon^2})
\]

where the constant \( \epsilon \) is given by, for example, \( \epsilon = 2/N \) and

\[
\lim_{N \to \infty} \int_{-1}^1 \delta_N^c(x)dx = 1.
\]

This approach yields more regularized \( \delta \)-function approximation on the collocation points. The parameter \( \epsilon \), however, can not be chosen arbitrarily small. The parameter \( \epsilon \) should be chosen properly so that \( \delta_N^c(x) \) can be smoothly
defined on the collocation points to guarantee the regularity within the approximation with given $N$. Furthermore, as we will show later in the following section, this approach yields only a first order convergence with $O(1)$ convergence near the jump discontinuity.

We can also consider the Galerkin approximation of the $\delta$ function. The Galerkin approach yields a relatively efficient algorithm when approximating the singular differential equation due to the properties of $\delta$ and its derivatives. Let $\delta_N^G(x - c)$ be the Chebyshev Galerkin approximation of the $\delta$-function, that is,

$$
\delta_N^G(x - c) = \sum_{l=0}^{N} \hat{\delta}_l T_l(x),
$$

where the expansion coefficients $\hat{\delta}_l$ is given by the Galerkin projection of $\delta$,

$$
\hat{\delta}_l = \frac{1}{c_l} \int_{-1}^{1} \frac{\delta(x - c)T_l(x)}{\sqrt{1 - x^2}} dx = \frac{1}{c_l \sqrt{1 - c^2}} T_l(c),
$$

where $c_l = \pi$ if $l = 0$ and $c_l = \frac{\pi}{2}$ otherwise. The above Galerkin approximation can be then evaluated at the collocation points $x_i$ when the given differential equations are approximated with the spectral collocation method. Alternatively, the Galerkin projection of the given differential equation onto $B_n$ can be also considered. As we will show in the following sections, however, such spectral Galerkin approximation also suffers from the Gibbs phenomenon and its convergence is only $O(1)$ near the local jump discontinuity.

Finally we consider the direct projection of the $\delta$-function onto $B_N$ at the collocation points $\{x_j\}_{j=0}^{N}$ with $x \in [-1, 1]$. By definition, we have

$$
\int_{-1}^{x} \delta(x - c) dx = H_c(x),
$$

where $c \in (-1, 1)$ and $H_c(x)$ is the Heaviside function defined by

$$
H_c(x) = \begin{cases} 
0, & x < c \\
1, & x > c
\end{cases}.
$$

Then $\delta(x - c)$ is given by

$$
\delta(x - c) = \frac{d}{dx} H_c(x).
$$

On the collocation points, $H_c(x_j) = 1$ if $x_j > c$ and $H_c(x_j) = 0$ if $x_j < c$. Here we consider the case that $c$ is not one of the collocation points to avoid any
ambiguity. Then the direct projection of $\delta$ onto $B_N$ at the collocation points $x_i$ is given by

$$\delta_N(x_i - c) = \sum_{j=0}^{N} D_{ij} H_c(x_j).$$

From above, we know that

$$\delta_N(x_i - c) = \sum_{j=k}^{N} l_j'(x_i),$$

where $l_j'(x_i), \forall i = 0, \cdots, N$, is the derivative of $l_j(x)$ on the collocation points and $k$ is an index such that $x_k > c$. This approximation does not satisfy the relation $\int_{-1}^{1} \delta_N dx = 1$ in the quadrature sense. In this note, we mainly consider this direct projection approach for the approximation of singularly perturbed differential equations.

In Figure 1, four different $\delta$-function approximations introduced above are shown. The number of collocation points is $N = 63$. The top left figure shows the approximation by $l_j^{N}$, the top right the Gaussian distribution with $\epsilon = \frac{2}{N}$, the bottom left the Galerkin approximation on the collocation points and the bottom right the direct projection of the Heaviside function. Here $l_j^{N}$ is the approximation of $\delta(x - x_{(N+1)/2})$ and the others are of $\delta(x)$. As shown in the figures, the Galerkin and direct projection approximations are high oscillatory.

![Fig. 1. Approximations of $\delta(x)$ with $N = 63$. Top left: $l_j^{N}(x)$ for $\delta(x - x_{(N+1)/2})$. Top right: Gaussian distribution. Bottom left: Galerkin projection on the collocation points. Bottom right: Direct collocation projection.](image)

As we will show in the following sections, the direct projection approach yields accurate results for some differential equations and it recovers spectral accu-
racy even though there exists a local jump discontinuity in the solution. Such performance seems to be possible due to the cancellation of the oscillations when the δ-function is approximated in a consistent manner with the given differential equations. We emphasize that this approach is limited to some differential equations. The application of the direct projection method, however, can be found in many realistic problems if the given problem has a nice property. In [4], we are using the direct projection method for the study of the test particle moving around the Schwarzschild black hole. If the test particle is treated as a perturbation of the Schwarzschild spacetime, the Einstein equations can be reduced into a simple linear wave equation with a singular source term known as the Zerilli equation [6,11]. Our preliminary results show that the direct projection method provides better results than the finite difference methods such as the Lax-Wendroff method.

