STRONG MORITA EQUIVALENCE OF HIGHER-DIMENSIONAL NONCOMMUTATIVE TORI

HANFENG LI

ABSTRACT. We show that matrices in the same orbit of the $SO(n, n|\mathbb{Z})$ action on the space of $n \times n$ skew-symmetric matrices give strongly Morita equivalent noncommutative tori, both at the C^*-algebra level and at the smooth algebra level. This proves a conjecture of Rieffel and Schwarz.

1. Introduction

Let $n \geq 2$ and $\mathcal{T}_n$ be the space of $n \times n$ real skew-symmetric matrices. For each $\theta \in \mathcal{T}_n$ the corresponding n-dimensional noncommutative torus A_θ is defined as the universal C^*-algebra generated by unitaries $U_1, \cdots, U_n$ satisfying the relation

$$U_k U_j = e(\theta_{kj}) U_j U_k,$$

where $e(t) = e^{2\pi i t}$. Noncommutative tori are one of the canonical examples in noncommutative differential geometry [12, 2].

One may also consider the smooth version A_θ^∞ of a noncommutative torus, which is the algebra of formal series

$$\sum c_{j_1, \cdots, j_n} U_1^{j_1} \cdots U_n^{j_n},$$

where the coefficient function $\mathbb{Z}^n \ni (j_1, \cdots, j_n) \mapsto c_{j_1, \cdots, j_n}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ i.e. the space of $\mathbb{C}$-valued functions on $\mathbb{Z}^n$ which vanish at infinity more rapidly than any polynomial grows. This is the space of smooth elements of A_θ for the canonical action of $\mathbb{T}^n$ on A_θ.

The notion of (strong) Morita equivalence of C^*-algebras was introduced by Rieffel [8, 10]. Strongly Morita equivalent C^*-algebras share a lot of important properties such as equivalent categories of modules, isomorphic K-groups, etc., and hence are usually thought to have the same geometry. In [14] Schwarz also introduced the notion of complete Morita equivalence of smooth noncommutative tori (see Section 2 below), which is stronger than strong Morita equivalence and has important application in M(atrix) theory [14, 4].

A natural question is to classify noncommutative tori up to strong Morita equivalence. Such results have important application to physics [3, 14]. For $n = 2$ this was done by Rieffel [9]. In this case there is a (densely defined) action of the group $GL(2, \mathbb{Z})$ on $\mathcal{T}_2$, and two matrices in $\mathcal{T}_2$ give strongly Morita equivalent noncommutative tori if and only if they are in the same orbit of this action. The higher dimensional case is much more complicated. In [13] Rieffel and Schwarz found a (densely defined) action of $SO(n, n|\mathbb{Z})$ on $\mathcal{T}_n$ generalizing the above $GL(2, \mathbb{Z})$-action. Here $O(n, n|\mathbb{R})$ is the group of linear transformations of the space $\mathbb{R}^{2n}$ preserving

Date: December 5 , 2003.
2000 *Mathematics Subject Classification.* Primary 46L87; Secondary 58B34.

the quadratic form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$, and $SO(n, n|\mathbb{Z})$ is the subgroup of $O(n, n|\mathbb{R})$ consisting of matrices with integer entries and determinant 1.

Following [13] we write the elements of $O(n, n|\mathbb{R})$ in 2×2 block form:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here A, B, C, D are $n \times n$ matrices satisfying

$$A^tC + C^tA = 0 = B^tD + D^tB, \quad A^tD + C^tB = I. \tag{1}$$

The action of $SO(n, n|\mathbb{Z})$ is then defined as

$$g\theta = (A\theta + B)(C\theta + D)^{-1}, \tag{2}$$

whenever $C\theta + D$ is invertible. For each $g \in SO(n, n|\mathbb{Z})$ this action is defined on a dense open subset of $\mathcal{T}_n$.

Rieffel and Schwarz conjectured that if two matrices in $\mathcal{T}_n$ are in the same orbit of this action then they give strongly Morita equivalent noncommutative tori, both at the C^*-algebra level and at the smooth algebra level. They proved it for matrices restricted to a certain subset of $\mathcal{T}_n$ of second category. They also showed that the converse of their conjecture at the C^*-algebra level fails for $n = 3$ [13, page 297], in contrast to the case $n = 2$, using the classification results of G. A. Elliott and Q. Lin [6].

