

STRICT QUANTIZATIONS OF ALMOST POISSON MANIFOLDS

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ABSTRACT. We show the existence of (non-Hermitian) strict quantization for every almost Poisson manifold.

1. INTRODUCTION

In the passage from classical mechanics to quantum mechanics, smooth functions on symplectic manifolds (more generally, Poisson manifolds) are replaced by operators on Hilbert spaces, and the Poisson bracket of smooth functions are replaced by commutators of operators. When one thinks of classical mechanics as limits of quantum mechanics, the Poisson bracket becomes limits of commutators. Based on the general theory of formal deformations of algebras [9], F. Bayer et al. [1] initiated the study of *deformation quantization* of Poisson manifolds.

Let M be a Poisson manifold. Denote $C^\infty(M)$ the space of smooth \mathbb{C} -valued functions on M , and denote $C^\infty(M)[[\hbar]]$ the space of formal power series with coefficients in $C^\infty(M)$. Recall that a *star product* on M is a $\mathbb{C}[[\hbar]]$ -bilinear associative multiplication $*$ on $C^\infty(M)[[\hbar]]$ of the form

$$f * g = \sum_{r=0}^{\infty} C_r(f, g) \hbar^r, \quad \text{for } f, g \in C^\infty(M),$$

where $C_0(f, g) = fg$, $f * g - g * f \equiv \{f, g\} i \hbar \pmod{\hbar^2}$, and each $C_r(\cdot, \cdot)$ is a bidifferential operator. The algebra $(C^\infty(M)[[\hbar]], *)$ is called a *deformation quantization* of M . The existence of deformation quantizations for any symplectic manifold was proven first by De Wilde and Lecomte [7]. The general case of Poisson manifolds was proven by Kontsevich [12].

In deformation quantizations \hbar is only a formal parameter, and elements in $C^\infty(M)[[\hbar]]$ are not operators on Hilbert spaces. In order to study quantizations in a stricter sense, Rieffel introduced *strict deformation quantization* of Poisson manifolds [23, 24], and showed that noncommutative tori arise naturally as strict deformation quantizations of certain Poisson brackets on the ordinary torus. Later, Landsman introduced a weaker notion *strict quantization* to accommodate some other interesting examples such as Berezin-Toeplitz quantization of Kähler manifolds. Recall the definition of strict quantization as formulated in [27, 13]:

Definition 1.1. Let M be a Poisson manifold, and let $C_\infty(M)$ be the algebra of continuous functions on M vanishing at ∞ . By a *strict quantization* of M we mean a dense $*$ -subalgebra A of $C_\infty(M)$ closed under the Poisson bracket, together with a continuous field of C^* -algebras \mathcal{A}_\hbar over a closed subset I of the real line containing 0 as a non-isolated point, and linear maps $\pi_\hbar : A \rightarrow \mathcal{A}_\hbar$ for each $\hbar \in I$, such that

- (1) $\mathcal{A}_0 = C_\infty(M)$ and π_0 is the canonical inclusion of A into $C_\infty(M)$,

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- (2) the section $(\pi_{\hbar}(f))$ is continuous for every $f \in A$,
(3) for all $f, g \in A$ we have

$$\lim_{\hbar \rightarrow 0} \| [\pi_{\hbar}(f), \pi_{\hbar}(g)]/(i\hbar) - \pi_{\hbar}(\{f, g\}) \| = 0.$$

If each π_{\hbar} is injective, we say that the strict quantization is *faithful*. If $A \supseteq C_c^\infty(M)$, the space of compactly supported smooth functions on M , we also say that the strict quantization is *flabby*. If $(\pi_{\hbar}(f))^* = \pi_{\hbar}(f^*)$ for all $\hbar \in I$ and $f \in A$, we say that the strict quantization is *Hermitian*. When a Lie group has an smooth action α on M preserving the Poisson bracket, if G also has an continuous action β_{\hbar} on each \mathcal{A}_{\hbar} such that $\beta_0 = \alpha^*$ and the maps π_{\hbar} are all G -equivariant, we say that the strict quantization is *G -equivariant*. When the strict quantization is faithful and $\pi_{\hbar}(A)$ is a $*$ -subalgebra of \mathcal{A}_{\hbar} for each \hbar , it's called a *strict deformation quantization*.

Strict quantizations have been constructed for several classes of Poisson manifolds such as Poisson manifolds coming from actions of \mathbb{R}^d [26], quantizable compact Kähler manifolds [2], dual of integrable Lie algebroids [25, 15], compact Riemannian surfaces of genus ≥ 2 [11, 17, 18] etc. These constructions are all global, and the resulting strict quantizations are Hermitian. However, the progress of the study of strict quantizations is much slower compared with that of deformation quantizations—so far there is even no existence result for general symplectic manifolds. Recently Natsume et al. [19] constructed strict quantizations for every compact symplectic manifold M satisfying the topological conditions that $\pi_1(M)$ is exact and $\pi_2(M) = 0$. Roughly speaking, they use partition of unity to reduce M to Darboux charts, where they can use the Moyal-Weyl product. Thus their construction is local. It turns out that the resulting strict quantizations are not Hermitian.

Recall that an *almost Poisson manifold* is a smooth manifold M equipped with some $\Pi \in \Gamma(\wedge^2 TM)$ [6]. In this case, we can still define a bracket $\{f, g\} = \Pi(df, dg)$ for $f, g \in C^\infty(M)$, which is bilinear and skew-symmetric, and satisfies the Leibniz rule. And the bracket satisfies the Jacobi identity if and only if M equipped with this bracket is actually a Poisson manifold. Clearly we can also talk about strict quantizations of almost Poisson manifolds. The main result of this paper is the following:

Theorem 1.2. *Let (M, Π) be an almost Poisson manifold, and let α be a smooth action of a Lie group G on M preserving the bracket. If M has a G -invariant Riemannian metric, then M has a G -equivariant faithful flabby strict quantization over $I = [0, 1]$ with $A = C_c^\infty(M)$. In particular, taking $G = \{e\}$ we see that M has a faithful flabby strict quantization over $[0, 1]$ with $A = C_c^\infty(M)$.*

Our construction is also local, but different from the one in [19]. Actually we shall construct a locally trivial C^* -algebra bundle over M in a canonical way, thus don't need local charts and partition of unity. But our strict quantizations are not Hermitian either.

This paper is organized as follows. Though our construction for strict quantizations of almost Poisson manifolds is only slightly more complicated for that of symplectic manifolds, the idea is most natural in the case of symplectic manifolds. So we prove Theorem 1.2 for symplectic manifolds first in Section 2. Then we prove Theorem 1.2 for the general case in Section 3. Our construction depends on the

choice of an inner product on the vector bundle $T^*M \oplus TM$. We define *homotopy* of strict quantizations in Section 4, and show that the homotopy class of our strict quantizations doesn't depend on the choice of the inner products. In Section 5 we define *local strict quantizations*, and show that they can't be Hermitian. We also show that our strict quantizations can't be restricted to a $*$ -subalgebra of A to get a strict deformation quantization of M unless $\Pi = 0$. This gives a negative answer to a question of Rieffel [27, Question 25]. In Section 6 we discuss certain functorial properties of our construction. All of our construction is based on the existence of *asymptotic representations of Heisenberg commutation relations* (Definition 2.1). We prove the existence of such asymptotic representations in Section 7.

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2. STRICT QUANTIZATIONS OF SYMPLECTIC MANIFOLDS

Throughout this paper, for a continuous field D of C^* -algebras $\{D_{\hbar,x}\}_{x \in X}$ over a locally compact Hausdorff space X we denote $\Gamma_\infty(D)$ the algebra of continuous sections of D vanishing at ∞ [8].