The paper is structured as follows. In Section 2, we examine the direct projection of the δ-function and explain the consistency of such formulation with the given differential equations. In Section 3, we provide the numerical experiments for both the elliptic and hyperbolic equations. Example for the nonlinear equation is also provided. In Section 4, we summarize the results and discuss future research projects.

2 Consistent approximation of singular differential equations

To explain the direct projection of the Heaviside function for the approximation of the δ-function, we first consider the consistent approximation of singularly perturbed differential equations with the spectral method.

Consider the following simple linear differential equation

\[ \mathcal{L}u = \delta(x), \quad x \in \Omega = [-1, 1], \]

(1)

where \( \mathcal{L} = \frac{d}{dx} \) is the first order differential operator. The source term \( \delta(x) \) in \( \Omega \) is defined with the following properties

\[ \int_{-1}^{1} \delta(x)dx = 1, \quad \text{and} \quad \int_{-1}^{1} u(x)\delta(x)dx = u(0). \]

The solution \( u(x) \) of Eq. (1) has a local jump discontinuity at \( x = 0 \). Suppose that

\[ \lim_{x \to -1} u(x) = 0. \]

With the above definition of the δ-function, the exact solution of Eq. (1) is
given by the Heaviside function \( H_c(x) \) with \( c = 0 \), i.e.

\[
    u(x) = H_0(x) = \begin{cases} 
        0, & x < 0 \\
        1, & x > 0 
    \end{cases}.
\] (2)

Now consider the spectral approximation of degree \( N \) of Eq. (1) with the projection operator \( P_N \),

\[
P_N \mathcal{L} P_N u_N = \delta_N,
\] (3)

where \( u_N \) and \( \delta_N \) are the spectral approximations of \( u(x) \) and \( \delta(x) \) residing in \( \mathcal{B}_N \). From Eqs. (1) and (2), we know that

\[
    \delta = \mathcal{L} H_c.
\] (4)

Using the above solution, let the projection of the \( \delta \)-function be

\[
    \delta_N = P_N \mathcal{L} P_N H_c = \mathcal{L}_N P_N H_c.
\]

By plugging the above equation into Eq. (3), we obtain

\[
P_N \mathcal{L} P_N (u_N - P_N H_c) = 0.
\] (5)

2.1 Spectral collocation method

For Eq. (1), the spectral collocation method solves the following equation

\[
    D_N u_N = \delta_N,
\]

where \( D_N \) is the differentiation matrix, \( u_N = (u_N(x_0), \ldots, u_N(x_N))^T \), and \( \delta_N \) the discretized \( \delta \)-function. The initial condition is \( u_N(x_0) = 0 \). Here we assume that we have a certain collocation point set \( \{x_i\}_{i=0}^N \) with \( x_0 = -1 \) and \( x_N = 1 \). Let \( H_N = (H_N(x_0), \ldots, H_N(x_N))^T \) denote the discretized version of the Heaviside function \( H_c(x) \) on the collocation points. The consistent formulation is that the discretized \( \delta_N \) is to be the derivative of the discretized Heaviside function \( H_N \) with the differentiation matrix \( D_N \) mimicking the continuous version in Eq. (4), i.e.

\[
    \delta_N = D_N \cdot H_N.
\] (6)
In fact, $\delta_N$ is highly oscillatory over the whole domain as it is the derivative of the discontinuous function. The Gibbs phenomenon appeared in Eq. (6) is inevitable. If we plug $\delta_N = D_N \cdot H_N$ in the collocation equation, however, the linear system yields the exact solution of $u_N$ at the collocation points, that is,

$$D_N u_N = \delta_N = D_N H_N,$$

and

$$D_N(u_N - H_N) = 0.$$ 

Use the initial condition to obtain

$$\bar{D}_N(\bar{u}_N - \bar{H}_N) = 0,$$

where the superscript $\bar{\cdot}$ denotes the quantity without the surface term involving the initial condition. For example, $\bar{D}_N$ is the inner matrix of $D_N$ without the first row and column. Since $\bar{D}_N$ is non-singular, it is obvious that

$$\bar{u}_N - \bar{H}_N = 0.$$ 

Since $u_N(-1) = 0 = H_N(-1)$, and $\bar{u}_N - \bar{H}_N = 0$, we obtain

$$u_N = H_N,$$  \hfill (7)

which is the same as the exact solution of the differential equation at the collocation points, $x_0, \cdots, x_N$.