The main goal of this paper is to prove their conjecture:

Theorem 1.1. *For any $\theta \in \mathcal{T}_n$ and $g \in SO(n, n|\mathbb{Z})$, if $g\theta$ is defined then A_θ and $A_{g\theta}$ are strongly Morita equivalent. Also A^∞_θ and $A^\infty_{g\theta}$ are completely Morita equivalent.*

Schwarz has proved that if two matrices in $\mathcal{T}_n$ give completely Morita equivalent smooth noncommutative tori then they are in the same orbit of the $SO(n, n|\mathbb{Z})$-action [14, Section 5]. Thus we get

Theorem 1.2. *Two matrices in $\mathcal{T}_n$ give completely Morita equivalent smooth noncommutative tori if and only if they are in the same orbit of the $SO(n, n|\mathbb{Z})$-action.*

We have learned recently that using classification theory N. C. Phillips has been able to show that two simple noncommutative tori A_θ and $A_{\theta'}$ are strongly Morita equivalent if and only if their ordered K_0-groups are isomorphic [7, Remark 7.9]. It would be interesting to see directly from the matrices why the ordered K_0-groups of A_θ and $A_{g\theta}$ are isomorphic.

This paper is organized as follows. Our proof of Theorem 1.1 is constructive, and we shall use the Heisenberg equivalence modules constructed by Rieffel in [11]. So we recall briefly Rieffel's construction first in Section 2. In order to apply Rieffel's construction we need to reduce an arbitrary matrix in $\mathcal{T}_n$ to one satisfying certain nice properties. This is done in Section 3. We prove Theorem 1.1 in Section 4.

Acknowledgments. I would like to thank Marc Rieffel for many helpful discussions and suggestions. I also thank Albert Schwarz for valuable discussions about complete Morita equivalence. I also thank Rolf Svegstrup for pointing out some misprints and the referee for helpful comments making the paper more readable.

2. Heisenberg Equivalence Modules

In this section we recall Schwarz's definition of complete Morita equivalence and Rieffel's construction of Heisenberg equivalence modules for noncommutative tori.

Let $L = \mathbb{R}^n$. We shall think of $\mathbb{Z}^n$ as the standard lattice in L^*, and θ as in $\wedge^2 L$. One may also describe A_θ as the universal C^*-algebra generated by unitaries $\{U_x\}_{x\in\mathbb{Z}^n}$ satisfying the relation

$$U_x U_y = \sigma_\theta(x, y) U_{x+y}, \tag{3}$$

where we write x, y as column vectors, and $\sigma_\theta(x, y) = e((x \cdot \theta y)/2)$. Under this description the smooth algebra A_θ^∞ becomes $\mathcal{S}(\mathbb{Z}^n, \sigma_\theta)$, the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ equipped with the convolution induced by (3). There is a canonical action of the Lie algebra L as derivations on A_θ^∞, which is induced by the canonical action of $\mathbb{T}^n$ on A_θ and is given explicitly by

$$\delta_X(U_x) = 2\pi i \langle X, x\rangle U_x$$

for all $X \in L$ and $x \in \mathbb{Z}^n$, where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between L and L^*.

Given a right A_θ^∞-module E, a *connection* on E is a linear map $\nabla : L \to Hom_{\mathbb{C}}(E)$ satisfying the Leibniz rule:

$$\nabla_X(fU_x) = (\nabla_X f)U_x + f \cdot \delta_X(U_x)$$

for all $X \in L$, $f \in E$ and $x \in \mathbb{Z}^n$. For each $X \in L$ the connection ∇ induces a derivation $\hat{\delta}_X$ on $End_{A_\theta^\infty}(E)$ by

$$(\hat{\delta}_X a)(f) = \nabla_X(af) - a \cdot \nabla_X f$$

for all $a \in End_{A_\theta^\infty}(E)$ and $f \in E$. If ∇ has *constant curvature*, i.e. there is skew-symmetric bilinear map $\Omega : L \times L \to \mathbb{C}$ such that $[\nabla_X, \nabla_Y] = \Omega(X, Y) \cdot 1$ for all $X, Y \in L$, then $X \mapsto \hat{\delta}_X$ is a Lie algebra homomorphism from L to the derivation space $Der(End_{A_\theta^\infty}(E))$ of $End_{A_\theta^\infty}(E)$. When E is equipped with an A_θ^∞-valued inner product, we shall consider only *Hermitian* connections, i.e. $\delta_X \langle f, g\rangle = \langle\nabla_X f, g\rangle + \langle f, \nabla_X g\rangle$ for $X \in L$ and $f, g \in A_\theta^\infty$.

We refer to [10] for the definition and standard facts about strong Morita equivalence of C^*-algebras. Let E be a strong Morita equivalence $A_{\theta'}^\infty$-A_θ^∞-bimodule. For clarity we let L_θ and $L_{\theta'}$ denote the space L for θ and θ' respectively. We say that E is a *complete Morita equivalence* $A_{\theta'}^\infty$-A_θ^∞-*bimodule* [14, page 729] if there is a constant-curvature connection ∇ on $E_{A_\theta^\infty}$ and a linear isomorphism $\phi : L_\theta \to L_{\theta'}$ such that the induced Lie algebra homomorphism $L_\theta \to Der(End_{A_\theta^\infty}(E)) = Der(A_{\theta'}^\infty)$ coincides with the composition homomorphism $L_\theta \xrightarrow{\phi} L_{\theta'} \to Der(A_{\theta'}^\infty)$. Intuitively, this means that the equivalence bimodule E is "smooth", i.e. it transfers the tangent spaces (L_θ and $L_{\theta'}$) of the noncommutative differentiable manifolds A_θ^∞ and $A_{\theta'}^\infty$ back and forth.

