STRONG MORITA EQUIVALENCE OF HIGHER-DIMENSIONAL NONCOMMUTATIVE TORI. II

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Abstract. We show that two C^*-algebraic noncommutative tori are strongly Morita equivalent if and only if they have isomorphic ordered K_0-groups and centers, extending N. C. Phillips's result in the case that the algebras are simple. This is also generalized to the twisted group C^*-algebras of arbitrary finitely generated abelian groups.

1. Introduction

Let $n \geq 2$ and denote by $\mathcal{T}_n$ the space of $n \times n$ real skew-symmetric matrices. For each $\theta \in \mathcal{T}_n$, the corresponding n-dimensional (C^*-algebraic) noncommutative torus A_θ is defined as the universal C^*-algebra generated by unitaries $U_1, \cdots, U_n$ satisfying the relations

$$U_k U_j = e(\theta_{kj}) U_j U_k,$$

where $e(t) = e^{2\pi i t}$. Noncommutative tori are one of the canonical examples in noncommutative differential geometry [32, 9].

One may also consider the smooth version A_θ^∞ of a noncommutative torus, which is the algebra of formal series

$$\sum c_{j_1,\cdots,j_n} U_1^{j_1} \cdots U_n^{j_n}$$

where the coefficient function $\mathbb{Z}^n \ni (j_1, \cdots, j_n) \mapsto c_{j_1,\cdots,j_n}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$. This is the space of smooth elements of A_θ for the canonical action of $\mathbb{T}^n$ on A_θ.

A notion of Morita equivalence of C^*-algebras (as an analogue of Morita equivalence of unital rings [1, Chapter 6]) was introduced by Rieffel in [27, 30]. This is now often called strong Morita equivalence or Rieffel-Morita equivalence. Strongly Morita equivalent C^*-algebras

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share a lot in common such as equivalent categories of Hilbert C^*-modules, isomorphic K-groups, etc., and hence are usually thought of as having the same geometry.

In [35] Schwarz introduced the notion of complete Morita equivalence of smooth noncommutative tori, which includes strong Morita equivalence of the corresponding C^*-algebras, but is stronger (as it also involves the smooth structure). This has important application in M(atrix) theory [35, 19].

A natural question is to classify noncommutative tori and their smooth counterparts up to the various notions of Morita equivalence. Such results are of some interest from the point of view of physics [10, 35]. In the case $n = 2$ this was done by Rieffel [29]. In this case it does not matter what kind of Morita equivalence one considers: there is a (densely defined) action of the group $\mathrm{GL}(2, \mathbb{Z})$ on $\mathcal{T}_2$, and two matrices in $\mathcal{T}_2$ give rise to Morita equivalent noncommutative tori or smooth noncommutative tori if and only if they are in the same orbit of this action, and also if and only if the ordered K_0-groups of the algebras are isomorphic. The higher dimensional case is much more complicated and there are examples showing that the smooth counterparts of two strongly Morita equivalent noncommutative tori may fail to be Morita equivalent (as unital $\mathbb{C}$-algebras) [14] (see also [33, 6]).

After the work of Rieffel, Schwarz, and the second-named author in [33, 35, 21] (see also [40]) it is now known that n-dimensional smooth noncommutative tori are classified up to complete Morita equivalence by the (densely defined) $\mathrm{SO}(n, n|\mathbb{Z})$ action on $\mathcal{T}_n$ introduced in [33], as a generalization of the $\mathrm{GL}(2, \mathbb{Z})$ action in the 2-dimensional case. In [14] we showed that, in the generic case, n-dimensional smooth noncommutative tori are classified up to Morita equivalence (as unital $\mathbb{C}$-algebras) in the same way.

Phillips showed that two simple noncommutative tori of dimension n are strongly Morita equivalent if and only if the exponential maps on $\Lambda^{\mathrm{even}}(\mathbb{Z}^n)$ associated to the skew-symmetric matrices have the same range up to a scalar multiplication [25, Theorem 3.11], or equivalently, their ordered K_0-groups are isomorphic (the equivalence of these two formulations follows from [13, Theorem 3.1] and Proposition 4.1 below). In this paper we shall complete the classification of noncommutative tori with respect to strong Morita equivalence. It turns out that in the general (nonsimple) case one needs to know, in addition to the ordered K_0-group of the algebra, only the center. (Since the center is isomorphic to $C(\mathbb{T}^k)$ for some nonnegative integer k, it is enough to know the dimension of $\mathbb{T}^k$. See Theorem 1.1 and Remark 4.4.)

We shall also do more than classify the noncommutative tori (as defined above), up to strong Morita equivalence. Note that n-dimensional noncommutative tori are exactly the twisted group C^*-algebras of $\mathbb{Z}^n$ (see Subsection 2.2). We shall obtain the classification, up to strong Morita equivalence, of the twisted group C^*-algebras of arbitrary finitely generated abelian groups. These are exactly the C^*-algebras admitting ergodic actions of compact abelian Lie groups [23].