We show the main idea of our construction first. Let (M, ω) be a symplectic manifold, and let $\Pi \in \Gamma(\wedge^2 TM)$ be the corresponding bivector field as usual. Let $f, g \in C_c^\infty(M)$. Since we think of \mathcal{A}_\hbar as deformations of $C_\infty(M)$, we would like to write $\pi_\hbar(f)$ as $f + \tau_\hbar(f)$, which makes sense when \mathcal{A}_\hbar contains $C_\infty(M)$ as a C^* -subalgebra, and assume that

$$(1) \quad \|\tau_\hbar(f)\| \rightarrow 0$$

as $\hbar \rightarrow 0$. Assume further that $C_\infty(M)$ lies in the center of \mathcal{A}_\hbar . Then

$$[\pi_\hbar(f), \pi_\hbar(g)]/(i\hbar) - \pi_\hbar(\{f, g\}) = [\tau_\hbar(f), \tau_\hbar(g)]/(i\hbar) - \{f, g\} - \tau_\hbar(\{f, g\}).$$

Thus the condition (3) in Definition 1.1 becomes $\|[\tau_\hbar(f), \tau_\hbar(g)]/(i\hbar) - \{f, g\}\| \rightarrow 0$. Notice that $\{f, g\}$ doesn't depend on f and g , but depends only on df and dg . So we would like to assume that $\tau_\hbar(f)$ depends only on df linearly. Then we attempt to write $\tau_\hbar(f)$ as $\hbar^{\frac{1}{2}}\varphi_\hbar(df)$, where $\varphi_\hbar : \Gamma(T^*M) \rightarrow \mathcal{A}_\hbar$ is a linear map. Now $\|[\tau_\hbar(f), \tau_\hbar(g)]/(i\hbar) - \{f, g\}\| \rightarrow 0$ becomes

$$(2) \quad \|[\varphi_\hbar(df), \varphi_\hbar(dg)] - \Pi(df, dg)i\| \rightarrow 0.$$

We assume further that $\mathcal{A}_\hbar = \Gamma_\infty(D)$ for some continuous field of C^* -algebras $\{D_{\hbar,x}\}_{x \in M}$ over M , and that $D_{\hbar,x}$ contains T^*M_x as a linear subspace with φ_\hbar being just pointwise embedding. Then (2) becomes

$$(3) \quad \|[u, v] - \Pi_x(u, v)i\|_{D_{\hbar,x}} \rightarrow 0.$$

for all $u, v \in T^*M_x$.

This leads to our definition of asymptotic representation of Heisenberg commutation relations and that of Heisenberg C^* -algebra \mathfrak{A}_{2n} (which will be $\mathcal{A}_{\hbar,x}$ for $2n = \dim M$) in Definition 2.1 below. In order to embed T^*M_x into \mathfrak{A}_{2n} without referring to local basis, we also need an action of the structure group of T^*M on \mathfrak{A}_{2n} . If we consider only Π , then the structure group is the symplectic linear group $Sp(2n)$ [5], which is too big. By adding a compatible almost complex structure on T^*M we can reduce the structure group to the unitary group $U(n)$ (see Lemma 2.5). Here we recall how $U(n)$ acts on T^*M_x .

Let V be a finite dimensional vector space over \mathbb{R} with a symplectic structure ω and a compatible almost complex structure J [5], *i.e.* $J : V \rightarrow V$ is linear satisfying that $J^2 = -1$ and $\langle u, v \rangle := \omega(u, Jv)$ is an inner product on V . Say $\dim V = 2n$. Then we can always find basis $u_1, \dots, u_n, v_1, \dots, v_n$ of V such that under this basis ω and J have matrix forms $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ respectively. We'll call such basis *unitary basis* of V . Notice that a unitary basis is an orthonormal basis under the induced inner product. If we make V into a complex vector space by J and identify matrix $X + iY \in M_{n \times n}(\mathbb{C})$, where $X, Y \in M_{n \times n}(\mathbb{R})$, with $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_{2n \times 2n}(\mathbb{R})$, then $U(n)$ is exactly the group of linear transformations on V taking unitary basis to unitary basis.

Definition 2.1. Let $n \in \mathbb{N}$, and let \mathbb{R}^{2n} be equipped with the standard symplectic vector space structure Ω and the standard compatible almost complex structure, *i.e.* for the standard basis e_1, \dots, e_{2n} being a unitary basis. By an *asymptotic representation of Heisenberg commutation relations*, we mean a unital C^* -algebra \mathfrak{A}_{2n} with a continuous action ρ of $U(n)$ and a $U(n)$ -equivariant \mathbb{R} -linear map $\varphi_{\hbar} : \mathbb{R}^{2n} \rightarrow \mathfrak{A}_{2n}$ for each $0 < \hbar \leq 1$ such that

- (1) for any $u, v \in \mathbb{R}^{2n}$ we have $[\varphi_{\hbar}(u), \varphi_{\hbar}(v)] \rightarrow \Omega(u, v)i$ as $\hbar \rightarrow 0$;
- (2) the map $(0, 1] \rightarrow B(\mathbb{R}^{2n}, \mathfrak{A}_{2n})$ given by $\hbar \mapsto \varphi_{\hbar}$ is continuous, where $B(\mathbb{R}^{2n}, \mathfrak{A}_{2n})$ is the Banach space of linear maps $\mathbb{R}^{2n} \rightarrow \mathfrak{A}_{2n}$;
- (3) $\hbar^{\frac{1}{2}} \|\varphi_{\hbar}\| \rightarrow 0$ as $\hbar \rightarrow 0$;
- (4) \mathfrak{A}_{2n} is generated by $\cup_{0 < \hbar \leq 1} \varphi_{\hbar}(\mathbb{R}^{2n})$.

The C^* -algebra \mathfrak{A}_{2n} will be called a *Heisenberg C^* -algebra of dimension $2n$* .

Remark 2.2. The condition (3) is not crucial. Given \mathfrak{A}_{2n} and φ_{\hbar} satisfying the other conditions we can always reparameterize φ_{\hbar} 's to make them satisfy (3).

The main technical part of our construction is the following:

Theorem 2.3. *For each $n \in \mathbb{N}$ there exists a Heisenberg C^* -algebra \mathfrak{A}_{2n} .*

Theorem 2.3 will be proved in Section 7. From now on we'll fix a Heisenberg C^* -algebra \mathfrak{A}_{2n} for each $n \in \mathbb{N}$ unless stated otherwise.

We'll construct a C^* -algebra bundle D over M with fibres all isomorphic to \mathfrak{A}_{2n} and a bundle map $T^*M \rightarrow D$. Actually we shall do this construction more generally for almost symplectic bundles, which will be useful in Section 3.

Definition 2.4. Let $E \rightarrow M$ be a real vector bundle, and let $\Pi \in \Gamma(\wedge^2 E^*)$. We call the pair (E, Π) an *almost Poisson bundle* over M . If Π is nondegenerate everywhere we call it an *almost symplectic bundle* over M . For two almost Poisson bundles (E, Π_E) and (F, Π_F) over M we call a bundle map $\psi : E \rightarrow F$ an *almost Poisson map* if $\Pi_F(\psi(f), \psi(g)) = \Pi_E(f, g)$ for all $f, g \in \Gamma(E)$.

As the case for symplectic manifolds [5], one sees easily that every almost symplectic bundle (E, Π) has compatible almost complex structures.

Lemma 2.5. *Let (E, Π) be an almost symplectic bundle over M with a compatible almost complex structure J . Say $\dim E = 2n$. Then for any $x \in M$ there exists a neighborhood U of x and sections $f_1, \dots, f_n, g_1, \dots, g_n$ of E on U such that they are a unitary basis at each point of U .*

Proof. It is easy to see that we can find sections $f_1, \dots, f_n, g_1, \dots, g_n$ near x such that under this basis, Π is of the matrix form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ at each point.