Next we recall Rieffel's construction of Heisenberg equivalence modules in [11, Sections 2-4]. Let M be a locally compact abelian group, let $\hat{M}$ be its dual group, and let $G = M \times \hat{M}$. There is a canonical Heisenberg cocycle on G defined by

$$\beta((m, s), (l, t)) = \langle m, t\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between M and $\hat{M}$. There is also a skew bicharacter, ρ, on G defined by

$$\rho(x,y) = \beta(x,y)\bar{\beta}(y,x). \tag{4}$$

We'll concentrate on the case $M = \mathbb{R}^p \times \mathbb{Z}^q \times W$, where $p, q \in \mathbb{Z}_{\geq 0}$ with $2p+q = n$ and W is a finite abelian group. Say $W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ for some $n_1, \cdots, n_k \in \mathbb{N}$. We shall write G as $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. Let

$$P_1 = diag(\frac{1}{n_1}, \cdots, \frac{1}{n_k}), \qquad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}, \tag{5}$$

$$J_2 = \begin{pmatrix} 0 & P_1 \\ -P_1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

Then J is a square matrix of size $n+q+2k$, and we shall think of it as a 2-form on $H^* := \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q \times \mathbb{R}^{*q} \times \mathbb{R}^k \times \mathbb{R}^{*k}$. Let J' be the matrix obtained by replacing negative entries of J by 0. Then $J = J' - (J')^t$. For any $x, y \in G$ we have

$$\beta(x,y) = e(x \cdot J'y) \quad \text{and} \quad \rho(x,y) = e(x \cdot Jy),$$

where we use the natural covering map $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$ to write x and y as column vectors in $\mathbb{R}^{n+q+2k}$ (notice that though $J'y$ depends on the choice of the representative of y in $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$, the values $e(x \cdot J'y)$ and $e(x \cdot Jy)$ do not depend on such choice).

Definition 2.1. [11, Definition 4.1] By an *embedding map* for $\theta \in \mathcal{T}_n$ we mean a linear map T from L^* to H^* such that:

(1) $T(\mathbb{Z}^n) \subseteq \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$. Then we can think of $T(\mathbb{Z}^n)$ as in G via composing $T|_{\mathbb{Z}^n}$ with the natural covering map $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$.

(2) $T(\mathbb{Z}^n)$ is a lattice in G.

(3) The form J on H^* is pulled back by T to the form θ on L^*, i.e. $T^t J T = \theta$.

The condition (2) above is equivalent to

(2') The map $\tilde{T} := \gamma \circ T : L^* \to \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q$ is invertible, where γ is the projection of H^* onto $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q$.

The bimodule Rieffel constructed is the Schwartz space $\mathcal{S}(M)$, i.e. the space of smooth functions on M which, together with all their derivatives, vanish at infinity more rapidly than any polynomial grows.

Proposition 2.2. [11, Theorem 2.15, Corollary 3.8] *Let $\theta, \theta' \in \mathcal{T}_n$, and let T, S be embedding maps of L^* into H^* for θ and $-\theta'$ respectively such that $S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp$, where $(T(\mathbb{Z}^n))^\perp = \{z \in G : \rho(z,y) = 1$ for all $y \in T(\mathbb{Z}^n)\}$. Let T' and T'' be the composition maps $\mathbb{Z}^n \xrightarrow{T} G \to M$ and $\mathbb{Z}^n \xrightarrow{T} G \to \hat{M}$ respectively. Define S' and S'' similarly. Fix a Haar measure on M. Then $\mathcal{S}(M)$ is a strong Morita equivalence $A^\infty_{\theta'}$-A^∞_θ-bimodule with the module structure and inner products defined by:*

$$\begin{aligned}
(fU_x)(m) &= e(-T(x) \cdot J'T(x)/2) \langle m, T''(x) \rangle f(m - T'(x)), \\
\langle f, g \rangle_{\mathcal{S}(\mathbb{Z}^n, \sigma_\theta)}(x) &= e(-T(x) \cdot J'T(x)/2) \int_G \langle m, -T''(x) \rangle g(m + T'(x)) \bar{f}(m) \, dm, \\
(V_x f)(m) &= e(-S(x) \cdot J'S(x)/2) \langle m, -S''(x) \rangle f(m + S'(x)), \\
{}_{\mathcal{S}(\mathbb{Z}^n, \sigma_{\theta'})}\langle f, g \rangle(x) &= K \cdot e(S(x) \cdot J'S(x)/2) \int_G \langle m, S''(x) \rangle f(m) \bar{g}(m + S'(x)) \, dm,
\end{aligned}$$