Theorem 1.1. *Two twisted group C^*-algebras of finitely generated abelian groups, in particular two noncommutative tori, are strongly Morita equivalent if and only if they have isomorphic ordered K_0-groups and centers.*

It is known that two unital C^*-algebras are strongly Morita equivalent if and only if they are Morita equivalent as unital $\mathbb{C}$-algebras [4, Theorem 1.8]. Thus Theorem 1.1 also classifies the twisted group C^*-algebras of finitely generated abelian groups up to Morita equivalence as unital $\mathbb{C}$-algebras.

The opposite algebra A^{op} of a C^*-algebra A is the algebra A with the multiplication reversed but all other operations, including the scalar multiplication, the same. (It is still a C^*-algebra.) A unital C^*-algebra may not be strongly Morita equivalent to its opposite algebra (see [24] for some interesting examples), and a smooth noncommutative torus may not be Morita equivalent to its opposite algebra (see the discussion after Theorem 1.1 in [14]). It is a long-standing open question whether every noncommutative torus is isomorphic to its opposite algebra. This is now known to be the case for simple noncommutative tori [25, Corollary 3.14]. Since the property of admitting an ergodic action of a fixed compact group is preserved under passing to the opposite algebra, the class of the twisted group C^*-algebras of finitely generated abelian groups is closed under the operation of taking the opposite algebra. As a consequence of Theorem 1.1, we obtain the answer to the easier version of the question above with strong Morita equivalence in place of isomorphism:

Corollary 1.2. *Any twisted group C^*-algebra of a finitely generated abelian group is strongly Morita equivalent to its opposite algebra.*

There are two main ingredients in our proof of Theorem 1.1. The first is that, geometrically speaking, every smooth noncommutative torus is completely Morita equivalent to the Cartesian product of a smooth simple noncommutative torus and an ordinary smooth torus. This result depends on Rieffel's construction of finitely generated projective modules over noncommutative tori in [31] and also on the $\mathrm{SO}(n, n|\mathbb{Z})$

action on $\mathcal{T}_n$ introduced by Rieffel and Schwarz in [33]. The second ingredient is Phillips's classification, up to strong Morita equivalence, of simple noncommutative tori, which as mentioned above depends on his structure theorem for these algebras [25, Theorem 3.8]. To extend Theorem 1.1 to the twisted group C^*-algebras of arbitrary (countable) discrete abelian groups, one might need to extend these results of Phillips, Rieffel, and Schwarz to the infinite dimensional case.

This paper is organized as follows. We review the basic definitions and facts concerning complete Morita equivalence and the twisted group algebras in Section 2. We prove the above-mentioned complete Morita equivalence result in Section 3 using the main result of [21]. Theorem 1.1 is proved in Section 4 in the case of noncommutative tori. In Section 5 we extend the methods and results of Sections 3 and 4 to the twisted group algebras of arbitrary finitely generated abelian groups, obtaining in particular Theorem 1.1 in full generality.

This work was carried out while H. Li was at the University of Toronto.

2. Preliminaries

2.1. **Complete Morita equivalence.** In this subsection we recall Schwarz's definition of complete Morita equivalence and note that it passes to quotients.

We refer the reader to [1, Sections 21 and 22] for details on algebraic Morita equivalence, to [27, 28, 30] for strong Morita equivalence, and to [35] and [19, Section 7.2] for complete Morita equivalence.

Let A and B be pre-C^*-algebras, i.e., dense sub-$*$-algebras of C^*-algebras. A strong Morita equivalence A-B-bimodule is an A-B-bimodule ${}_AE_B$ with an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ and a B-valued inner product $\langle\cdot,\cdot\rangle_B$ satisfying certain conditions (see [27, Definition 6.10] for detail; there an equivalence bimodule is called an imprimitivity bimodule). E has a norm defined by $\|x\| := \|{}_A\langle x,x\rangle\|^{1/2} = \|\langle x,x\rangle_B\|^{1/2}$ for $x \in E$ [28, Proposition 3.1]. The completion of E is a strong Morita equivalence bimodule between the completions of A and B. By [28, Theorem 3.1, Lemma 3.1] (note that the condition there that A and B be C^*-algebras is unnecessary), there is a bijective correspondence between the lattice of closed two-sided ideals J of B and the lattice of closed A-B-submodules Y of E via $Y = \overline{EJ} = \{y \in E : \langle x,y\rangle_B \in J \text{ for all } x \in E\}$ and $J = \overline{\langle E,Y\rangle}$. By symmetry a similar correspondence holds for the lattice of closed two-sided ideals of A. Moveover, if $K(J)$ is the ideal of A corresponding to Y (the submodule corresponding to J as above), then the A-valued inner product on E

drops to an $A/K(J)$-valued inner product on E/Y, and the B-valued inner product on E drops to a B/J-valued product on E/Y, so that E/Y becomes a strong Morita equivalence $A/K(Y)$-B/J-bimodule [28, Corollary 3.2]. A and B are strongly Morita equivalent if there exists a strong Morita equivalence A-B-bimodule. In particular, if A and B are strongly Morita equivalent, then so also are their completions.