Let $E^+ = \text{span}\{f_1, \dots, f_n\}$. Then near x this is a subbundle of E , and E_y^+ is a Lagrangian subspace of E_y at each point y . Let f'_1, \dots, f'_n be n sections of E^+ near x such that $f'_1(y), \dots, f'_n(y)$ is an orthonormal basis at each point y . Then $f'_1, \dots, f'_n, Jf'_1, \dots, Jf'_n$ satisfy the requirement. \square

Lemma 2.5 shows that the structure group of (E, Ω, J) is $U(n)$. Let $U(E)$ be the set of unitary basis of E at all points, then $U(E)$ is a principle $U(n)$ -bundle on M . As usual, the action ρ of $U(n)$ on \mathfrak{A}_{2n} induces a C^* -algebra bundle $D = U(E) \times_{U(n)} \mathfrak{A}_{2n}$ over M , which is the quotient of $U(E) \times \mathfrak{A}_{2n}$ by the relation $(a, T) \sim (ag, g^{-1}T)$ with $g \in U(n)$. Then D has all fibres isomorphic to \mathfrak{A}_{2n} . Notice that the induced vector bundle $E' := U(E) \times_{U(n)} \mathbb{R}^{2n}$ has an induced almost symplectic structure and an induced almost complex structure. Clearly E' equipped with these structures is isomorphic to (E, Ω, J) . For each $\hbar \in (0, 1]$ the $U(n)$ -equivariant linear map $\varphi_{\hbar} : \mathbb{R}^{2n} \rightarrow \mathfrak{A}_{2n}$ also induces a bundle map $E \cong E' \rightarrow D$, which we still denote by φ_{\hbar} .

Definition 2.6. Let (E, Π, J) be as in Lemma 2.5. We call the C^* -algebra bundle D constructed above the *quantization bundle* of (E, Ω, J) , and call the bundle maps $\varphi_{\hbar} : E \rightarrow D$ the *quantization maps*.

Conventions 2.7. Let E and D be a real vector bundle and a C^* -algebra bundle over M respectively. Every \mathbb{R} -linear bundle map $\varphi : E \rightarrow D$ extends to a \mathbb{C} -linear bundle map from the complexified bundle $E \otimes \mathbb{C}$ to D sending $f + ig$ to $\varphi(f) + i\varphi(g)$, where $f, g \in \Gamma(E)$. We'll denote this extended map still by φ .

Since \mathfrak{A}_{2n} is unital the bundle D contains the trivial bundle $M \times \mathbb{C}$ as a subbundle naturally. Thus $\Gamma_{\infty}(D)$ contains $C_{\infty}(M)$ as a subalgebra. We are ready to construct strict quantizations for M :

Theorem 2.8. *Let (M, Π) be a symplectic manifold. Let J be a compatible almost complex structure on T^*M , and let D and φ_{\hbar} be the quantization bundle and maps of (T^*M, Π, J) . Let $\mathcal{A}_{\hbar} = \Gamma_{\infty}(D)$ and $\pi_{\hbar}(f) = f + \hbar^{\frac{1}{2}}\varphi_{\hbar}(df)$ for all $0 < \hbar \leq 1$ and $f \in C_c^{\infty}(M)$. Also let $\mathcal{A}_0 = C_{\infty}(M)$, and let π_0 be the canonical embedding of $C_c^{\infty}(M)$ into $C_{\infty}(M)$. Let $\{\mathcal{A}_{\hbar}\}$ be the subfield of the trivial continuous field of C^* -algebras $[0, 1] \times \Gamma_{\infty}(D)$ over $[0, 1]$. Then $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ is a faithful flabby strict quantization of M over $I = [0, 1]$ with $A = C_c^{\infty}(M)$. If a Lie group G has an action on M preserving Π and J , then this strict quantization is G -equivariant.*

Proof. We verify the conditions in Definition 1.1. Condition (1) follows from our choice of (\mathcal{A}_0, π_0) . Condition (2) follows from Definition 2.1(2)(3). Condition (3) follows from Definition 2.1(1) and our discussion at the beginning of this section. Thus $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ is a strict quantization of M . Since $A = C_c^{\infty}(M)$ this strict quantization is flabby. The faithfulness follows from Lemma 2.9 below. In fact we need only $\varphi_{\hbar}(\mathbb{R}^{2n} \otimes \mathbb{C}) \cap \mathbb{C}1_{\mathfrak{A}_{2n}} = \{0\}$ here, but we shall need the full result of Lemma 2.9 later in Corollary 5.6.

Lemma 2.9. *Let \mathfrak{A}_{2n} be a Heisenberg algebra. Let $V_{\hbar} = \varphi_{\hbar}(\mathbb{R}^{2n} \otimes \mathbb{C}) + (\varphi_{\hbar}(\mathbb{R}^{2n} \otimes \mathbb{C}))^*$. Then $V_{\hbar} \cap \mathbb{C}1_{\mathfrak{A}_{2n}} = \{0\}$ for every $0 < \hbar \leq 1$.*

Proof. It suffices to show that the only $U(n)$ -fixed element in V_{\hbar} is 0. Endow $U(n)$ with the normalized Haar measure. Let σ be the canonical map $\mathfrak{A}_{2n} \rightarrow (\mathfrak{A}_{2n})^{U(n)}$ defined by $\sigma(a) = \int_{U(n)} \rho_{\hbar}(a) dh$. Similarly define $\tau : \mathbb{R}^{2n} \otimes \mathbb{C} \rightarrow (\mathbb{R}^{2n} \otimes \mathbb{C})^{U(n)}$. Then $\varphi_{\hbar} \circ \tau = \sigma \circ \varphi_{\hbar}$. Clearly $(\mathbb{R}^{2n})^{U(n)} = \{0\}$, and hence $\tau(\mathbb{R}^{2n} \otimes \mathbb{C}) = \{0\}$. Consequently $(V_{\hbar})^{U(n)} = \sigma(V_{\hbar}) = \{0\}$. \square

Finally, the assertions about G -action is clear. \square

The proof of Theorem 2.8 generalizes to Theorem 2.10 immediately:

Theorem 2.10. *Let (M, Π) be an almost Poisson manifold. Let (E, Π_E, J_E) be as in Lemma 2.5. Suppose that $\psi : T^*M \rightarrow E$ is an almost Poisson map (Definition 2.4). Let D and φ_{\hbar} be the quantization bundle and maps of (E, Π_E, J_E) . Let $\mathcal{A}_{\hbar} = \Gamma_{\infty}(D)$ and $\pi_{\hbar}(f) = f + \hbar^{\frac{1}{2}}(\varphi_{\hbar} \circ \psi)(df)$ for all $0 < \hbar \leq 1$ and $f \in C_c^{\infty}(M)$. Also let \mathcal{A}_0 and φ_0 be as in Theorem 2.8. Let $\{A_{\hbar}\}$ be the subfield of the trivial continuous field of C^* -algebras $[0, 1] \times \Gamma_{\infty}(D)$ over $[0, 1]$. Then $\{A_{\hbar}, \pi_{\hbar}\}$ is a faithful flabby strict quantization of M over $I = [0, 1]$ with $A = C_c^{\infty}(M)$. If a Lie group G has an action on M preserving Π and has an action on (E, Π_E, J_E) such that the projection $E \rightarrow M$ and the map ψ are G -equivariant, then this strict quantization is G -equivariant.*

3. STRICT QUANTIZATIONS OF ALMOST POISSON MANIFOLDS

Throughout the rest of this paper (M, Π) will be an almost Poisson manifold unless stated otherwise. To construct strict quantizations of M , by Theorem 2.10 it suffices to find (E, Π_E) and $\psi : T^*M \rightarrow E$ as in Theorem 2.10. In fact there is a canonical choice of such (E, Π_E) and ψ . This follows from Lemmas 3.1 and 3.2, for which we omit the proof.