where K is a positive constant and for clarity V_x denotes the unitary in $\mathcal{S}(\mathbb{Z}^n, \sigma_{\theta'})$. Moreover, there is a linear map $Q : \mathbb{R}^{*p} \times \mathbb{R}^p \times \mathbb{R}^{*q} \to Hom_{\mathbb{C}}(\mathcal{S}(M))$ such that $\nabla_X = Q_{(\tilde{T}^{-1})^*(X)}$ and $\nabla'_X = Q_{(\tilde{S}^{-1})^*(-X)}$ are connections with respect to $\mathcal{S}(M)_{A_\theta^\infty}$ and ${}_{A_{\theta'}^\infty}\mathcal{S}(M)$ respectively. The connection ∇ has constant curvature

$$\Omega = 2\pi i \tilde{T}^{-1}(\sum_{j=1}^{p} \bar{e}_j \wedge e_j),$$

where $e_1, \cdots, e_p$ are the standard basis of $\mathbb{R}^p$ and $\bar{e}_1, \cdots, \bar{e}_p$ are the dual basis of $\mathbb{R}^{*p}$. Thus $\mathcal{S}(M)$ is a complete Morita equivalence $A_{\theta'}^\infty$-A_θ^∞-bimodule. When completed with the norm $\| f \| := \| \langle f, f \rangle_{A_\theta^\infty} \|^{\frac{1}{2}} = \| {}_{A_{\theta'}^\infty}\langle f, f \rangle \|^{\frac{1}{2}}$, $\mathcal{S}(M)$ becomes a strong Morita equivalence $A_{\theta'}$-A_θ-bimodule.

Remark 2.3. (1) The definition of embedding maps in Definition 2.1 differs from that in [11, Definition 4.1] by a sign of θ. This is because Rieffel's A_θ is our $A_{-\theta}$ (see the discussion at the end of page 285 of [11]).

(2) In [11, Section 4] the definition of embedding maps and the part of Proposition 2.2 above concerning connections and curvatures are only given for the case $W = 0$ [11, Definition 4.1] [11, pages 290-291]. The general case was discussed there in terms of tensor products with finite dimensional representations [11, Section 5]. For our purpose it's better to deal with $\mathbb{R}^p \times \mathbb{Z}^q \times W$ directly. The proofs in [11, pages 290-291] for the case $W = 0$ are easily checked to hold for the general case.

3. Decomposition of Matrices

In Proposition 4.1 we shall use the construction in [1] to find the appropriate finite abelian group W. To this goal we need the matrix $g \in SO(n, n|\mathbb{Z})$ to be of the special form in Lemma 3.3 below. We shall prove in Lemma 3.5 that every g can be reduced to such a special one.

Lemma 3.1. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Then DC^t is skew-symmetric.*

Proof. Since $g \in O(n, n|\mathbb{R})$ we have that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Hence

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}.$$

Since $O(n, n|\mathbb{R})$ is a group we have that $g^{-1} \in O(n, n|\mathbb{R})$. By (1) the matrix $(D^t)^t C^t = DC^t$ is skew-symmetric. □

Using Lemma 3.1 simple calculations yield:

Lemma 3.2. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Let $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible. Then $(C\theta + D)^{-1}C$ is skew-symmetric.*

Lemma 3.3. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n|\mathbb{Z})$, and let $p \in \mathbb{Z}_{\geq 0}$. Then the following are equivalent:*

(i) there is some $\theta \in \mathcal{T}_n$ such that $(C\theta + D)^{-1}C$ is of the form $\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$ for some $F_{11} \in GL(2p|\mathbb{R})$;

(ii) there exists a $Z \in \mathcal{T}_{2p}$ such that

$$C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} -C_{11}Z & D_{12} \\ -C_{21}Z & D_{22} \end{pmatrix},$$

where $C_{11} \in M_{2p}(\mathbb{Z})$.

In this event, the matrix $\begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}$ is invertible. The matrix Z is unique, and its entries are all rational numbers. Also for any $\theta' \in \mathcal{T}_n$ in the block form $\begin{pmatrix} \theta'_{11} & \theta'_{12} \\ \theta'_{21} & \theta'_{22} \end{pmatrix}$, where θ'_{11} has size $2p \times 2p$, the matrix $C\theta' + D$ is invertible if and only if $\theta'_{11} - Z$ is invertible. In this case

$$(C\theta' + D)^{-1}C = \begin{pmatrix} (\theta'_{11} - Z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Proof. (i)⇒(ii). From the assumption we have $C\begin{pmatrix} I_{2p} & 0 \\ 0 & 0 \end{pmatrix} = C$. Thus C has the desired form in (ii). Notice that

$$\begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} = C = (C\theta + D)\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (C_{11}\theta_{11} + D_{11})F_{11} & 0 \\ (C_{21}\theta_{11} + D_{21})F_{11} & 0 \end{pmatrix},$$

where we are writing both θ and D in block forms. Thus $C_{j1} = (C_{j1}\theta_{11} + D_{j1})F_{11}$ for $j = 1, 2$. Let $Z = \theta_{11} - (F_{11})^{-1}$. Then $D_{j1} = -C_{j1}Z$. By Lemma 3.2 the matrix F_{11} is skew-symmetric. Then so is Z.