Throughout the rest of this section we shall assume further that A and B are unital and spectrally invariant, i.e., an element of A (resp. B) is invertible in A (resp. B) if it is invertible in the completion of A (resp. B). Let ${}_AE_B$ be a strong Morita equivalence A-B-bimodule. Then ${}_A\langle E, E\rangle = A$ and $\langle E, E\rangle_B = B$. Furthermore, ${}_AE_B$ is an algebraic Morita equivalence A-B-bimodule (see the proof of [4, Theorem 1.8]); that is, $A = \mathrm{End}(E_B)$, $B = \mathrm{End}({}_AE)$, and ${}_AE$ and E_B are finitely generated projective modules and are generators in the sense that ${}_AA$ and B_B are direct summands of direct sums of finitely many copies of ${}_AE$ and E_B respectively. It is also easily checked that in the correspondence between closed two-sided ideals J of B and closed A-B-submodules Y of E described above, one has $Y = EJ$ (using the fact that E_B is a finitely generated projective B-module) and $J = \langle E, Y\rangle_B$ (using $Y = EJ$ and $\langle E, E\rangle_B = B$).

Suppose that a Lie algebra L_B (resp. L_A) acts on B (resp. A) as $*$-derivations. A (Hermitian) connection on E_B [8] is a linear map $\nabla : L_B \to \mathrm{Hom}_{\mathbb{C}}(E)$ satisfying the Leibniz rule:

$$\begin{aligned} \nabla_X(yb) &= (\nabla_X y)b + y(\delta_X b), \\ \delta_X \langle x, y\rangle_B &= \langle \nabla_X x, y\rangle_B + \langle x, \nabla_X y\rangle_B, \end{aligned}$$

for all $X \in L_B$, $b \in B$ and $x, y \in E$, where δ_X is the derivation of B corresponding to X. A connection ∇ is said to have constant curvature if $[\nabla_X, \nabla_{X'}] - \nabla_{[X,X']}$ is a scalar multiplication for all $X, X' \in L_B$. E is said to be a complete Morita equivalence bimodule [35] (this is called a gauge Morita equivalence bimodule in [38, 19, 37]) between (A, L_A) and (B, L_B) if there are constant curvature connections for ${}_AE$ and E_B respectively and a Lie algebra isomorphism from L_B onto L_A such that the diagram

$$\begin{array}{ccc} L_A & \longleftarrow & L_B \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_{\mathbb{C}}(E) & \end{array}$$

commutes. Let J, Y, and $K(J)$ be as above. If a Lie subalgebra $L_{B/J}$ of L_B leaves J invariant, then one checks easily that $L_{B/J}$ and $L_{A/K(J)}$ also leave Y and $K(J)$ respectively invariant, where $L_{A/K(J)}$ denotes the image of $L_{B/J}$ in L_A, and that the actions of L_B and L_A on B and A drop to actions of $L_{B/J}$ and $L_{A/K(J)}$ on B/J and $A/K(J)$ respectively such that E/Y is a complete Morita equivalence bimodule between $(A/K(J), L_{A/K(L)})$ and $(B/J, L_{B/J})$. (A, L_A) and (B, L_B) are said to be completely Morita equivalent if there exists a complete Morita equivalence bimodule between them (see [36, 37] for a more general notion called Morita equivalence of Q-algebras).

2.2. **Twisted group algebras.** In this subsection we shall recall basic definitions and facts about the twisted group C^*-algebras [42] and smooth twisted group algebras of finitely generated abelian groups.

Let G be a finitely generated abelian group, and let σ be a 2-cocycle on G (with values in the unit circle group $\mathbb{T}$), i.e., a map $G \times G \to \mathbb{T}$ such that $\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_3)\sigma(g_2, g_3)$ for all $g_1, g_2, g_3 \in G$. The twisted group C^*-algebra $C^*(G;\sigma)$ is the universal C^*-algebra generated by unitaries u_g for $g \in G$ subject to the condition $u_g \cdot u_h = \sigma(g, h)u_{gh}$. Since G is finitely generated and abelian, the dual group $\hat{G}$ is a Lie group. Because of the universal property of $C^*(G;\sigma)$, $\hat{G}$ has a canonical strongly continuous action α on $C^*(G;\sigma)$ determined by $\alpha_x(u_g) = x(g)u_g$. This action is ergodic in the sense that the fixed point elements are exactly the scalar multiples of the unit. The smooth twisted group algebra $S(G;\sigma)$ is the algebra of smooth elements of $C^*(G;\sigma)$ with respect to this action. It is a spectrally invariant dense sub-$*$-algebra of $C^*(G;\sigma)$. The Lie algebra $\mathrm{Lie}(\hat{G})$ acts on $S(G;\sigma)$ as $*$-derivations. Throughout the rest of this paper, we shall use this Lie algebra action on $S(G;\sigma)$, and when we talk about complete Morita equivalence between two $(S(G;\sigma), \mathrm{Lie}(\hat{G}))$'s we shall simply say complete Morita equivalence between the $S(G;\sigma)$'s.