### 2.2 Spectral Galerkin method

To compare with the spectral collocation method, consider the spectral Galerkin approximation of Eq. (1). From Eq. (5), the spectral solution $u_N$ with the Galerkin method is given by the Galerkin projection of $H_c(x)$ to $B_N$,

$$u_N(x) = \mathcal{P}_N H_c(x).$$

The Galerkin method seeks an approximation $u_N(x) \in B_N$ as

$$u_N(x) = \sum_{l=0}^{N} \hat{u}_l \psi_l(x),$$  \hfill (8)

where $\psi_l(x)$ is the orthogonal polynomial basis function of degree $l$ and $\hat{u}_l$ are the expansion coefficients computed by
\[ \hat{u}_l = \frac{1}{h_l} \int_{-1}^{1} w(x) H_c(x) \psi_l(x) dx. \] (9)

Here \( h_l \) and \( w(x) \) are the normalization factor and the weight function, respectively such that \( \int_{-1}^{1} w(x) \psi_l(x) \psi_l(x) dx = h_l \delta_{l\ell} \) and \( \delta_{l\ell} \) is the usual Kronecker delta. Since \( H_c(x) \) is a discontinuous function, \( u_N(x) \) is highly oscillatory near the singularity at \( x = 0 \). In practice, the Galerkin method for Eq. (3) is as follows. From Eq. (1), the Galerkin projection yields

\[ \int_{-1}^{1} w \psi_l u'_N dx = \int_{-1}^{1} w \psi_l \delta dx, \quad \forall l = 0, \ldots, N, \] (10)

where the superscript \(^'\) denotes the first order derivative with respect to \( x \). Using integration by parts and with Eq. (8), we obtain

\[ w \psi_l u_N |_{x=1} - \int_{-1}^{1} (w \psi_l)' u_N dx = w \psi_l u_N |_{x=-1} + \int_{-1}^{1} w \psi_l \delta dx, \]

and

\[ \sum_{l'=0}^{N} \hat{u}_{l'} \left( w \psi_l \psi_{l'} |_{x=1} - \int_{-1}^{1} (w \psi_l)' \psi_{l'} dx \right) = w \psi_l u |_{x=-1} - w \psi_l(0), \]

where the last term in the RHS is evaluated using the definition of the \( \delta \)-function. If we let the stiffness matrix \( S \) be

\[ S_{l\ell} = w \psi_l \psi_{l'} |_{x=1} - \int_{-1}^{1} (w \psi_l)' \psi_{l'} dx, \]

and the load vector \( b \)

\[ b_l = w \psi_l u_N |_{x=-1} - w \psi_l(0), \]

then the expansion coefficients \( \hat{u}_l \) can be evaluated by solving the following linear system

\[ S \cdot \hat{u} = b, \] (11)

where \( \hat{u} = (\hat{u}_0, \ldots, \hat{u}_N)^T \). The Galerkin formulation is easily implemented due to the fact that any integration involving the \( \delta \)-function can be evaluated easily and exactly. The solution \( \hat{u} \) obtained by Eq. (11) is not necessarily the same as that by Eq. (9).
For the consistent formulation, one should use $P_N \mathcal{L} \mathcal{P} H_c$ for the RHS of Eq. (10) such that

$$\int_{-1}^{1} w \psi_l u'_N \, dx = \int_{-1}^{1} w \psi_l \left( \sum_{k=0}^{N} \psi_k \int_{-1}^{1} w \psi_k H'_c \, ds \right) \, dx, \quad \forall l = 0, \cdots, N.$$ 

Then we have

$$\int_{-1}^{1} w \psi_l u'_N \, dx = \int_{-1}^{1} w \psi_k H'_c \, dx, \quad \forall l = 0, \cdots, N.$$ 

Thus

$$\int_{-1}^{1} w \psi_l (u'_N - H'_c) \, dx = 0, \quad \forall l = 0, \cdots, N.$$ 

Since the above integral vanishes for $\forall l = 0, \cdots, N$, we obtain

$$u'_N = H'_c.$$ 

This is basically the same equation as Eq. (1), which also implies that the Galerkin approximation is not exact even with the consistent formulation.

### 2.3 Higher order derivatives and lower order terms

The exactness of the solution Eq. (7) is not always obtained if higher order derivatives appear or the lower order terms are also involved in the given differential equation.

First consider the following differential equation with high order derivatives

$$\frac{d^m}{dx^m} u(x) = \delta^{(\alpha)}(x), \quad x \in [-1, 1],$$

where $m$ and $\alpha$ denote the $m$th and $\alpha$th derivatives respectively. In the previous section, we considered $m = 1$ and $\alpha = 0$. Suppose that $m \leq \alpha$. Then the spectral collocation method with the consistent formulation $\delta_N = D_N \mathcal{P} H_N$ solves the following equation with proper boundary conditions,

$$D_N \cdots D_N u_N = D_N \cdots D_N H_N.$$ 

Then we have

$$D_N \cdots D_N (u_N - D_N \cdots D_N H_N) = 0.$$ 

Thus the spectral approximation $u_N$ is given by the $\alpha + 1 - m$ order derivative of the Heaviside function and the Gibbs oscillations appear in $u_N$. For example,
if $m = 1$ and $\alpha = 1$, the exact solution is $u(x) = \delta(x)$ and the discrete spectral approximation is to be $u_N = D_N H_N$, which is obviously oscillatory.