(ii)⇒(i). For any $\theta' \in \mathcal{T}_n$ we have

$$C\theta' + D = \begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}\begin{pmatrix} \theta'_{11} - Z & \theta'_{12} \\ 0 & I \end{pmatrix}.$$

Take $\theta \in \mathcal{T}_n$ such that $C\theta + D$ is invertible. Then $\begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix}$ is invertible. Therefore $C\theta' + D$ is invertible if and only if $\theta'_{11} - Z$ is invertible. In this case simple computations yield (6). In particular $(C\theta + D)^{-1}C$ has the form described in (i). By varying θ slightly we may assume that θ is rational, *i.e.* the entries of θ are all rational numbers. Then so are F_{11} and $Z = \theta_{11} - (F_{11})^{-1}$. □

Notation 3.4. For any $R \in GL(n|\mathbb{Z})$ we denote by $\rho(R)$ the matrix $\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$. For any $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$ we denote by $\mu(N)$ the matrix $\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$.

Notice that the noncommutative tori corresponding to the matrices $\rho(R)\theta = R\theta R^t$ and $\mu(N)\theta = \theta + N$ are both isomorphic to A_θ.

Lemma 3.5. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$. Then there exists an $R \in GL(n|\mathbb{Z})$ such that $g \cdot \rho(R)$ satisfies the condition (1) in Lemma 3.3 for some $p \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $V = \{X \in \mathbb{R}^n | CX = 0\}$, and let $K = V \cap \mathbb{Z}^n$. Since the entries of C are all integers, K spans V. By the elementary divisors theorem [5, page 153, Theorem III.7.8] we can find a basis $\beta_1, \cdots, \beta_n$ of $\mathbb{Z}^n$, some integer $1 \le k \le n$ and positive integers $c_k, \cdots, c_n$ such that K is generated by $c_k\beta_k, \cdots, c_n\beta_n$. Then $V = \mathrm{span}(\beta_k, \cdots, \beta_n)$. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{Z}^n$. Then $(\beta_1, \cdots, \beta_n) = (e_1, \cdots, e_n)R$ for some $R \in GL(n|\mathbb{Z})$. Let

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rho(R) \in SO(n, n|\mathbb{Z}).$$

Choose $\theta \in \mathcal{T}_n$ such that $C\theta + D$ is invertible. Let $\theta' = R^{-1}\theta(R^{-1})^t \in \mathcal{T}_n$. Now we need

Lemma 3.6. *$(C'\theta' + D')^{-1}C'$ is of the form $\begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$ for some $F_{11} \in GL(k-1|\mathbb{R})$.*

Proof. In view of Lemma 3.2 this is clearly equivalent to saying that the vectors $X = (x_1, \cdots, x_n)^t$ in $\mathbb{R}^n$ satisfying $(C'\theta' + D')^{-1}C'X = 0$ are exactly those with $x_1 = \cdots = x_{k-1} = 0$. Notice that $(C'\theta' + D')^{-1}C' = R^t(C\theta + D)^{-1}CR$. Hence $(C'\theta' + D')^{-1}C'X = 0$ if and only if $CRX = 0$, if and only if $RX \in V$, if and only if $(\beta_1, \cdots, \beta_n)X \in V$, if and only if $x_1 = \cdots = x_{k-1} = 0$. □

Back to the proof of Lemma 3.5. By Lemma 3.2 the matrix F_{11} is skew-symmetric. Since $F_{11} \in GL(k-1|\mathbb{Z})$ we see that $k - 1$ is even. This completes the proof of Lemma 3.5. □

4. Strong Morita Equivalence

In this section we prove Theorem 1.1. We shall employ the notation in Section 2 and Lemma 3.3. In view of Proposition 2.2 the key is to find embedding maps. This is established in the following

Proposition 4.1. *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$ satisfying the conditions (1) and (2) in Lemma 3.3 for some $p \in \mathbb{Z}_{\ge 0}$. Then there exist an $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$, an $R \in GL(n|\mathbb{Z})$, a $g' \in SO(n, n|\mathbb{Z})$ and a finite abelian group W such that $g = \nu(N)\rho(R)g'$ and for any $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible there are embedding maps $T, S : L^* \to H^*$ for θ and $-g'\theta$ respectively satisfying $S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp$ (see Definition 2.1(1) and Proposition 2.2 for the meaning of $(T(\mathbb{Z}^n))^\perp$).*