Two 2-cocycles σ and σ' on G are said to be cohomologous if there is a 1-cochain λ on G, i.e., a map λ from G to $\mathbb{T}$, such that $\sigma'(g, h) = \lambda_g\lambda_h\lambda_{gh}^{-1}\sigma(g, h)$ for all $g, h \in G$. In this case, there is a natural $\hat{G}$-equivariant $*$-isomorphism from $C^*(G;\sigma)$ onto $C^*(G;\sigma')$. A map σ from $G \times G$ to $\mathbb{T}$ is said to be a bicharacter if $\sigma(g, \cdot)$ and $\sigma(\cdot, g)$ are homomorphisms from G to $\mathbb{T}$ for each $g \in G$. Clearly, every bicharacter is a 2-cocycle. Conversely, every 2-cocycle is cohomologous to a bicharacter [18, Theorem 7.1].

A bicharacter σ' on G is said to be skew-symmetric if $\sigma(g, g) = 1$ for all $g \in G$, which implies that $\sigma(g, h)\sigma(h, g) = 1$ for all $g, h \in G$. Associated to any 2-cocycle σ on G, there is a skew-symmetric bicharacter σ^* on G defined by $\sigma^*(g, h) = \sigma(g, h)\sigma(h, g)^{-1}$ for all $g, h \in G$. Two 2-cocycles σ and σ' on G are cohomologous exactly if $\sigma^* = (\sigma')^*$

[23, Proposition 3.2]. Associated to a 2-cocycle σ on G, there is also a homomorphism β_σ from G to $\hat{G}$ defined by $\beta_\sigma(g)(\cdot) = \sigma^*(g, \cdot)$ for all $g \in G$. Denote by H_σ the kernel of β_σ. The center of $C^*(G;\sigma)$ (resp. $S(G;\sigma)$) is the closed linear span of u_g, $g \in H_\sigma$, in $C^*(G;\sigma)$ (resp. $S(G;\sigma)$) and is isomorphic to the algebra of ($\mathbb{C}$-valued) continuous (resp. smooth) functions on $\widehat{H_\sigma}$ via the Fourier transform. σ is said to be nondegenerate if $H_\sigma = \{0\}$. It is known that σ is nondegenerate exactly if $C^*(G;\sigma)$ is simple [39, Theorem 3.7], and also exactly if $S(G;\sigma)$ is simple (see [5, Lemma 3.2] and Theorem 13 of [17] and the remark following).

When $G = \mathbb{Z}^n$ for $n \geq 0$, any $\theta \in \mathcal{T}_n$ gives rise to a skew-symmetric bicharacter σ_θ on $\mathbb{Z}^n$ defined by $\sigma_\theta(g,h) = e^{\pi i g\theta h^t}$. One has $A_\theta = C^*(\mathbb{Z}^n;\sigma_\theta)$ and $A^\infty_\theta = S(\mathbb{Z}^n;\sigma_\theta)$ through the identification of U_j with u_{e_j} for $1 \leq j \leq n$, where $U_1, \cdots, U_n$ are the canonical generators of A_θ and $e_1, \cdots, e_n$ are the canonical basis elements of $\mathbb{Z}^n$. On the other hand, any bicharacter σ on $\mathbb{Z}^n$ may be written as $\sigma(g,h) = e^{2\pi i g\Theta h^t}$ for some $n \times n$ real matrix Θ. Set $\theta = \Theta - \Theta^t$. Then θ is in $\mathcal{T}_n$. Note that σ and σ_θ are cohomologous via the 1-cochain λ given by $\lambda_g = e^{\pi i g\Theta g^t}$. Therefore, every 2-cocycle on $\mathbb{Z}^n$ is cohomologous to σ_θ for some $\theta \in \mathcal{T}_n$. Thus, noncommutative tori and smooth noncommutative tori are exactly all the twisted group C^*-algebras and smooth twisted group algebras, respectively, of torsion-free finitely generated abelian groups.

3. Complete Morita equivalence

As well as the case $n \geq 2$, we shall consider also 1-dimensional and 0-dimensional noncommutative tori below, though they are actually commutative. An element θ in $\mathcal{T}_n$ for $n > 0$ is said to be *nondegenerate* if any element X in $\mathbb{Z}^n$ with $X\theta \in \mathbb{Z}^n$ is 0. As a convention, let us say that θ is nondegenerate if $n = 0$. Then θ is nondegenerate exactly if the 2-cocycle σ_θ defined in Subsection 2.2 is nondegenerate, and also exactly if A_θ is simple, and also exactly if A^∞_θ is simple. The main result of this section is the following

Theorem 3.1. *For any $\theta \in \mathcal{T}_n$, the smooth noncommutative torus A^∞_θ is completely Morita equivalent to $A^\infty_{\theta'}$ for some $\theta' \in \mathcal{T}_n$ such that*

$$\theta' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix},$$

where $\tilde{\theta}$ belongs to $\mathcal{T}_k$ for some $0 \leq k \leq n$ and is nondegenerate.