Lemma 3.1. *Let $E \rightarrow M$ be a vector bundle, and let E^* be its dual bundle. Then $\Omega_x((u_1, v_1^*), (u_2, v_2^*)) = \langle u_1, v_2^* \rangle - \langle u_2, v_1^* \rangle$ defines an almost symplectic structure on $E \oplus E^*$, where $u_j \in E_x, v_j^* \in E_x^*$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing between E and E^* .*

Lemma 3.2. *Let (E, Π_E) be an almost Poisson bundle over M , and let $\sigma : E \rightarrow E^*$ be the induced bundle map defined by $\sigma_x(u) = \Pi_{E,x}(\cdot, u)$. Let $E \oplus E^*$ be endowed with the canonical almost symplectic structure defined in Lemma 3.1. Then the bundle map $\psi : E \rightarrow E \oplus E^*$ defined by $\psi_x(u) = (\frac{1}{\sqrt{2}}u, \frac{1}{\sqrt{2}}\sigma_x(u))$ is an almost Poisson map.*

By Theorem 2.10 we have:

Theorem 3.3. *Let Ω_M be the canonical almost symplectic structure on $T^*M \oplus TM$ defined in Lemma 3.1 for $E = T^*M$. Let $\psi : T^*M \rightarrow T^*M \oplus TM$ be the bundle map defined in Lemma 3.2 for (T^*M, Π) . Let J be a compatible almost complex structure for $(T^*M \oplus TM, \Omega_M)$, and let D and φ_{\hbar} be the quantization bundle and maps of $(T^*M \oplus TM, \Omega_M, J)$. Let $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ be as in Theorem 2.10. Then $\{A_{\hbar}, \pi_{\hbar}\}$ is a faithful flabby strict quantization of M over $I = [0, 1]$ with $A = C_c^{\infty}(M)$. If a Lie group G has a smooth action on M preserving Π and J , then this strict quantization is G -equivariant.*

We are ready to prove Theorem 1.2:

Proof of Theorem 1.2. Given any Riemannian metric on M , the induced isomorphism $TM \rightarrow T^*M$ induces an inner product on T^*M , and hence induces an inner product on $T^*M \oplus TM$ by requiring T^*M and TM to be perpendicular to each other. Therefore $T^*M \oplus TM$ has a G -invariant inner product. Notice that given an inner product on an almost symplectic bundle (E, Ω) there is a canonical way to construct a compatible almost complex structure J on E [5]. Thus $T^*M \oplus TM$ has a G -invariant compatible almost complex structure. Now the assertions follow from Theorem 3.3. \square

Corollary 3.4. *For any smooth action of a compact Lie group G on M preserving Π , there is a G -equivariant faithful strict quantization of M over $I = [0, 1]$ with $A = C_c^\infty(M)$.*

Proof. For any smooth action of a compact Lie group, the manifold admits an invariant Riemannian metric by integrating any given Riemannian metric. \square

Rieffel showed [23] that there is no strict deformation quantization of the rotationally invariant symplectic structure on S^2 respecting the action of $SO(3)$. So this gives us some sign on how more restrictive strict deformation quantizations are than strict quantizations.

Corollary 3.5. *If a Lie group G has a bi-invariant Riemannian metric, then \mathfrak{g}^* equipped with the Lie-Poisson bracket [6] admits a faithful strict quantization with $A = C_c^\infty(\mathfrak{g}^*)$ equivariant under the coadjoint action of G , where \mathfrak{g} is the Lie algebra of G and \mathfrak{g}^* is the dual.*

Proof. Identify the cotangent space of \mathfrak{g}^* at each point with \mathfrak{g} . Then for any $\xi \in \mathfrak{g}^*$ and $g \in G$, the isomorphism $T_\xi \mathfrak{g}^* \rightarrow T_{Ad_g^*(\xi)} \mathfrak{g}^*$ is exactly $Ad_g : \mathfrak{g} \rightarrow \mathfrak{g}$. In particular considering $\xi = 0$, we see that \mathfrak{g}^* admits an invariant Riemannian metric if and only if the vector space \mathfrak{g} has an inner product invariant under the adjoint action of G , if and only if G has a bi-invariant Riemannian metric. \square

Example 3.6. Assume that TM is trivial. Let $X_1, \dots, X_m \in \Gamma(TM)$ giving the trivialization of TM . Let $X_1^*, \dots, X_m^* \in \Gamma(T^*M)$ be the dual basis. Define J on $T^*M \oplus TM$ by $J(X_k^*) = X_k$ and $J(X_k) = -X_k^*$ for $1 \leq k \leq m$. Then the quantization bundle D is the trivial bundle $M \times \mathfrak{A}_{2m}$, and thus $\Gamma_\infty(D) = C_\infty(M, \mathfrak{A}_{2m}) = C_\infty(M) \otimes \mathfrak{A}_{2m}$. Let

$$\beta_{jk} = \Pi(X_j^*, X_k^*)$$

be the *structure functions*. Then the bundle map $\sigma : T^*M \rightarrow TM$ is determined by $\sigma(X_k^*) = \sum_{j=1}^m \beta_{jk} X_j$. Thus for any $f, g \in C_c^\infty(M)$ we have

$$\begin{aligned} \{f, g\} &= \sum_{1 \leq j, k \leq m} \beta_{jk} X_j(f) X_k(g), \\ \pi_{\hbar}(f) &= f \otimes 1 + \frac{1}{\sqrt{2}} \hbar^{\frac{1}{2}} \left(\sum_{1 \leq j \leq m} X_j(f) \otimes \varphi_{\hbar}(e_j) + \sum_{1 \leq j, k \leq m} \beta_{jk} X_k(f) \otimes \varphi_{\hbar}(e_{m+j}) \right), \end{aligned}$$

where e_j and φ_{\hbar} are as in Definition 2.1.

Example 3.7. Let M be the m -torus \mathbb{T}^m , and let x_1, \dots, x_m be the standard coordinates. Let θ be a real skew-symmetric $m \times m$ matrix. Define a Poisson

bracket $\{\cdot, \cdot\}$ on M by

$$\{f, g\} = \frac{1}{2} \sum_{1 \leq j, k \leq m} \theta_{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}.$$

Let $X_j = \frac{\partial}{\partial x_j}$. Let β_{jk} and J be as in Example 3.6. Then $\beta_{jk} = \frac{1}{2}\theta_{jk}$. Thus

$$\pi_{\hbar}(f) = f \otimes 1 + \frac{1}{\sqrt{2}} \hbar^{\frac{1}{2}} \left(\sum_{1 \leq j \leq m} \frac{\partial f}{\partial x_j} \otimes \varphi_{\hbar}(e_j) + \frac{1}{2} \sum_{1 \leq j, k \leq m} \theta_{jk} \frac{\partial f}{\partial x_k} \otimes \varphi_{\hbar}(e_{m+j}) \right).$$

This is very different from Rieffel's Moyal product approach [23, 26], which leads to the noncommutative torus A_{θ} .

Example 3.8. Let \mathfrak{g} be a Lie algebra, and let M be the dual \mathfrak{g}^* equipped with the Lie-Poisson bracket. Let v_1, \dots, v_m be a basis of \mathfrak{g} , and let μ_1, \dots, μ_m be the dual basis of \mathfrak{g}^* . Let c_{jkl} be the *structure constants* satisfying $[v_j, v_k] = \sum_l c_{jkl} v_l$. We may take X_j in Example 3.6 to be μ_j . Then $X_j^* = v_j$, and $\beta_{jk} = \sum_l c_{jkl} v_l$. Thus

$$\pi_{\hbar}(f) = f \otimes 1 + \frac{1}{\sqrt{2}} \hbar^{\frac{1}{2}} \left(\sum_{1 \leq j \leq m} \mu_j(f) \otimes \varphi_{\hbar}(e_j) + \sum_{1 \leq j, k, l \leq m} c_{jkl} v_l \mu_k(f) \otimes \varphi_{\hbar}(e_{m+j}) \right).$$

4. HOMOTOPY

Our construction in Theorem 3.3 depends on the choice of J . We define homotopy of strict quantizations first, then show that the homotopy class of our construction is independent of the choice of J (Proposition 4.3). The definition of homotopy of strict quantizations is similar to the usual definition of homotopy of homomorphisms between C^* -algebras:

Definition 4.1. Let $\{\mathcal{A}_{\hbar}^j, \pi_{\hbar}^j\}$ be strict quantizations of (M, Π) over I for A , where $j = 0, 1$. By a *homotopy* of these two strict quantizations, we mean a continuous field of C^* -algebras $\{\mathcal{A}_{\hbar, t}\}$ over $I \times [0, 1]$ and linear maps $\pi_{\hbar, t} : A \rightarrow \mathcal{A}_{\hbar, t}$ such that

(1) the restriction of this field on $I \times \{t\}$ gives a strict quantization of (M, Π) over A for each $t \in [0, 1]$ in a uniform way, *i.e.* for all $f, g \in A$ we have

$$\lim_{\hbar \rightarrow 0} \sup_{0 \leq t \leq 1} \|\pi_{\hbar, t}(f), \pi_{\hbar, t}(g)\| / (i\hbar) - \pi_{\hbar, t}(\{f, g\})\| = 0,$$

(2) for $t = 0, 1$ the restriction of this field gives the strict quantizations $\{\mathcal{A}_{\hbar}^0, \pi_{\hbar}^0\}$ and $\{\mathcal{A}_{\hbar}^1, \pi_{\hbar}^1\}$ respectively.