Proof. Let Z be as in Lemma 3.3 for g. Then Z is rational, and hence there is some $m \in \mathbb{N}$ such that mZ is integral. Thinking of mZ as a bilinear alternating form on $\mathbb{Z}^n$, by [5, page 598, Exercise XV.17] we can find an $R \in GL(2p|\mathbb{Z})$, some integer $1 \le k \le p$ and integers $h_1, \cdots, h_k$ such that

$$mZ = R^t \begin{pmatrix} 0 & P & 0 \\ -P & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R,$$

where $P = diag(h_1, \cdots, h_k)$. Let $m_j/n_j = h_j/m$ with $(m_j, n_j) = 1$ and $n_j > 0$ for each $1 \le j \le k$. Set $W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

Let $\theta \in \mathcal{T}_n$ with $C\theta + D$ invertible. We are ready to construct an embedding map for θ now. Our method is similar to that in the proof of the proposition in [13]. But our situation is more complicated since we have to deal with the torsion part W. Write θ in block form as in Lemma 3.3. By Lemmas 3.3 and 3.2 the matrix $\theta_{11} - Z$ is invertible and skew-symmetric. So we can find a $T_{11} \in GL(2p|\mathbb{R})$ such that $T_{11}^t J_0 T_{11} = \theta_{11} - Z$, where J_0 is defined in (5). Let $T_{31} = \theta_{21}$, and let T_{32} be any $q \times q$ matrix such that $T_{32} - T_{32}^t = \theta_{22}$, where $q = n - 2p$. Let $P_2 = diag(m_1, \cdots, m_k)$, and let

$$T_1 = \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix}, T_2 = \begin{pmatrix} P_2 & 0 & 0 \\ 0 & I_k & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & I_q \end{pmatrix}, T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}.$$

Then T_1, T_2 and T have sizes $(n+q) \times n$, $2k \times n$ and $(n+q+2k) \times n$ respectively. Simple calculations yield $T^t J T = \theta$, where J is defined in (5). Notice that as a linear map from $L^* = \mathbb{R}^{*n}$ to $H^* = \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{R}^q \times \mathbb{R}^{*q} \times \mathbb{R}^k \times \mathbb{R}^{*k}$, T carries the lattice $\mathbb{Z}^n = \mathbb{Z}^{2p} \times \mathbb{Z}^q$ into $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{R}^{*q} \times \mathbb{Z}^k \times \mathbb{Z}^k$. Also observe that $\tilde{T}$ (see Definition 2.1(2')) is given by the invertible matrix $\begin{pmatrix} T_{11} & 0 \\ 0 & I_q \end{pmatrix}$. Thus the conditions in Definition 2.1 are satisfied and hence T is an embedding map for θ.

Let $\mathcal{D} = T(\mathbb{Z}^n)$. By Definition 2.1(1) we may think of $\mathcal{D}$ as in $G = \mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. We need to find some embedding map of $\mathbb{Z}^n$ into G with image being exactly $\mathcal{D}^\perp = \{z \in G : \rho(z, y) = 1 \text{ for all } y \in \mathcal{D}\}$, where ρ is defined in (4).

For any $x \in G$, it is in $\mathcal{D}^\perp$ exactly if $x \cdot JTz \in \mathbb{Z}$ for all $z \in \mathbb{Z}^n$, exactly if $T^t Jx \in \mathbb{Z}^n$. Let $T_3 = \begin{pmatrix} 0 \\ -I_q \end{pmatrix}$ be an $(n+q) \times q$ matrix. Let $T_4 = diag(n_1, \cdots, n_k, n_1, \ldots, n_k)$. Set

$$\bar{T} = \begin{pmatrix} T_1 & T_3 & 0 \\ T_2 & 0 & T_4 \end{pmatrix},$$

a square matrix of size $n + q + 2k$. It is easy to check that $T^t Jx \in \mathbb{Z}^n$ exactly if $\bar{T}^t Jx \in \mathbb{Z}^{n+q+2k}$. Also it is easy to see that $\bar{T}$ is invertible. Thus

$$\mathcal{D}^\perp = (\bar{T}^t J)^{-1}(\mathbb{Z}^{n+q+2k}).$$

Recall the matrices J_0 and J_1 defined in (5). Straight-forward calculations show that

$$(\bar{T}^t J)^{-1} = \begin{pmatrix} -J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} & J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} \begin{pmatrix} T_2^t \\ 0 \end{pmatrix} T_4^{-1} \\ 0 & \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \end{pmatrix},$$

and

$$J_1 \begin{pmatrix} T_1^t \\ T_3^t \end{pmatrix}^{-1} = \begin{pmatrix} J_0 (T_{11}^t)^{-1} & 0 & -J_0 (T_{11}^t)^{-1} T_{31}^t \\ 0 & 0 & I_q \\ 0 & -I_q & T_{32}^t \end{pmatrix}.$$