Remark 3.2. Note that A_θ^∞ has a natural Fréchet topology (see for instance [14, Example 3.1]). It turns out that $A_{\theta'}^\infty$ is the topological tensor product [41, Chapter 43] of the Fréchet algebras $A_{\tilde{\theta}}^\infty$ and $C^\infty(\mathbb{T}^{n-k})$, where $C^\infty(\mathbb{T}^{n-k})$ is the algebra of smooth functions on $\mathbb{T}^{n-k}$. Thus, in geometric language, Theorem 3.1 says that every smooth noncommutative torus is completely Morita equivalent to the Cartesian product of a smooth simple noncommutative torus and an ordinary smooth torus.

Denote by $\mathrm{O}(n, n|\mathbb{R})$ the group of linear transformations of the space $\mathbb{R}^{2n}$ preserving the quadratic form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$, and by $\mathrm{SO}(n, n|\mathbb{Z})$ the subgroup of $\mathrm{O}(n, n|\mathbb{R})$ consisting of matrices with integer entries and determinant 1.

Following [33] let us write the elements of $\mathrm{O}(n, n|\mathbb{R})$ in 2×2 block form:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then A, B, C, and D are arbitrary $n \times n$ matrices satisfying

$$A^tC + C^tA = 0 = B^tD + D^tB, \quad A^tD + C^tB = I.$$

The group $\mathrm{SO}(n, n|\mathbb{Z})$ has a partial action on $\mathcal{T}_n$ [33], defined by

$$g\theta = (A\theta + B)(C\theta + D)^{-1} \tag{1}$$

whenever $C\theta + D$ is invertible. For each $g \in \mathrm{SO}(n, n|\mathbb{Z})$ this action is defined on a dense open subset of $\mathcal{T}_n$ (see the discussion before [33, Theorem]). (The set of maximal homeomorphisms between dense open subsets of a Hausdorff space can be made into a group, with the natural composition law consisting of taking ordinary composition, to the extent defined (always on a dense open subset), and then extending to the largest open subset on which this map is a homeomorphism onto another open subset (the largest such open subset always exists). The map from $\mathrm{SO}(n, n|\mathbb{Z})$ to the group of what might be called (maximal essential) partial homeomorphisms of $\mathcal{T}_n$ is a group homomorphism.)

Theorem 3.1 follows immediately from Proposition 3.3 below and [21, Theorem 1.1].

Proposition 3.3. *Let $\theta \in \mathcal{T}_n$. Then there exists $g \in \mathrm{SO}(n, n|\,\mathbb{Z})$ such that $g\theta$ is defined and*

$$g\theta = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix},$$

where $\tilde{\theta}$ belongs to $\mathcal{T}_k$ for some $0 \le k \le n$ and is nondegenerate.

Example 3.4. Let γ be a real number and m be a nonzero integer. Consider the matrix

$$\theta = \begin{pmatrix} 0 & -3/m & -2/m \\ 3/m & 0 & \gamma \\ 2/m & -\gamma & 0 \end{pmatrix} \in \mathcal{T}_3.$$

Consider $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(3,3|\,\mathbb{Z})$ where

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & -2 & 3 \\ 0 & -m & m \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $g\theta$ is defined, and

$$g\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m\gamma \\ 0 & -m\gamma & 0 \end{pmatrix}.$$

Let us establish some preliminary results to prepare for the proof of Proposition 3.3. Associated to the quadratic form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$ there is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$ given by

$$\langle (x_1, \cdots, x_{2n}),\, (y_1, \cdots, y_{2n}) \rangle = \sum_{j=1}^{n} (x_j y_{n+j} + x_{n+j} y_j).$$

The elements of $\mathrm{O}(n, n|\mathbb{R})$ are exactly those linear transformations of $\mathbb{R}^{2n}$ preserving $\langle \cdot, \cdot \rangle$. Let us say that a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ for $\mathbb{Z}^{2n}$ is compatible with the form $\langle \cdot, \cdot \rangle$ if

$$\langle e_i,\, e_j \rangle = \langle f_i,\, f_j \rangle = 0 \quad \text{and} \quad \langle e_i, f_j \rangle = \delta_{i,j}$$

for all $1 \le i,\, j \le n$. The standard basis of $\mathbb{Z}^{2n}$ is a compatible one.

Lemma 3.5. *Let M be a direct summand of $\mathbb{Z}^{2n}$ such that M is isotropic, i.e., $\langle M, M \rangle = 0$. Then any basis for M extends to a basis for $\mathbb{Z}^{2n}$ compatible with $\langle \cdot, \cdot \rangle$.*

Proof. Note that the pairing between M and $0^n \oplus \mathbb{Z}^n$ under $\langle \cdot, \cdot \rangle$ induces a homomorphism $0^n \oplus \mathbb{Z}^n \to \mathrm{Hom}(M, \mathbb{Z})$. Denote by W the kernel of this homomorphism. Then $M + W$ is also isotropic.