Remark 4.2. One may also define a weaker notion of homotopy without requiring the convergence in (1) to be uniformly. We adopt the stronger one because the homotopies we construct here all satisfy the uniform condition.

Clearly homotopy is an equivalence relation between strict quantizations of M .

Proposition 4.3. *The homotopy class of the strict quantization in Theorem 3.3 does not depend on the choice of the compatible almost complex structure J on $T^*M \oplus TM$.*

Proof. Let J_0 and J_1 be two compatible almost complex structures on $T^*M \oplus TM$. Let $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ be the induced inner products. Let $\langle \cdot, \cdot \rangle_t = t \langle \cdot, \cdot \rangle_1 + (1-t) \langle \cdot, \cdot \rangle_0$ for $0 \leq t \leq 1$. The canonical way of constructing compatible almost complex structure from given inner product [5] is continuous. Thus we get a continuous family of compatible almost complex structures J_t on

$T^*M \oplus TM$. Let $(D_t, \varphi_{\hbar,t})$ be the corresponding quantization bundle and maps. Denote $\Gamma_c(T^*M \oplus TM)$ the space of compactly supported sections of $T^*M \oplus TM$. Then the sections $(\varphi_{\hbar,t}(f))_{0 \leq t \leq 1}$ for $0 < \hbar \leq 1$ and $f \in \Gamma_c(T^*M \oplus TM)$ generate a continuous field of C^* -algebras over $[0, 1]$ with fibre $\Gamma_\infty(D_t)$ at t . Now it is easy to see that the strict quantizations in Theorem 3.3 associated with J_t 's combine together to give a homotopy of the ones associated with J_0 and J_1 . \square

For given (M, Π, A, I) we don't know whether there is only one homotopy class of strict quantizations. But this is the case when $\Pi = 0$ and I is an interval. Notice first that for $\Pi = 0$ there is a canonical trivial strict quantization over any I , namely $\mathcal{A}_\hbar = C_\infty(M)$ and π_\hbar is the canonical inclusion of A into $C_\infty(M)$.

Proposition 4.4. *When $\Pi = 0$ and I is an interval, every strict quantization over I is homotopic to the canonical one.*

Proof. Let $\{\mathcal{A}_\hbar, \varphi_\hbar\}$ be a strict quantization. Define a map $\gamma : I \times [0, 1] \rightarrow I$ by $\gamma(\hbar, t) = t\hbar$. Then the pull back of the field $\{\mathcal{A}_\hbar\}$ under γ is a continuous field over $I \times [0, 1]$ with fibre $\mathcal{A}_{\hbar,t} = \mathcal{A}_{t\hbar}$ at (\hbar, t) . For each $f \in A$ let $\{\pi_{\hbar,t}(f)\}$ be the pull back of the section $\{\pi_\hbar(f)\}$ under γ , namely $\pi_{\hbar,t}(f) = \pi_{t\hbar}(f)$. Then clearly this is a homotopy between the canonical strict quantization and $\{\mathcal{A}_\hbar, \varphi_\hbar\}$. \square

5. LOCAL STRICT QUANTIZATIONS

There are two different meanings for a strict quantization to be local. The first one is an intuitive one, meaning that the construction is local in the sense that we construct strict quantizations for open subsets of M first, then gluing them together to get strict quantization for M . This includes our construction in Theorem 2.10 and the construction in [19]. The second one means that the algebras and maps $\{\mathcal{A}_\hbar, \pi_\hbar\}$ are local in the sense that $\mathcal{A}_\hbar \subseteq \mathcal{D}_\hbar$ for some (upper-semi)continuous field of C^* -algebras \mathcal{D}_\hbar and the maps $\pi_\hbar : C_\infty(M) = \Gamma_\infty(M \times \mathbb{C}) \rightarrow \mathcal{A}_\hbar \hookrightarrow \Gamma_\infty(\mathcal{D}_\hbar)$ are fibrewise. Here we'll concentrate on the second meaning.

Let X be a locally compact Hausdorff space. Recall that an $C_\infty(X)$ -algebra is a C^* -algebra \mathcal{A} with an injective nondegenerate homomorphism $\gamma : C_\infty(X) \rightarrow \mathcal{M}(\mathcal{A})$ such that $\gamma(C_\infty(X))$ being contained in the center $\mathcal{ZM}(\mathcal{A})$ of the multiplier algebra $\mathcal{M}(\mathcal{A})$ [10]. This is equivalent to saying that \mathcal{A} is the global section algebra of an upper-semicontinuous field of C^* -algebras over X [21]. Under this correspondence the fibre algebra of the field at $x \in X$ is $\mathcal{A}/\overline{\gamma(I_x)\mathcal{A}}$, where $I_x = \{h \in C_\infty(X) : h(x) = 0\}$. This motivates our definition of local strict quantizations:

Definition 5.1. Let $\{\mathcal{A}_\hbar, \pi_\hbar\}$ be a strict quantization of (M, Π) on I . We call $\{\mathcal{A}_\hbar, \pi_\hbar\}$ *local* if each \mathcal{A}_\hbar is a $C_\infty(M)$ -algebra with $\gamma_\hbar : C_\infty(M) \hookrightarrow \mathcal{ZM}(\mathcal{A}_\hbar)$ such that $\|\pi_\hbar(f) - \gamma_\hbar(f)\| \rightarrow 0$ as $\hbar \rightarrow 0$ for every $f \in A$.

Clearly the strict quantizations in Theorem 2.10 are local. But the ones in [19] are not.

Proposition 5.2. *Let $\{\mathcal{A}_\hbar, \pi_\hbar\}$ be a local strict quantization of (M, Π) on I . Let $f, g \in A_{sa}$. If $\pi_\hbar(f), \pi_\hbar(g) \in (\mathcal{A}_\hbar)_{sa}$ for all $\hbar \in I$, then $\{f, g\} = 0$.*

Proof. Using the embeddings γ_\hbar in Definition 5.1 we'll identify $C_\infty(M)$ as a subalgebra of $\mathcal{M}(\mathcal{A}_\hbar)$. Then $\|\pi_\hbar(\{f, g\}) - \{f, g\}\| \rightarrow 0$ as $\hbar \rightarrow 0$. Thus Definition 1.1(3) becomes

$$(4) \quad \lim_{\hbar \rightarrow 0} \|\pi_\hbar(f), \pi_\hbar(g)\| / (i\hbar) - \{f, g\} = 0.$$

For $x \in M$ let $I_x = \{h \in C_\infty(M) : h(x) = 0\}$, and let $\mathcal{A}_{\hbar,x} = \mathcal{A}_{\hbar}/\overline{I_x \mathcal{A}_{\hbar}}$. Let $\beta_{\hbar,x} : \mathcal{A}_{\hbar} \rightarrow \mathcal{A}_{\hbar,x}$ be the quotient map. Notice that the identity of $\mathcal{M}(\mathcal{A}_{\hbar,x})$ is $\beta_{\hbar,x}(h)$ for any $h \in C_\infty(M)$ with $h(x) = 1$. Taking $\beta_{\hbar,x}$ on (4) we get

$$(5) \quad \lim_{\hbar \rightarrow 0} \|\beta_{\hbar,x}(\pi_{\hbar}(f)), \beta_{\hbar,x}(\pi_{\hbar}(g))\|/(i\hbar) - \{f, g\}(x) = 0.$$