Thus

$$(\bar{T}^t J)^{-1}(0^{2p} \times \mathbb{Z}^q \times 0^q \times 0^{2k}) = 0^{2p} \times 0^q \times \mathbb{Z}^q \times 0^{2k},$$

which is 0 in G. So $(\bar{T}^t J)^{-1}(\mathbb{Z}^{n+q+2k}) = (\bar{T}^t J)^{-1}(\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k})$. Let $\triangle$ be the set of all vectors $y = \begin{pmatrix} y_1 \\ 0 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^q \times \mathbb{Z}^{2k}$ satisfying $y_3 \in n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \times$

$n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z}$ and $\begin{pmatrix} y_1 \\ 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} T_2^t \\ 0 \end{pmatrix} T_4^{-1} y_3$. Then $(\bar{T}^t J)^{-1}(\triangle) = 0$ in G. For each $1 \le j \le k$ since $(m_j, n_j) = 1$ we can find $c_j, d_j \in \mathbb{Z}$ such that $c_j m_j + d_j n_j = 1$. Now we need

Lemma 4.2. *Let φ_1 be the embedding $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k}$ sending $(x_1, \cdots, x_n)^t$ to*

$$(-d_1 x_1, \cdots, -d_k x_k, 0, \cdots, 0, x_{2k+1}, \cdots, x_{2p})^t \times 0^q \times (x_{2p+1}, \cdots, x_n, c_1 x_1, \cdots, c_k x_k, x_{k+1}, \ldots, x_{2k})^t.$$

Let φ be the composition of φ_1 and $\begin{pmatrix} R^t & 0 \\ 0 & I_{2q+2k} \end{pmatrix} : \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} \to \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k}$. Then $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n)$.

Proof. Let $y = \begin{pmatrix} y_1 \\ 0 \\ y_2 \\ y_3 \end{pmatrix}$ in $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^q \times \mathbb{Z}^{2k}$ satisfying $y_3 \in n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \times n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z}$. Say $y_3 = (n_1 z_1 \cdots, n_k z_k, n_1 z_{k+1}, \cdots, n_k z_{2k})^t$. Then it is easy to see that $y \in \triangle$ exactly if

$$(R^t)^{-1} y_1 \quad = \quad (m_1 z_1, \cdots, m_k z_k, z_{k+1}, \cdots, z_{2k}, 0, \cdots, 0)^t \quad \text{and} \quad y_2 = 0.$$

It is clear from this that $\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n)$. □

Back to the proof of Proposition 4.1. Putting $\varphi : \mathbb{Z}^n \to (\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k})$ and $(\bar{T}^t J)^{-1} : \mathbb{Z}^{n+q+2k} \to H^*$ together, we get a map $S := (\bar{T}^t J)^{-1} \circ \varphi : \mathbb{Z}^n \to H^*$ with $S(\mathbb{Z}^n) = \mathcal{D}^\perp$. Let

$$Q_1 = diag(d_1, \cdots, d_k), \quad Q_2 = diag(c_1, \cdots, c_k).$$

A routine calculation shows that

$$S = \begin{pmatrix} W_1 & W_2 \\ \begin{pmatrix} 0 & -I_k & 0 \\ Q_2 & 0 & 0 \end{pmatrix} & 0 \end{pmatrix},$$

where W_1 and W_2 have sizes $(n+q) \times 2p$ and $(n+q) \times q$ respectively:

$$W_1 = \begin{pmatrix} J_0 (T_{11}^t)^{-1} R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -J_0 (T_{11}^t)^{-1} T_{31}^t \\ I_q \\ T_{32}^t \end{pmatrix}.$$

Clearly S satisfies Definition 2.1(1)(2'). Then S is an embedding map for

$$-\theta' = S^t J S = -\begin{pmatrix} \theta'_{11} & \theta'_{12} \\ \theta'_{21} & \theta'_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\theta'_{11} &= \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} RF_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -Q_2P_1 & 0 \\ Q_2P_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta'_{12} &= \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} RF_{11}\theta_{12}, \\
\theta'_{21} &= -\theta_{21}F_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix}, \\
\theta'_{22} &= -\theta_{21}F_{11}\theta_{12} + \theta_{22}.
\end{aligned}$$

Proposition 2.2 tells us that $\mathcal{S}(M)$ is a complete Morita equivalence $A^{\infty}_{\theta'}$-A^{∞}_{θ}-bimodule. Clearly the dual $\phi^* : L^*_{\theta'} \to L^*_{\theta}$ of $\phi : L_\theta \to L_{\theta'}$ is just $-\tilde{T}^{-1} \circ \tilde{S}$. A routine calculation shows that ϕ^* is given by the matrix

$$\mathscr{A} = -\begin{pmatrix} -F_{11}R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} & -F_{11}\theta_{12} \\ 0 & I_q \end{pmatrix}.$$