Denote by π the projection of $\mathbb{Z}^{2n}$ onto $\mathbb{Z}^n \oplus 0^n$. Similarly, the pairing between $\pi(M)$ and $0^n \oplus \mathbb{Z}^n$ under $\langle \cdot, \cdot \rangle$ induces a surjective homomorphism $0^n \oplus \mathbb{Z}^n \to \mathrm{Hom}(\pi(M), \mathbb{Z})$. Note that the kernel of this homomorphism is also W. Thus, $n = \mathrm{rank}(\pi(M)) + \mathrm{rank}(W)$.

Since $\pi(M)$ is a free abelian group, we can find a homomorphism $\psi : \pi(M) \to M$ such that $\pi \circ \psi$ is the identity map on $\pi(M)$. Then the restriction of π to $\psi(\pi(M))$ is injective, and hence $\psi(\pi(M)) \cap (0^n \oplus \mathbb{Z}^n) = \psi(\pi(M)) \cap \ker \pi = \{0\}$. In particular, $\psi(\pi(M)) \cap W = \{0\}$. Consequently,

$$\begin{aligned} \mathrm{rank}(M + W) &\geq \mathrm{rank}(\psi(\pi(M))) + \mathrm{rank}(W) \\ &= \mathrm{rank}(\pi(M)) + \mathrm{rank}(W) = n. \end{aligned}$$

Consider the set $\tilde{M}$ of elements of $\mathbb{Z}^{2n}$ some multiple of which by a nonzero integer is in $M + W$. Then $\tilde{M}$ is an isotropic subgroup of $\mathbb{Z}^{2n}$ with rank at least n. By the elementary divisor theorem [20, Theorem III.7.8], $\tilde{M}$ is a direct summand of $\mathbb{Z}^{2n}$, and any basis for M extends to a basis for $\tilde{M}$. Replacing M by $\tilde{M}$, we may assume that $\mathrm{rank}(M) \geq n$.

The pairing between M and $\mathbb{Z}^{2n}$ under $\langle \cdot, \cdot \rangle$ induces a homomorphism $\varphi : \mathbb{Z}^{2n} \to \mathrm{Hom}(M, \mathbb{Z})$. Note that for any $x \in \mathbb{Z}^{2n}$, if x/m is not in $\mathbb{Z}^{2n}$ for every integer $m \geq 2$, then there exists $y \in \mathbb{Z}^{2n}$ with $\langle x, y \rangle = 1$. Using again the elementary divisor theorem, one sees easily that φ is surjective. Since M is contained in the kernel of φ, we obtain

$$\begin{aligned} 2n &= \mathrm{rank}(\ker(\varphi)) + \mathrm{rank}(\mathrm{Hom}(M, \mathbb{Z})) \\ &\geq \mathrm{rank}(M) + \mathrm{rank}(\mathrm{Hom}(M, \mathbb{Z})) = 2 \cdot \mathrm{rank}(M). \end{aligned}$$

Therefore, $\mathrm{rank}(\ker(\varphi)) = \mathrm{rank}(M) = n$. Using the elementary divisor theorem one more time, one sees that M is a direct summand of $\ker(\varphi)$. It follows that the kernel of φ is exactly M.

Let $e_1, \cdots, e_n$ be a basis for M. Choose $h_1, \cdots, h_n$ in $\mathbb{Z}^{2n}$ such that $\varphi(h_1), \cdots, \varphi(h_n)$ is the dual basis of $e_1, \cdots, e_n$. Then the subgroup P of $\mathbb{Z}^{2n}$ generated by $h_1, \cdots, h_n$ maps isomorphically onto $\mathrm{Hom}(M, \mathbb{Z})$ under φ, and hence $\mathbb{Z}^{2n} = M \oplus P$. In other words, $e_1, \cdots, e_n, h_1, \cdots, h_n$ is a basis for $\mathbb{Z}^{2n}$. Note that for any $x \in \mathbb{Z}^{2n}$, $\langle x, x \rangle$ is an even integer. Define $f_j \in \mathbb{Z}^{2n}$ inductively by

$$f_j = h_j - \frac{1}{2} \langle h_j, h_j \rangle e_j - \sum_{k=1}^{j-1} \langle h_j, f_k \rangle e_k.$$

Then, clearly, $e_1, \cdots, e_n, f_1, \cdots, f_n$ is a basis for $\mathbb{Z}^{2n}$ compatible with $\langle \cdot, \cdot \rangle$. □

Remark 3.6. Let R be a principal entire ring [20, page 86] with characteristic not equal to 2. Then Lemma 3.5 holds with $\mathbb{Z}$ replaced by R.