If $m > \alpha$, we have

$$D_{\alpha+1} \cdots D_{\alpha} \left( D_{m-\alpha-1} \cdots D_{m} u_N - H_N \right) = 0.$$  

Then this problem is close to the one solving $\frac{d^{(m-\alpha-1)}}{dx^{(m-\alpha-1)}} u(x) = H_c(x)$. If $m - \alpha - 1 = 0$, we know that the spectral approximation is exact, i.e. $u_N = H_c(x)$. If $m - \alpha - 1 \neq 0$, we know that the exact solution $u(x)$ is continuous and $u(x) \in C^{m-\alpha-2}$. The convergence near $x = 0$ is rather slower than in the region away from $x = 0$, but $u_N$ can be obtained accurately without the Gibbs oscillations as $N$ increases.

Now consider the differential equation with the lower order terms. For simplicity, consider the following differential equation

$$\begin{cases}
\frac{du(x)}{dx} + au(x) = \delta(x), & x \in [-1, 1] \\
u(-1) = \exp(a),
\end{cases} \quad (12)$$

where $a$ is a real constant. The exact solution of Eq. (12) is given by $u^e(x) = \exp(-ax)(\int_{-1}^{x} \exp(at)\delta(t)dt + 1)$. The exact solution at each collocation point $x_i$ is also given by

$$u^e(x_i) = \exp(-ax_i)(\int_{-1}^{x_i} \exp(at)\delta(t)dt + 1) = \exp(-ax_i)(H_N(x_i) + 1).$$

But note that the derivative is not necessarily the same as the exact one. The consistent formulation yields

$$(D_N + aI)u_N = D_N H_N,$$

where $I$ is the $N \times N$ identity matrix. Thus $u_N$ is not exact, i.e. $u_N$ is not necessarily the same as $u(x)$ at the collocation points. Let $E$ denote the truncation error defined on the collocation points such as

$$E = (D_N + aI)u - D_N H_N,$$

where we also use the notation of $u$ as $u = (u(x_0), \ldots, u(x_N))^T$ and $E = (E(x_0), \ldots, E(x_N))^T$. As long as it is not confused with the notation, let the notation defined in the continuous space be also used in the discrete space. For example, let $\exp(x)\delta(x)$ also denote $(\exp(x_0)\delta(x_0), \ldots, \exp(x_N)\delta(x_N))^T$ in
the following analysis. Then using the exact solution, the truncation error is given by

$$E = (D_N + aI)(e^{-ax}(\int_{-1}^{1} e^{at} \delta(t) dt + 1)) - D_N H_N.$$ 

Rearranging the RHS, we have

$$\|E\| \leq \| (D_N + aI)e^{-ax} \| + \| (D_N + aI)e^{-ax} \int_{-1}^{1} e^{at} \delta(t) dt - D_N H_N \|, \quad (14)$$

where $\| \cdot \|$ denotes the vector norm, such as $\| \cdot \| = | \cdot |_{\infty}$. Since $e^{-at} \in C^\infty$, the first term in the RHS of Eq. (14) decays exponentially with $N$. In the second term of the RHS, the discretized version of the integral term involving the $\delta$-function on the collocation points $x_i$ is exactly given by

$$\exp(-ax) \int_{-1}^{1} \exp(at) \delta(t) dt \big|_{x=x_i} = \exp(-ax_i) H_N(x_i).$$

Here note that the exact form of the derivative of the solution is given by

$$\frac{d}{dx} (\exp(-ax) \int_{-1}^{1} \exp(at) \delta(t) dt \big|_{x=x_i}) = -a \exp(-ax) \int_{-1}^{1} \exp(at) \delta(t) dt + \delta(x)$$

$$= -a \exp(-ax_i) H_N(x_i) + \delta(x_i),$$

but this relation does not hold with the discretized version, for which the consistent formulation does not yield fast convergence as we will see shortly. With the exact discretized version of the integral term in the RHS, we have

$$\|E\| \leq \| (D_N + aI)e^{-ax} \| + \| (D_N + aI)(e^{-ax}(H_N)) - D_N H_N \|,$$

$$\|E\| \leq \| (D_N + aI)e^{-ax} \| + \| D_N((e^{-ax} - 1)H_N) + aIe^{-ax}H_N \|,$$