It is also easy to see that the matrix form of the normalized curvature $\frac{1}{2\pi i}\Omega$ is

$$\Phi = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Now that we have the matrices $\theta, \theta', \mathscr{A}$, and Ω, Schwarz [14, page 733] has shown how to find $g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in SO(n, n|\mathbb{Z})$ such that $\theta' = g'\theta$. Actually we have the formulas:

$$\begin{aligned}
C' &= \mathscr{A}^{-1}\Phi, \quad D' = \mathscr{A}^{-1} - C'\theta, \\
A' &= \mathscr{A}^t + \theta'C', \quad B' = \theta'\mathscr{A}^{-1} - A'\theta.
\end{aligned} \tag{7}$$

Our formulas (7) are exactly the equation (53) of [14], in slightly different form. Straight-forward calculations yield

$$C' = \begin{pmatrix} \begin{pmatrix} T_4 & 0 \\ 0 & -I_{2p-2k} \end{pmatrix} (R^t)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D' = \begin{pmatrix} \begin{pmatrix} 0 & -P_2 & 0 \\ P_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R & 0 \\ 0 & I_q \end{pmatrix},$$

$$A' = \begin{pmatrix} \begin{pmatrix} 0 & -Q_2 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (R^t)^{-1} & 0 \\ 0 & I_q \end{pmatrix}, \quad B' = \begin{pmatrix} \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} := g(g')^{-1}.$$

Then $\tilde{g} \in SO(n,n|\mathbb{Z})$. A routine calculation shows that $\tilde{C} = 0$. By (1) we have $I = \tilde{A}^t\tilde{D} + \tilde{C}^t\tilde{B} = \tilde{A}^t\tilde{D}$. Then $\tilde{A}$ is invertible. Recall the matrix $\rho(\tilde{A})$ in Notation 3.4. We get

$$\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix} = \begin{pmatrix} I & \tilde{B}\tilde{A}^t \\ 0 & I \end{pmatrix} \rho(\tilde{A}).$$

Hence $\begin{pmatrix} I & \tilde{B}\tilde{A}^t \\ 0 & I \end{pmatrix} = \tilde{g}(\rho(\tilde{A}))^{-1} \in SO(n,n|\mathbb{Z})$. By (1) the matrix $I^t(\tilde{B}\tilde{A}^t) = \tilde{B}\tilde{A}^t$ is skew-symmetric. So we get

$$\tilde{g} = \nu(\tilde{B}\tilde{A}^t)\rho(\tilde{A}), \quad g = \tilde{g}g' = \nu(\tilde{B}\tilde{A}^t)\rho(\tilde{A})g'.$$

Notice that $g', \tilde{B}\tilde{A}^t$ and $\tilde{A}$ do not depend on θ. This finishes the proof of Proposition 4.1. □

Remark 4.3. (1) We would like to point out that the argument right after (52) of [14] is not quite complete. When $n = 2$ the fact that W transforms the integral lattice $\wedge^{ev}(D^*)$ into itself does not imply that $\tilde{W}$ has integral entries. In other words, (7) above may not give integral matrices when $n = 2$. So the case $n = 2$ in [14] has to be dealt separately, and it does follow from [9]. In our situation we don't need to separate the case $n = 2$ since g' obviously has integral entries.

(2) Given g' explicitly, one can also check directly that $g' \in SO(n,n|\mathbb{Z})$ and $\theta' = g'\theta$: straight-forward calculations show that g' satisfies (1) and $\theta' = g'\theta$. Then g', $\tilde{g} \in O(n,n|\mathbb{Z})$ and hence we still have $I = \tilde{A}^t\tilde{D}$. Thus $det(g') = det(\tilde{g}^{-1}) = 1$. Therefore $g' \in SO(n,n|\mathbb{Z})$.

Proof of Theorem 1.1. We may think of A_θ as the universal C^*-algebra generated by unitaries $\{U_{x,\theta}\}_{x\in\mathbb{Z}}$ satisfying the relation $U_{x,\theta}U_{y,\theta} = e(x\cdot\theta y)U_{y,\theta}U_{x,\theta}$. For any $R \in GL(n|\mathbb{Z})$ and $\theta \in \mathcal{T}_n$ there is a natural isomorphism $A^\infty_\theta \to A^\infty_{\rho(R)\theta} = A^\infty_{R\theta R^t}$ given by $U_{x,\theta} \mapsto U_{(R^{-1})^t x,\rho(R)\theta}$. Under this isomorphism $\delta_{X,\theta}$ becomes $\delta_{RX,\rho(R)\theta}$ for any $X \in L^*$. Similarly, for any $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$ and $\theta \in \mathcal{T}_n$ there is a natural isomorphism $A^\infty_\theta \to A^\infty_{\mu(N)\theta} = A^\infty_{\theta+N}$ given by $U_{x,\theta} \mapsto U_{x,\mu(N)\theta}$, under which $\delta_{X,\theta}$ becomes $\delta_{X,\mu(N)\theta}$ for any $X \in L^*$. Now Theorem 1.1 follows from Lemma 3.5 and Propositions 4.1 and 2.2. □

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Department of Mathematics, University of Toronto, Toronto ON M5S 3G3, CANADA
E-mail address: `hli@fields.toronto.edu`