Lemma 3.7. *For any $n \times n$ matrix A with entries in $\mathbb{C}$, there exists a function $\zeta : \{1, \cdots, n\} \to \{1, -1\}$ such that*

$$\det(A - \mathrm{diag}(\zeta(1), \cdots, \zeta(n))) \neq 0.$$

Proof. We prove the assertion by induction on n. The case $n = 1$ is trivial. Suppose that the assertion holds for $n = k$ and A is a $(k+1) \times (k+1)$ matrix. Denote by B the $k \times k$ upper left corner of A. Then we can find a function $\zeta : \{1, \cdots, k\} \to \{1, -1\}$ such that $\det(B - \mathrm{diag}(\zeta(1), \cdots, \zeta(k))) \neq 0$. Define functions $\zeta^+, \zeta^- : \{1, \cdots, k+1\} \to \{1, -1\}$ extending ζ with $\zeta^\pm(k+1) = \pm 1$. Observing that the matrices $A - \mathrm{diag}(\zeta^+(1), \cdots, \zeta^+(k+1))$ and $A - \mathrm{diag}(\zeta^-(1), \cdots, \zeta^-(k+1))$ differ at only one entry, we have

$$\begin{aligned} & \det(A - \mathrm{diag}(\zeta^-(1), \cdots, \zeta^-(k+1))) - \\ & \det(A - \mathrm{diag}(\zeta^+(1), \cdots, \zeta^+(k+1))) \\ = \; & 2 \cdot \det(B - \mathrm{diag}(\zeta(1), \cdots, \zeta(k))) \neq 0. \end{aligned}$$

Thus at least one of ζ^+ and ζ^- satisfies the requirement. This finishes the induction step. □

Lemma 3.8. *Let $e_1, \cdots, e_n, f_1, \cdots, f_n$ be a basis for $\mathbb{Z}^{2n}$ compatible with $\langle \cdot, \cdot \rangle$. Let V be an n-dimensional isotropic linear subspace of $\mathbb{R}^{2n}$, i.e., $\langle V, V \rangle = 0$. Then we can choose η_j from e_j and f_j for each $1 \leq j \leq n$ in such a way that* $\mathrm{span}_{\mathbb{R}}(\eta_1, \cdots, \eta_n) \cap V = \{0\}$.

Proof. Set $u_j = e_j + f_j$ and $v_j = e_j - f_j$ for each j. Set $W_1 = \mathrm{span}_{\mathbb{R}}(u_1, \cdots, u_n)$ and $W_2 = \mathrm{span}_{\mathbb{R}}(v_1, \cdots, v_n)$. With respect to the basis $u_1, \cdots, u_n, v_1, \cdots, v_n$, the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$ gives rise to the quadratic form

$$\mathbb{R}^{2n} \ni \sum x_j u_j + \sum y_j v_j \mapsto \sum x_j^2 - \sum y_j^2.$$

Thus, the restriction of $\langle \cdot, \cdot \rangle$ to W_2 is negative definite. Since V is isotropic, $V \cap W_2 = \{0\}$. Therefore, $V \subseteq \mathbb{R}^{2n} = W_1 \oplus W_2$ is the graph of a linear map $\varphi : W_1 \to W_2$. Denote by A the matrix of φ with respect to the bases $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$. Choose ζ as in Lemma 3.7 applied to A. Then $\varphi(u_1) - \zeta(1)v_1, \cdots, \varphi(u_n) - \zeta(n)v_n$ are linearly independent. Note that $u_1 + \varphi(u_1), \cdots, u_n + \varphi(u_n)$ is a basis for V. It follows easily that $\mathrm{span}_{\mathbb{R}}(u_1 + \zeta(1)v_1, \cdots, u_n + \zeta(n)v_n) \cap V = \{0\}$. Now we may just take η_j to be $\frac{1}{2}(u_j + \zeta(j)v_j)$ for each j. □

The next lemma is a consequence of [40, Corollary 2.3]. For the convenience of the reader, we give a direct proof here.

Lemma 3.9. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{O}(n, n|\,\mathbb{Z})$. *If* $C\theta + D$ *is invertible for some* $\theta \in \mathcal{T}_n$, *so that* $g\theta$ *is defined in the sense of (1), then* g *is in* $\mathrm{SO}(n, n|\,\mathbb{Z})$.

Proof. Suppose that $C\theta+D$ is invertible and g is not in $\mathrm{SO}(n, n|\,\mathbb{Z})$. Let φ denote the linear transformation of $\mathbb{R}^{2n}$ exchanging the 1st and the $(n+1)$-st coordinates. Then φ preserves the quadratic form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$ and has determinant -1. Thus the matrix h corresponding to φ (with respect to the standard basis of $\mathbb{R}^{2n}$) is in $\mathrm{O}(n, n|\,\mathbb{Z})$ but not in $\mathrm{SO}(n, n|\,\mathbb{Z})$. Therefore, hg is in $\mathrm{SO}(n, n|\,\mathbb{Z})$, and in particular hg acts on a dense open subset of $\mathcal{T}_n$. Perturbing θ slightly, we may assume then that $(hg)(\theta)$ is defined and that $g(\theta)$ is still defined in the sense of (1). Set $\theta' = g(\theta)$. Then (by matrix algebra) $\theta = g^{-1}(\theta')$ and hence (in the same way) $hg(g^{-1}(\theta'))$ is defined in the sense of (1). It follows (in the same way) that $h(\theta')$ is defined in the sense of (1). But it is easy to see that h does not act on any element of $\mathcal{T}_n$ in the sense of (1). Thus we get a contradiction. Therefore, g is in $\mathrm{SO}(n, n|\,\mathbb{Z})$. □