$$= \| (D_N + aI)e^{-ax} \| + \| D_N(g(x)H_N) - Ig'(x)H_N \|, \quad (15)$$

where $g(x) = (e^{-ax} - 1)H(x)$ and $g(x)H(x)$ are continuous with a jump in the first derivative at $x = 0$. The discretized version of $(g(x)H(x))'$ is the same as $(g(x)H(x))' = g'(x)H(x)$. Since $g(x)H(x) \in C^0$, we know that the second term in the RHS of Eq. (15) vanishes as $N \to \infty$

$$\lim_{N \to \infty} \|E\|_{\infty} = 0,$$

although its overall convergence is not exponential. This implies that the Gibbs oscillations do not appear as $N \to \infty$ in the spectral collocation solution of Eq. (12) even though the solution is discontinuous.
To verify the above statements, consider Eq. (12). We consider two methods, the Chebyshev collocation method and the Chebyshev Galerkin method. For the Galerkin method, we seek a solution in the form of

\[ u(x) = \sum_{l=0}^{N} g_l T_l(x), \]

where \( g_l \) and \( T_l(x) \) are the expansion coefficients and the Chebyshev polynomial of degree \( l \), respectively. The Galerkin formulation of Eq. (12) is obtained by multiplying each side of Eq. (12) by \( T_l(x) \) and taking an integral over the whole domain for every \( l = 0, \cdots, N \),

\[ \int_{-1}^{1} u(x) T_l(x) dx = \int_{-1}^{1} \delta(x) T_l(x) dx. \]

The RHS of the above equation is easily obtained using the definition of the \( \delta \)-function as

\[ \int_{-1}^{1} \delta(x) T_l(x) dx = T_l(0) = \begin{cases} (-1)^{l/2}, & \text{if } l \text{ is even} \\ 0, & \text{if } l \text{ is odd} \end{cases}. \]

Use integration by parts for the first term in the LHS and rearrange the equation to obtain

\[ S \cdot g = b, \]

where

\[ S_{lk} = 1 - \int_{-1}^{1} T'_l(x) T_k(x) dx + a \int_{-1}^{1} T_l(x) T_k(x) dx, \]

\[ b_l = u(-1)(-1)^l + T_l(0), \]

and

\[ g = (g_0, \cdots, g_N)^T. \]

where \( u(-1) \) is the initial condition, \( u(-1) = \exp(a) \). We use the quadrature rules to compute each element of \( S \). For \( T'_l(x) \), we use \( T'_l(x) = l U_{l-1}(x) \), for \( l = 1, 2, 3, \cdots, N \), where \( U_l(x) \) is the Chebyshev polynomial of second kind of degree \( l \).

For the numerical example, we use \( a = 1.3 \) with \( N = 15, 31, 63, 127, \) and \( 253 \). For fair comparison, we compute the pointwise errors for each case on the Gauss-Lobatto collocation points, i.e. \( x_j = -\cos(j\pi/N) \), for \( j = 0, \cdots, N \). Figure 2 shows the results of each method. The top figures show the pointwise errors with the Chebyshev collocation method (left) and the Chebyshev Galerkin method (right) with \( N = 15, 31, 63, \) and \( 127 \). As explained above, the pointwise errors decay at every collocation point with the Chebyshev collocation method with \( N \). That is, the \( L_\infty \) error decays with \( N \) although it decays slow. The \( L_\infty \) error of the Galerkin method does not decay with \( N \) as the right figure clearly shows. The bottom figures of Figure 2 show the approximations at the collocation points with the Chebyshev collocation method (left) and the Chebyshev Galerkin method (right) with \( N = 63 \). The symbols
° and × represent the exact solution and the approximation respectively. The left figure shows that the collocation method yields an accurate result without any significant Gibbs oscillations while the Galerkin approximation at the two collocation points around \( x = 0 \) shows a clear difference between the exact solution and the approximation at these points. Note that the solution is discontinuous at \( x = 0 \).

![Fig. 2. Top: Pointwise errors with the Chebyshev collocation method (left) and the Chebyshev Galerkin method (right) for \( N = 15, 31, 63, \) and 127. Bottom: The exact solution (solid line marked with °) and the approximation (×) of Eq. (12) with the collocation (left) and the Galerkin (right) methods for \( N = 63 \).](image)

### 3 Numerical examples

In this section, several numerical examples are provided with the direct projection method based on the consistent formulation. We use the Chebyshev spectral method with the Gauss-Lobatto collocation points, \( x_j = -\cos(j\pi/N) \), \( x_0 = -1, x_N = 1 \) in the domain \( \Omega = [-1, 1] \).
3.1 A singular 2nd order ODE

We consider the following 2nd order differential equation with the singular source term given by the derivative of the $\delta$-function

\[ u'' - \alpha^2 u = -\beta \delta'(x - \frac{1}{6}), \]  