Notice that $[\beta_{\hbar,x}(\pi_{\hbar}(f)), \beta_{\hbar,x}(\pi_{\hbar}(g))\|/(i\hbar)$ is a *self-commutator*, i.e. of the form $[S^*, S]$ for some S (for instance $S = (2\hbar)^{-1/2}\beta_{\hbar,x}(\pi_{\hbar}(f)) - i(2\hbar)^{-1/2}\beta_{\hbar,x}(\pi_{\hbar}(g))$). It is known that self-commutators can't be invertible [22, Corollary 1]. Thus $\{f, g\}(x) = 0$. \square

Corollary 5.3. *An almost Poisson manifold (M, Π) admits a Hermitian local strict quantization if and only if $\Pi = 0$.*

Proof. Assume that $\Pi \neq 0$ and that M admits a Hermitian local strict quantization. Then we can find a covector field $Y^* \in \Gamma(T^*M)$ with $\sigma(Y^*) \neq 0$, where σ is as in Lemma 3.2 for $E = T^*M$. We claim that there is a vector field $X \neq 0$ such that $X(df) = 0$ for all $f \in A$. If $\sigma(dg) \neq 0$ for some $g \in A_{sa}$, by Proposition 5.2 we may take $X = \sigma(dg)$. Otherwise we may take $X = \sigma(Y^*)$. Let Z be a nonconstant integral curve of X . Then the restriction of every $f \in A$ on Z is constant, which contradicts A being dense in $C_\infty(M)$. This proves the "only if" part. The "if" part is trivial. \square

Remark 5.4. We don't know when a local strict quantization is homotopic to a Hermitian strict quantization. As a comparison, a star product on a symplectic manifold is equivalent to a Hermitian one if and only if its characteristic class is Hermitian [20]. Also every Poisson manifold has Hermitian star products [4].

Thus the strict quantizations in Theorem 2.10 are not Hermitian unless $\Pi = 0$. In fact we can say more:

Proposition 5.5. *Let $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ be a local strict quantization of (M, Π) on I . Identify $C_\infty(M)$ as a subalgebra of $\mathcal{M}(\mathcal{A}_{\hbar})$ via γ_{\hbar} . Let $\tau_{\hbar}(f) = \pi_{\hbar}(f) - f$ for $0 < \hbar \leq 1$ and $f \in A$. Assume that*

$$(6) \quad A \cap (\tau_{\hbar}(A) + (\tau_{\hbar}(A))^*) = \{0\}$$

for every $0 < \hbar \leq 1$. If this is a strict deformation quantization, then it is Hermitian and $\Pi = 0$.

Proof. Let $f \in A$ and $0 < \hbar \leq 1$. Then $(\pi_{\hbar}(f))^* = \pi_{\hbar}(g)$ for some $g \in A$. Thus $f^* + (\tau_{\hbar}(f))^* = g + \tau_{\hbar}(g)$. By our assumption $f^* = g$. Thus this strict quantization is Hermitian. By Corollary 5.3 $\Pi = 0$. \square

Corollary 5.6. *Let $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ be a local strict quantization of (M, Π) on I with $\Pi \neq 0$. Let τ_{\hbar} be as in Proposition 5.5 and assume that (6) holds for all $0 < \hbar \leq 1$. Then $\{\mathcal{A}_{\hbar}, \pi_{\hbar}\}$ can't be restricted to a dense $*$ -subalgebra of A to get a strict deformation quantization of (M, Π) . In particular, the strict quantizations in Theorem 2.10 can't be restricted to a dense $*$ -subalgebra of $C_c^\infty(M)$ to get a strict deformation quantization unless $\Pi = 0$.*

Proof. By Lemma 2.9 the strict quantizations in Theorem 2.10 satisfy (6). \square

Corollary 5.7. *For any (M, Π) there is a faithful flabby strict quantization, which can't be restricted to any dense $*$ -subalgebra of A to get a strict deformation quantization.*

Proof. The case $\hbar \neq 0$ follows from Theorem 3.3 and Corollary 5.6. The case $\hbar = 0$ is settled in [16]. \square

Corollary 5.7 gives Question 25 in [27] a negative answer, which asks whether there is an example of a faithful strict quantization of a Poisson manifold such that it's impossible to restrict it to some dense $*$ -subalgebra to get a strict deformation quantization. This leaves the question whether we can require the strict quantization to be Hermitian.

6. FUNCTORIAL PROPERTIES

It is unlikely that there is a universal way to construct a canonical strict quantization for each Poisson manifolds such that it gives a contravariant functor from the category of Poisson manifolds with (proper) Poisson maps to the category of continuous field of C^* -algebras over I [14]. Instead, Landsman proposed other categories closely related to Morita equivalence, and showed that there is such a functor on the subcategory of dual of integrable Lie algebroids.

Though our construction in Theorem 3.3 doesn't give a contravariant functor, it does has some properties similar to functors. In this section we discuss two questions:

(1) fixing a strict quantization $\{\mathcal{A}_\hbar^M, \pi_\hbar^M\}$ of (M, Π_M) on I , for any proper Poisson map ϕ from M to another almost Poisson manifold (N, Π_N) can we find a strict quantization $\{\mathcal{A}_\hbar^N, \pi_\hbar^N\}$ of (N, Π_N) on I with a "homomorphism" of these two strict quantizations extending $\phi^* : C_\infty(N) \rightarrow C_\infty(M)$, i.e. a homomorphism $\xi_\hbar : \mathcal{A}_\hbar^N \rightarrow \mathcal{A}_\hbar^M$ for each $\hbar \in I$ such that these maps $\{\phi_\hbar\}$ send continuous sections to continuous ones and $\xi_\hbar \circ \pi_\hbar^N = \pi_\hbar^M \circ \phi^*$?

(2) the similar question but fixing the strict quantization of N instead.

The first question has a positive answer because of Theorem 1.2 and the following Proposition, whose proof is just routine verification.

Proposition 6.1. *Let $\{\mathcal{A}_\hbar^M, \pi_\hbar^M\}$ and $\{\mathcal{A}_\hbar^N, \pi_\hbar^N\}$ be strict quantizations of (M, Π_M) and (N, Π_N) on I for A^M and A^N respectively. Let $\phi : M \rightarrow N$ be a proper Poisson map with $\phi^*(A^N) \subseteq A^M$. Then the sections $\{(f, g) : f \in \Gamma(\{\mathcal{A}_\hbar^M\}), g \in \Gamma(\{\mathcal{A}_\hbar^N\}), f_0 = \phi^*(g_0)\}$ determine a continuous field of C^* -algebras over I with fibre $\mathcal{A}_\hbar^M \oplus \mathcal{A}_\hbar^N$ at $\hbar \neq 0$ and fibre $C_\infty(M)$ at $\hbar = 0$. And $(\pi_\hbar^M \circ \phi^*) \oplus \pi_\hbar^N$ for $\hbar \neq 0$ give a strict quantization for (N, Π_N) on I for A^N .*

Remark 6.2. (1) When both $\{\mathcal{A}_\hbar^M, \pi_\hbar^M\}$ and $\{\mathcal{A}_\hbar^N, \pi_\hbar^N\}$ are local in the sense of Definition 5.1, so is $\{\mathcal{A}_\hbar^M \oplus \mathcal{A}_\hbar^N, (\pi_\hbar^M \circ \phi^*) \oplus \pi_\hbar^N\}$;

(2) When $(M, \Pi_M) = (N, \Pi_N)$ and $\phi = id_M$, Proposition 6.1 shows that the set of isomorphism classes of strict quantizations of (M, Π) over I for A has a natural abelian semigroup structure. Clearly the addition is compatible with homotopy defined in Definition 4.1. Thus the set of homotopy classes of strict quantizations of (M, Π) over I for A is also an abelian semigroup.

