SOLUTION FOR HOMEWORK #11

1. Section 11.8

4. $\lim_{n\to\infty} |\frac{(-1)^{n+1}x^{n+1}/(n+2)}{(-1)^n x^n/(n+1)}| = \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+2} = |x|$. By the Ratio Test, the radius of convergence is 1. When $x = 1$, the series converges by the Alternating Series Test; when $x = -1$, the series diverges because it is the harmonic series. Thus the interval of convergence is $(-1, 1]$.

6. $\lim_{n\to\infty} |\frac{\sqrt{n+1}x^{n+1}}{\sqrt{n}x^n}| = \lim_{n\to\infty} |x| \cdot \sqrt{1 + \frac{1}{n}} = |x|$. By the Ratio Test, the radius of convergence is 1. When $x = \pm 1$, $\lim_{n\to\infty} |\sqrt{n}x^n| = \infty$, so the series diverge by the Test for Divergence. Thus the interval of convergence is $(-1, 1)$.

8. $\lim_{n\to\infty} |n^n x^n|^{\frac{1}{n}} = \lim_{n\to\infty} n|x| = \infty$ unless $x = 0$. By the Ratio Test, the radius of convergence is 0, and the interval of convergence is $\{0\}$.

14. $\lim_{n\to\infty} |\frac{(-1)^{n+1}x^{2(n+1)}/(2(n+1))!}{(-1)^n x^{2n}/(2n)!}| = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)} = 0$. By the Ratio Test, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

16. $\lim_{n\to\infty} |\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}| = \lim_{n\to\infty} |x-5| \cdot (1 + \frac{1}{n})^3 = |x-5|$. By the Ratio Test, the radius of convergence is 1. When $x = 6$ or 4, $\lim_{n\to\infty} |n^3(x-5)^n| = \infty$, so the series diverge by the Test for Divergence. Thus the interval of convergence is $(4, 6)$.

24. $\lim_{n\to\infty} |\frac{(n+1)^2 x^{n+1}/(2\cdot 4 \cdot \cdots \cdot (2n+2))}{n^2 x^n/(2\cdot 4 \cdot \cdots \cdot (2n))}| = \lim_{n\to\infty} |x| \cdot \frac{n+1}{2n^2} = 0$. By the Ratio Test, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

2. Section 11.9

4. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ exactly for all x with $|x| < 1$ and the power series converges exactly on $(-1, 1)$, we have $f(x) = \frac{3}{1-x^4} = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$ for all x such that $|x^4| < 1 \Leftrightarrow |x| < 1$ and the interval of convergence for this power series is $(-1, 1)$.

8. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ exactly for all x with $|x| < 1$ and the power series converges exactly on $(-1, 1)$, we have $f(x) = \frac{x}{4x+1} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-4)^n x^{n+1}$ for all x such that $|-4x| < 1 \Leftrightarrow |x| < \frac{1}{4}$ and the interval of convergence for this power series is $(-\frac{1}{4}, \frac{1}{4})$.

24. Since $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for all t with $|t| < 1$ and the radius of convergence for the power series is 1, we have $\frac{\ln(1-t)}{t} = -\sum_{n=0}^{\infty} \frac{t^n}{n+1}$ for all t with $|t| < 1$ and the radius of convergence for this power series is also 1. Thus, we have $\int \frac{\ln(1-t)}{t}\,dt = C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2} = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$ for all t with $|t| < 1$ and C being a constant and the radius of convergence for this power series is also 1.

3. Section 11.10

4. $f^{(2n)}(x) = (-1)^n 2^{2n} \sin 2x$ and $f^{(2n+1)}(x) = (-1)^n 2^{2n+1} \cos 2x$ for all nonnegative integers n. Thus $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$. So the Maclaurin series of $f(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$. Since

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}2^{2n+3}x^{2n+3}/(2n+3)!}{(-1)^n 2^{2n+1}x^{2n+1}/(2n+1)!}\right| = \lim_{n\to\infty} \frac{4x^2}{(2n+2)(2n+3)} = 0,$$

the radius of convergence for this power series is ∞ be the Ratio Test.

6. $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ for all positive integers n. Thus $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ for all positive integers n. So the Maclaurin series of $f(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$. Since

$$\lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1}/(n+1)}{(-1)^{n-1}x^n/n}\right| = \lim_{n\to\infty} \frac{n|x|}{n+1} = |x|,$$

the radius of convergence for this power series is 1 be the Ratio Test.

8. $f^{(n)}(x) = (x+n)e^x$ for all nonnegative integers n. Thus $f^{(n)}(0) = n$. So the Maclaurin series of $f(x)$ is $\sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n$. Since

$$\lim_{n\to\infty} \left|\frac{x^{n+1}/n!}{x^n/(n-1)!}\right| = \lim_{n\to\infty} \frac{|x|}{n} = 0,$$

the radius of convergence for this power series is ∞ be the Ratio Test.

12. $f(x) = x^3 = ((x+1)-1)^3 = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$ for all x. Thus $-1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$ must be the Taylor series of f centered at -1.

16. $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$ for all nonnegative integers n. Thus $f^{(2n)}(\frac{\pi}{2}) = (-1)^n$ and $f^{(2n+1)}(\frac{\pi}{2}) = 0$. So the Taylor series of $f(x)$ centered at $\frac{\pi}{2}$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x - \frac{\pi}{2})^{2n}$.

40. Since $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, we have $\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}$ for all x. Thus, we have $\int \frac{\sin x}{x}\, dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!\cdot(2n+1)}$ for all x and C being a constant.

56. Since $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all x, we have $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.