(16)

with $u(-1) = 0 = u(1)$ [10]. Here the superscript $'$ denotes the derivative with respect to $x$. The exact solution is given by

\[ u(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} - \frac{\beta}{2} H_{\frac{1}{6}}(x) (e^{\alpha(x-\frac{x}{6})} + e^{-\alpha(x-\frac{x}{6})}), \]

where $c_2 = -\frac{\beta}{2} (e^{5\alpha/6} + e^{-5\alpha/6})/(e^{3\alpha} - e^{-\alpha})$ and $c_1 = -c_2 e^{2\alpha}$. For the numerical experiment, we use $\alpha = 6$ and $\beta = 12$. The left figure of Figure 3 shows the exact solution (○) and the Chebyshev collocation approximation (×). The right figure shows the pointwise errors for various $N = 32, 64, 128,$ and $256$. The pointwise errors are supposed to decay with $N$. As the left figure shows, the approximation is accurate away from the singularity $x = \frac{1}{6}$ while the approximation is not accurately determined near the singularity. As the right figure shows, however, the decaying pointwise errors with $N$ imply that the approximation converges to the exact solution although its convergence rate is slow and the solution is discontinuous.

3.2 Singular hyperbolic equations

The 1st order advection equation: We first consider the 1D singular advection equation

Fig. 3. Left: The exact solution (○) and the approximation (×) for Eq. (16) for $N = 64$. Right: Pointwise errors in logarithmic scale for $N = 32, 64, 128,$ and $256$. Note that the pointwise errors decay as $N$ increases.
\[ \begin{align*}
&u_t + u_x = \delta(x), \quad x \in [-1, 1], t > 0, \\
&u(x, 0) = -\sin(\pi x), \\
&u(-1, t) = -\sin(\pi(-1 - t)).
\end{align*} \tag{17} \]

The exact solution for \( t > 0 \) is given by

\[
u(x, t) = -\sin(\pi(x - t)) + \int_{-1}^{x} \delta(x) \, dx,
\]

and at the collocation points it is represented by

\[
u(x_i, t) = \begin{cases} 
-\sin(\pi(x_i - t)), & i \in [0, \ldots, \frac{N+1}{2}], \\
1 - \sin(\pi(x_i - t)), & i \in [\frac{N+3}{2}, \ldots, N].
\end{cases}
\]

Here we assume that \( N \) is odd to avoid any ambiguity at \( x = 0 \) for \( \delta(x) \). Then \( x_{\frac{N+1}{2}} < 0 < x_{\frac{N+3}{2}} \). We use the TVD 3rd order Runge-Kutta method [7] for the time integration. For numerical experiments, the final time, \( t_f = 10.3 \), the time step \( dt = 5 \times 10^{-4} \) and \( N = 15, 31, 63 \) are used.

**Fig. 4.** Chebyshev collocation approximations with \( N = 33 \) at \( t = 10.3 \) for Eq. (17). Left: The exact solution (◦) and the approximation (×). Right: Pointwise errors in logarithmic scale for \( N = 15(\times), 31(\cdot), \) and 63(◦).

The left figure of Figure 4 shows the Chebyshev approximation and the exact solution at \( t = t_f \) with \( N = 33 \). The symbols ◦ and × represent the exact solution and the Chebyshev approximation respectively. The solid line represents the exact solution as well. At \( t = t_f \), the \( L_2 \) and \( L_\infty \) errors are given by \( L_2 = 5.15 \times 10^{-10} \), and \( L_\infty = 9.27 \times 10^{-10} \) respectively. The right figure shows the pointwise errors for various \( N = 15(\times), 33(\cdot), \) and 63(◦). In the few initial time steps, there are the Gibbs oscillations around the discontinuity due to the singular source term. The numerical solution, however, soon becomes smooth without the Gibbs oscillations although there exists a jump discontinuity. Figure 4 shows that the Chebyshev collocation method with the
direct projection method based on the consistent formulation of \( \delta(x) \) yields an accurate result. Note that the solution in the figure is not a steady-state solution but it changes with time. The numerical results show that no Gibbs oscillations are seen in the solution for \( \forall t > T \) where \( T \) is the time interval needed for the initial Gibbs oscillation to leave the domain.

Table 1

\( L_2 \) and \( L_\infty \) errors with various approximations of \( \delta(x) \) at \( t = 10.3 \) for Eq. (17).

<table>
<thead>
<tr>
<th>Method (( \delta(x) ))</th>
<th>( L_2 ) error</th>
<th>( L_\infty ) error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinal function</td>
<td>1.68</td>
<td>2.4</td>
</tr>
<tr>
<td>Gaussian distribution</td>
<td>4.14(-2)</td>
<td>2.34(-1)</td>
</tr>
<tr>
<td>Galerkin method</td>
<td>1.18(-1)</td>
<td>9.386(-1)</td>
</tr>
<tr>
<td>Direct projection of ( H(x) )</td>
<td>5.15(-10)</td>
<td>9.27(-10)</td>
</tr>
</tbody>
</table>

\((n) = 10^n\)

Table 1 shows the \( L_2 \) and \( L_\infty \) errors at \( t = t_f \) when different approximations of \( \delta(x) \) introduced in Section 1 are used. These are the cardinal function approximation \( \delta_N^c(x) \), the Gaussian distribution \( \delta_N^\epsilon(x) \), the spectral Galerkin method and the direct derivative projection of \( H(x) \). The table shows that the best approximation is obtained with the direct derivative projection of \( H(x) \).