For the second question we have a partial positive answer:

Proposition 6.3. *Let (N, Π_N) be an almost symplectic manifold, and let $k \geq \dim N$. Then there is a strict quantization $\{\mathcal{A}_\hbar^N, \pi_\hbar^N\}$ of (N, Π_N) as constructed in Theorem 3.3 such that for any proper Poisson map $\phi : (M, \Pi_M) \rightarrow (N, \Pi_N)$ with $k \geq \dim M$ there is a strict quantization $\{\mathcal{A}_\hbar^M, \pi_\hbar^M\}$ of (M, Π_M) as constructed in*

Theorem 3.3 and homomorphisms $\xi_{\hbar} : \mathcal{A}_{\hbar}^N \rightarrow \mathcal{A}_{\hbar}^M$ sending continuous sections to continuous ones with $\xi_{\hbar} \circ \pi_{\hbar}^N = \pi_{\hbar}^M \circ \phi^$.*

Proof. Let $n = \dim N$, and let $n \leq m \leq k$. We'll choose a special asymptotic representation of Heisenberg commutation relations of dimension $2m$. Let e_1, \dots, e_{2m} and e'_1, \dots, e'_{2k} be the standard basis of \mathbb{R}^{2m} and \mathbb{R}^{2k} respectively. Then the linear map $\eta : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2k}$ defined by $\eta(e_j) = e'_j, \eta(e_{j+m}) = e'_{j+k}$ for $1 \leq j \leq m$ preserves the standard symplectic structure and the standard compatible almost complex structure. Thus $U(m)$ can be thought of as the subgroup of $U(k)$ fixing e'_l, e'_{l+k} for $m < l \leq k$. Let $(\mathfrak{A}_{2k}, \varphi_{\hbar})$ be an asymptotic representation of Heisenberg commutation relations of dimension $2k$. Let \mathfrak{A}_{2m} be the C^* -subalgebra generated by $\cup_{0 < \hbar \leq 1} (\varphi_{\hbar} \circ \eta)(\mathbb{R}^{2m})$. Then $(\mathfrak{A}_{2m}, \varphi_{\hbar} \circ \eta)$ is an asymptotic representation of Heisenberg commutation relations of dimension $2m$.

Fix a compatible almost complex structure J^N on $T^*N \oplus TN$. Let $x \in M$. Since (N, Π_N) is almost symplectic, $(T^*\phi)_{\phi(x)}$ is injective. Thus $n \leq \dim M$. Let $(D^N, \sigma^N, \varphi_{\hbar}^N)$ and σ^M be as in Theorem 3.3 and Lemma 3.2 for N and M respectively. Since (N, Π_N) is almost symplectic, σ^N is invertible. Then we have a linear map $\theta_x := \sigma_x^M \circ (T^*\phi)_{\phi(x)} \circ (\sigma_{\phi(x)}^N)^{-1} : TN_{\phi(x)} \rightarrow TM_x$. Let $\zeta_x := (T^*\phi)_{\phi(x)} \oplus \theta_x : T^*N_{\phi(x)} \oplus TN_{\phi(x)} \rightarrow T^*M_x \oplus TM_x$. Easy computation shows that $TM_x \circ \theta_x$ is the identity map on $TN_{\phi(x)}$, and that ζ_x preserves the canonical symplectic structure on $T^*N_x \oplus TN_x$. Then $\{\zeta_x(T^*N_{\phi(x)} \oplus TN_{\phi(x)})\}$ is an almost symplectic subbundle of $T^*M \oplus TM$, which we'll denote by E . Let $F_x = (E_x)^{\perp}$ with respect to the almost symplectic structure. Then $\{F_x\}$ is also an almost symplectic subbundle of $T^*M \oplus TM$, which we'll denote by F . Clearly $T^*M \oplus TM = E \oplus F$, and J^N induces a compatible almost complex structure J^E on E . Take a compatible almost complex structure J^F on F . Then $J^M := J^E \oplus J^F$ is a compatible almost complex structure on $T^*M \oplus TM$. Let (D^M, φ_{\hbar}^M) be as in Theorem 3.3 for (M, Π_M, J^M) .

Say $m = \dim M$. Notice that

$$\zeta_x(u_1), \dots, \zeta_x(u_n), \mu_1, \dots, \mu_{m-n}, \zeta_x(v_1), \dots, \zeta_x(v_n), \gamma_1, \dots, \gamma_{m-n}$$

is a unitary basis of $T^*M_x \oplus TM_x$ for any unitary basis $u_1, \dots, u_n, v_1, \dots, v_n$ of $T^*N_{\phi(x)} \oplus TN_{\phi(x)}$ and any unitary basis $\mu_1, \dots, \mu_{m-n}, \gamma_1, \dots, \gamma_{m-n}$ of F_x . Because of our choice of \mathfrak{A}_{2m} and \mathfrak{A}_{2n} the map ζ_x determines a unital C^* -algebra embedding $\xi_x : D_{\phi(x)}^N \rightarrow D_x^M$ such that $\varphi_{\hbar, x}^M \circ \zeta_x = \xi_x \circ \varphi_{\hbar, \phi(x)}^N$ for all $0 < \hbar \leq 1$. Since ϕ is proper, the ξ_x 's combine to give a homomorphism $\xi : \Gamma_{\infty}(D^N) \rightarrow \Gamma_{\infty}(D^M)$ whose restriction on $C_{\infty}(N)$ is ψ^* . Clearly $\xi_{\hbar} := \xi$ satisfy the requirement. \square

Remark 6.4. (1) When both M and N have G -equivariant Riemannian metrics there is an obvious G -equivariant version of Proposition 6.3;

(2) If we can find a C^* -algebra \mathfrak{A}_{∞} with an action of $U(\infty) := \cup_{n \in \mathbb{N}} U(n)$ and linear maps $\varphi_{\hbar}^n : \mathbb{R}^{2n} \rightarrow \mathfrak{A}_{\infty}$ for all $n \in \mathbb{N}$ compatible with the embedding $\eta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2k}$ in the proof of Proposition 6.3 such that for each n these maps give an asymptotic representation of Heisenberg commutation relations, then we can use \mathfrak{A}_{∞} instead of \mathfrak{A}_{2k} in the above proof and hence throw away the requirement $k \geq \dim M$. But we don't know whether such infinite dimensional asymptotic representation of Heisenberg commutation relations exists or not.

7. ASYMPTOTIC REPRESENTATION OF HEISENBERG COMMUTATION RELATIONS

In this section we prove Theorem 2.3.

Lemma 7.1. *Let H be a separable Hilbert space with orthonormal basis $\{e_j\}_{j=1}^\infty \cup \{e'_j\}_{j=1}^\infty$. Then there exist norm continuous paths $T(\hbar), S(\hbar)$ of operators in $B(H)$ for $0 < \hbar \leq 1$ such that*

- (1) $[T(\hbar), S(\hbar)](e_{2j-1}) = (1 + \hbar)ie_{2j-1}$,
- (2) $[T(\hbar), S(\hbar)](e_{2j}) = (1 - \hbar)ie_{2j}$,
- (3) $[T(\hbar), S(\hbar)](e'_j) = ie'_j$,
- (4) $\hbar \|T(\hbar)\|$ and $\hbar \|S(\hbar)\|$ are bounded uniformly in \hbar .

Brown and Pearcy [3] proved that for a separable Hilbert space H an operator $R \in B(H)$ is a commutator if and only if it is not a non-zero scalar modulo compact operators. For the "if" part, their proof is constructive. Since we need $T(\hbar)$ and $S(\hbar)$ to be continuous, and want some control on their norms, and the construction in [3] depends on some choices of isomorphisms of Hilbert spaces, we write down the proof of Lemma 7.1 here, though it is just following the construction in [3].