For the Chebyshev Galerkin method, the \( L_2 \) and \( L_\infty \) errors away from the discontinuity in \( x > 0.1243 \), for example, are slightly better than those given in the table as \( L_2 \sim 1.2 \times 10^{-3} \) and \( L_\infty \sim 1.24 \times 10^{-1} \) but they are still large compared to the collocation method. For the Chebyshev Galerkin method, we use the Galerkin formulation of the given PDE, Eq (17) as

\[
\int_{-1}^{1} u_i(x,t)T_l(x)dx + \int_{-1}^{1} T_l(x)u(x)dx = \int_{-1}^{1} \delta(x)T_l(x)dx, \quad \forall l = 0, \ldots, N,
\]

which ends up with the following linear equation

\[
S \cdot \frac{db}{dt} = R \cdot b + u(-1)h + d,
\]

where \( S_{ij} = \int_{-1}^{1} T_i(x)T_j(x)dx \), \( R_{ij} = \int_{-1}^{1} T_i'(x)T_j(x)dx - 1 \), \( h_i = T_i(-1) \) and \( d_i = T_i(0) \). For \( R \), we also use the quadrature rules with the Chebyshev polynomials of second kind as in the previous section. Here \( b \) is the expansion coefficient vector, \( b = (b_0, \ldots, b_N)^T \) for the Galerkin approximation \( u(x,t) \) given by \( u(x,t) = \sum_{i=0}^{N} b_i(t)T_i(x) \). The \( L_2 \) and \( L_\infty \) errors are obtained at the Gauss-Lobatto collocation points for the comparison with other methods.

The 2nd order wave equation: We also consider the 2nd order linear wave equation with the singular source term.
\begin{equation}
\left\{
\begin{array}{l}
    u_{tt} - u_{xx} = -\delta'(x), \ x \in [-1, 1], \ t > 0, \\
    u(x, 0) = 0, \\
    u_t(x, 0) = 0,
\end{array}
\right.
\tag{18}
\end{equation}

where note that the RHS is given by the derivative of the \( \delta \)-function. The accurate result is also obtained with the consistent formulation when the RHS contains both \( \delta(x) \) and \( \delta'(x) \). The exact solution \( u(x, t) \) for \( t \to \infty \) is given by

\[
    u(x, t) = \begin{cases} 
    -\frac{1}{2} & x < 0 \\
    \frac{1}{2} & x > 0
    \end{cases}
\]

The derivative of the source term, \( \delta'_N(x) \), is calculated as

\[
    \delta'_N(x) = \frac{d\delta_N(x)}{dx} := D \cdot D \cdot H_N = D^2 \cdot H_N.
\]

At the boundaries \( x = -1 \) and \( x = 1 \), non-reflecting boundary conditions are used. For the numerical experiment, \( N = 33, \ dt = 0.005 \) and \( t_f = 20 \) are used. At \( t = t_f \), the \( L_2 \) and \( L_\infty \) errors are obtained by

\[
    L_2 = 3.271 \times 10^{-14}, \quad L_\infty = 5.429 \times 10^{-14}.
\]

During the initial stages of the numerical evaluation of Eq. (18), there exist the Gibbs oscillations in the approximation due to \( \delta'(x) \) at the origin. These oscillations are moving towards the boundaries and disappear through the boundaries as non-reflecting boundary conditions are applied at the boundaries. Once the oscillations entirely leave the domain, the discontinuity at \( x = 0 \) does not induce any further oscillations and the solution reaches the steady-state solution in an accurate manner. Figure 5 shows the exact solution(\( \circ \)) and the approximation(\( \times \)) at \( t = t_f \) with \( N = 33 \) (left) and the pointwise errors with different \( N = 15, \) and 33 (right). As shown in the figures, the approximation with \( N = 33 \) is as accurate as machine accuracy.

### 3.3 A nonlinear hyperbolic equation

For a nonlinear problem, we consider the following Burgers’ equation

\[
    u_t + (1/2u^2)_x = \delta(x + 0.1), \quad x \in [-1, 1], \quad t > 0,
\]

with

\[
    u(x, 0) = \beta, \quad u(-1, t) = \beta.
\]
Fig. 5. The Chebyshev collocation approximation with \(N = 33\) at \(t = 20\). Left: The exact solution (○) and the approximation (×). Right: Pointwise errors in logarithmic scale for \(N = 15(\times)\), and \(33(\cdot)\) at \(t = 20\).

For nonlinear problems, the direct projection approach seems to be highly sensitive to the oscillatory profile of \(\delta_N(x)\). For example, if we choose the initial condition \(u(x,0) = 0\), the oscillations of \(\delta_N(x)\) which changes its sign at the collocation points and makes \(u(x,t)\) also change its sign point by point. This induces the numerical instability quickly and the numerical solution soon blows up. Furthermore, our numerical experiments indicate that the direct projection method is not applicable for general initial conditions. For this reason, let us choose the initial condition so that the initial oscillations have the same sign at all collocation points, for example, \(u(x,0) = \beta > 0\). If there exists the steady-state solution, \(u^s(x,t)\), it is given by

\[
    u^s(x,t) = (\alpha - \beta)H_{-0.1}(x) + \beta,
\]

where the constant \(\alpha\) is given by \(\alpha = \sqrt{2 + \beta^2}\). Here note that the steady-state solution is a discontinuous function with the local jump discontinuity at \(x = -0.1\).

For the numerical examples, \(N = 64\) and \(dt = 5 \times 10^{-5}\) are used. Figure 6 shows solutions for different \(\beta = 2\) (left) and \(\beta = 10\) (right). The top figures show the normalized solutions defined as \((u(x,t) - \beta)/(\alpha - \beta)\) at \(t = 1\). The jump magnitudes of the steady-state solutions are \([u^s] \sim 0.4495\) and \([u^s] \sim 0.0995\) for \(\beta = 2\) and \(\beta = 10\), respectively. \([u^s]\) for \(\beta = 10\) is about 5 times smaller than that for \(\beta = 2\). The figures show that the solution with \(\beta = 10\) is oscillatory with the larger jump magnitude with \(\beta = 2\) while no significant Gibbs oscillations are seen in the solution. The bottom figures show the series of the normalized solutions with time. As shown in the figures, the magnitude of the Gibbs oscillations with \(\beta = 2\) observed in the initial stages decrease with time but according to our numerical experiments, these oscillations do not seem to decay even though \(t\) goes to \(\infty\) but remain as the same as those at \(t = 1\). The magnitude of the Gibbs oscillations observed in the initial stages with \(\beta = 10\) are, however, smaller than those with \(\beta = 2\). Furthermore, these
oscillations leave the domain more quickly than those with $\beta = 2$ and the solution becomes smooth around the jump discontinuity.

![Graph](image)

Fig. 6. The normalized Chebyshev collocation approximations for Eq. (19) with $N = 64$ for $\beta = 2$ (left) and $\beta = 10$ (right). Top: The normalized solutions, $(u(x,t) - \beta)/(\alpha - \beta)$ at $t = 1$. Bottom: Series of the normalized solutions with time.

4 Summary and future works

In this brief note, we consider the spectral collocation approximation of singularly perturbed differential equations. In order to approximate singular source terms of the $\delta$-function and its derivatives, the direct projection of the Heaviside function $H_c(x)$ on the collocation points is used. That is, the collocation derivative operator is applied to the discrete Heaviside function on the collocation points. Since the Heaviside function is itself discontinuous, its collocation derivative is highly oscillatory over the whole domain. Accordingly the approximation of the $\delta$-function is also highly oscillatory. We show in this note that such approach can be used for certain types of differential equations in order to obtain a non-oscillatory solution with a better convergence. We considered simple ODEs as well as hyperbolic equations. The numerical results show that the spectral collocation approximation with the direct projection approach yields better results than the spectral Galerkin method and the so-
olution does not have the Gibbs oscillations. For the linear wave equations with
the singular source, spectral accuracy is recovered. For time-dependent prob-
lems, the numerical solution shows the Gibbs oscillations in the beginning
but there is a cancellation of the Gibbs oscillations if time becomes large due
to the consistent formulation of the given differential equations. This direct
approach is limited to certain types of differential equations. This method,
however, can be applied to more general types of differential equations when
the given problem has a nice property such as the existence of the steady-state
solution or symmetry. For example, the spherical symmetry of the two body
problem composed of the static massive black hole and the single star or less
massive black hole moving around it leads to the 1D wave equation with the
singular source term similar to Eq. (18). The singular source term represents
the motion of the test particle towards the central black hole and it is also
a function of time as \( \delta(x,t) \). In this case, the local jump discontinuity is not
static but moving with time. In [4], such system is considered with the direct
projection method and a better result has been obtained than those with the
finite difference methods.

In our future work, the direct projection method will be investigated when
the domain decomposition method is applied. For the multi-domain spectral
collocation method, one can confine the location of the singular source term in
a separate domain and minimize the approximation errors of the \( \delta \)-function.
The direct projection method will be also studied in higher dimensions and
applied to more general types of singular differential equations.

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References


the head-on collision of black holes, to be submitted. Also presented at the
ICOSAHOM 2007.