Proof. Let $\eta_j = \frac{1}{\sqrt{2}}(e_{2j-1} + e_{2j})$, $\eta'_j = \frac{1}{\sqrt{2}}(e_{2j-1} - e_{2j})$. Define $R(\hbar) \in B(H)$ by $R(\hbar)(e_{2j-1}) = (1 + \hbar)ie_{2j-1}$, $R(\hbar)(e_{2j}) = (1 - \hbar)ie_{2j}$ and $R(\hbar)(e'_j) = ie'_j$. Define $Z(\hbar) \in B(H)$ by $Z(\hbar)(\eta_j) = \eta_j$, $Z(\hbar)(e'_i) = e'_i$ and $Z(\hbar)(\eta'_j) = \eta_j + \hbar\eta'_j$. Then on $\text{span}_{\mathbb{C}}\{e_{2j-1}, e_{2j}\}$, with η_j, η'_j as basis, $Z(\hbar)$ and $Z(\hbar)^{-1}$ are $\begin{pmatrix} 1 & 1 \\ 0 & \hbar \end{pmatrix}$ and $\hbar^{-1} \begin{pmatrix} \hbar & -1 \\ 0 & 1 \end{pmatrix}$ respectively. Therefore $\|Z(\hbar)\| < 3$ and $\|Z(\hbar)^{-1}\| < 3\hbar^{-1}$. Hence it suffices to find continuous paths $T(\hbar), S(\hbar)$ such $[T(\hbar), S(\hbar)] = Z(\hbar)^{-1}R(\hbar)Z(\hbar)$ and $\|T(\hbar)\|, \|S(\hbar)\|$ are bounded uniformly in \hbar .

If we identify H with the closure span of $\{e'_i\}_1^\infty, \{\eta'_i\}_1^\infty, \{\eta_i\}_1^\infty$ respectively and hence identify H with $H \oplus H \oplus H$, simple calculation shows that

$$Z(\hbar)^{-1}R(\hbar)Z(\hbar) = i \begin{pmatrix} I & 0 & 0 \\ 0 & 2I & I \\ 0 & (\hbar^2 - 1)I & 0 \end{pmatrix}.$$

If we identify H with the closure span of $e'_1, \eta'_1, e'_2, \eta'_2, \dots$, and hence identify H with $H \oplus H$, we see that $Z(\hbar)^{-1}R(\hbar)Z(\hbar)$ is of the form $\begin{pmatrix} A & W \\ B & 0 \end{pmatrix}$, where A, W don't depend on \hbar and W is an isometry with $\dim(\ker(W^*)) = \infty$ and B is continuous for $0 < \hbar \leq 1$ with $\|B\| \leq 1$. By Lemma 5.1 of [3] we can find $X \in B(H)$ such that $A + WX = [B_1, B_2]$ for some $B_1, B_2 \in B(H)$ and $XW = 0$. Replacing B_1 by some $B_1 + \lambda I$ we may assume that $B_1 - I$ is invertible. By the similarity transformation

$$\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A & W \\ B & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} A + WX & W \\ -XA - XWX + B & 0 \end{pmatrix}$$

it suffices to find continuous paths $T(\hbar), S(\hbar)$ such that

$$[T(\hbar), S(\hbar)] = \begin{pmatrix} A + WX & W \\ -XA - XWX + B & 0 \end{pmatrix}$$

and $\|T(\hbar)\|, \|S(\hbar)\|$ are bounded uniformly in \hbar . Simple calculation shows that

$$T(\hbar) = \begin{pmatrix} B_1 & 0 \\ 0 & I \end{pmatrix}, S(\hbar) = \begin{pmatrix} B_2 & (B_1 - I)^{-1}W \\ (-XA - XWX + B)(I - B_1)^{-1} & 0 \end{pmatrix}$$

satisfy the requirements. \square

Lemma 7.2. *Let H be a separable Hilbert space. Then for any $n \in \mathbb{N}$ there exist norm continuous paths $T_1(\hbar), T_2(\hbar), \dots, T_n(\hbar), S_1(\hbar), S_2(\hbar), \dots, S_n(\hbar)$ of operator in $B(H)$ for $0 < \hbar \leq 1$ such that*

- (1) $\lim_{\hbar \rightarrow 0} \|[T_j(\hbar), S_k(\hbar)] - \delta_k^j i\| = 0$,
- (2) $[T_j(\hbar), T_k(\hbar)] = [S_j(\hbar), S_k(\hbar)] = 0$,
- (3) $\hbar^{-1/3} \|T_j(\hbar)\|$ and $\hbar^{-1/3} \|S_j(\hbar)\|$ are bounded uniformly in \hbar .

Proof. Let $T(\hbar)$ and $S(\hbar)$ be as in Lemma 7.1. Let

$$\begin{aligned} T_j(\hbar) &= I \otimes I \otimes \dots \otimes I \otimes T(\hbar^{1/3}) \otimes I \otimes \dots \otimes I, \\ S_j(\hbar) &= I \otimes I \otimes \dots \otimes I \otimes S(\hbar^{1/3}) \otimes I \otimes \dots \otimes I, \end{aligned}$$

where $T(\hbar^{1/3}), S(\hbar^{1/3})$ are at the j -th place. Identify H with $H^{\otimes n}$. Then clearly $T_j(\hbar), S_j(\hbar)$ satisfy the conditions. \square

Proof of Theorem 2.3. Let H and $T_j(\hbar), S_j(\hbar)$ be as in Lemma 7.2. For each $0 < \hbar \leq 1$ define a \mathbb{R} -linear map $\phi_\hbar : \mathbb{R}^{2n} \rightarrow B(H)$ by $\phi_\hbar(e_j) = T_j(\hbar), \phi_\hbar(e_{j+n}) = S_j(\hbar)$ for $1 \leq j \leq n$, where e_1, \dots, e_{2n} is the standard basis of \mathbb{R}^{2n} . Clearly $\{\phi_\hbar\}$ satisfy the conditions (1)-(3) in Definition 2.1.

Denote the action of $U(n)$ on \mathbb{R}^{2n} by σ . Consider the product C^* -algebra $\prod_{\hbar \in U(n)} B(H)$, whose elements are bounded maps $f : U(n) \rightarrow B(H)$. There is a natural (discontinuous) action ρ of $U(n)$ on $\prod_{\hbar \in U(n)} B(H)$ given by $\rho_g(f)(h) = f(g^{-1}h)$. For each $0 < \hbar \leq 1$ define a \mathbb{R} -linear map $\varphi_\hbar : \mathbb{R}^{2n} \rightarrow \prod_{\hbar \in U(n)} B(H)$ by $\varphi_\hbar(u)(h) = \phi_\hbar(\sigma_{h^{-1}}(u))$ for $u \in \mathbb{R}^{2n}$. Clearly φ_\hbar is $U(n)$ -equivariant. Let \mathfrak{A}_{2n} be the C^* -subalgebra of $\prod_{\hbar \in U(n)} B(H)$ generated by $\cup_{0 < \hbar \leq 1} \varphi_\hbar(\mathbb{R}^{2n})$. Then the restriction of ρ on \mathfrak{A}_{2n} is continuous. Clearly $\|\varphi_\hbar\| = \|\phi_\hbar\|$ and $\|\varphi_\hbar - \varphi_{\hbar'}\| = \|\phi_\hbar - \phi_{\hbar'}\|$, which verify the conditions (2) and (3) of Definition 2.1. Using Lemma 7.2(2) simple calculation shows that

$$\max_{1 \leq j < k \leq 2n} \|[\varphi_\hbar(e_j), \varphi_\hbar(e_k)] - \delta_{k-n}^j i\| \leq \max_{1 \leq j, k \leq n} \|[T_j(\hbar), S_k(\hbar)] - \delta_k^j i\|.$$

Then Definition 2.1(1) is also satisfied. In particular, when \hbar is small enough we have $\|[\varphi_\hbar(e_1), \varphi_\hbar(e_{n+1})] - i\| < 1$. Then $[\varphi_\hbar(e_1), \varphi_\hbar(e_{n+1})]$ is invertible in $\prod_{\hbar \in U(n)} B(H)$ and hence \mathfrak{A}_{2n} contain the identity of $\prod_{\hbar \in U(n)} B(H)$. \square

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