We are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Denote by $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ the standard basis of $\mathbb{R}^{2n}$. Denote by φ the linear map $0^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus 0^n$ whose matrix with respect to the bases $\beta_1, \cdots, \beta_n$ and $\alpha_1, \cdots, \alpha_n$ is θ. Since θ is skew-symmetric, the graph V of φ is an n-dimensional isotropic linear subspace of $\mathbb{R}^{2n}$. Denote by M the intersection of V and $\mathbb{Z}^{2n}$. Using the elementary divisor theorem [20, Theorem III.7.8], one sees easily that M is a direct summand of $\mathbb{Z}^{2n}$. By Lemma 3.5 we can find a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ for $\mathbb{Z}^{2n}$ compatible with $\langle\cdot,\cdot\rangle$ such that $f_1, \cdots, f_{n-k}$ is a basis for M. Observing that $f_1, \cdots, f_{n-k} \in V$, by Lemma 3.8 we may assume that $\mathrm{span}_{\mathbb{R}}(e_1, \cdots, e_n) \cap V = \{0\}$. Then V is the graph of a linear map $\psi : \mathrm{span}_{\mathbb{R}}(f_1, \cdots, f_n) \to \mathrm{span}_{\mathbb{R}}(e_1, \cdots, e_n)$. Denote by θ' the matrix of ψ with respect to the bases $f_1, \cdots, f_n$ and $e_1, \cdots, e_n$. Since $f_1, \cdots, f_{n-k} \in V$, we have $\psi(f_1) = \cdots = \psi(f_{n-k}) = 0$, and hence the first $n-k$ columns of θ' are 0. Since V is isotropic and $e_1, \cdots, e_n, f_1, \cdots, f_n$ is a basis for $\mathbb{Z}^{2n}$ compatible with $\langle\cdot,\cdot\rangle$, one sees easily that θ' is skew-symmetric. Therefore, $\theta' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix}$ for some $\tilde{\theta} \in \mathcal{T}_k$. Set g to be the $2n \times 2n$ matrix such that

$$(e_1, \cdots, e_n, f_1, \cdots, f_n)g = (\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n).$$

Since g takes a compatible basis to another one, it is in $\mathrm{O}(n, n|\,\mathbb{Z})$. Using the expressions of elements of V in terms of θ and θ' respectively, a simple calculation shows that $g\theta$ is defined in the sense of (1) and $g\theta = \theta'$. By Lemma 3.9, g is in $\mathrm{SO}(n, n|\,\mathbb{Z})$.

It remains to show that $\tilde{\theta}$ is nondegenerate. Let $(y_{n-k+1}, \cdots, y_n) \in \mathbb{Z}^k$ be such that $\tilde{\theta}(y_{n-k+1}, \cdots, y_n)^t$ has integral entries. Then

$$\sum_{j=n-k+1}^{n} y_j f_j + \psi(\sum_{j=n-k+1}^{n} y_j f_j) \in V \cap \mathbb{Z}^{2n} = M.$$

Since $f_1, \cdots, f_{n-k}$ is a basis for M, we get $y_{n-k+1} = \cdots = y_n = 0$. Therefore, $\tilde{\theta}$ is nondegenerate. □

4. Morita equivalence

In this section we prove Theorem 1.1 in the case of noncommutative tori.

We discuss first how to see whether two noncommutative tori have isomorphic ordered K_0-groups. (See [2, Section 6] for basics on ordered K_0-groups.) For any $\theta \in \mathcal{T}_n$, A_θ has a canonical tracial state τ_θ given by the integration over the canonical action of $\widehat{\mathbb{Z}^n}$. By [13, Lemma 2.3], all tracial states on A_θ induce the same homomorphism from $K_0(A_\theta)$ to $\mathbb{R}$, which we denote by ω_θ. By [13, Theorem 3.1], $\omega_\theta(K_0(A_\theta))$ is the subgroup of $\mathbb{R}$ generated by 1 and the numbers $\sum_\xi (-1)^{|\xi|} \prod_{s=1}^{m} \theta_{j_{\xi(2s-1)} j_{\xi(2s)}}$ for $1 \leq j_1 < j_2 < \cdots < j_{2m} \leq n$, where the sum is taken over all elements ξ of the permutation group S_{2m} such that $\xi(2s-1) < \xi(2s)$ for all $1 \leq s \leq m$ and $\xi(1) < \xi(3) < \cdots < \xi(2m-1)$. θ is said to be rational if its entries are all rational numbers; otherwise it is said to be nonrational. Clearly θ is rational if and only if $\omega_\theta(K_0(A_\theta))$ has rank 1.

Proposition 4.1. *Let $\theta_j \in \mathcal{T}_{n_j}$ for $j = 1, 2$. Then A_{θ_1} and A_{θ_2} have isomorphic ordered K_0-groups if and only if $\omega_{\theta_2}(K_0(A_{\theta_2})) = \mu\omega_{\theta_1}(K_0(A_{\theta_1}))$ for some real number $\mu > 0$ and either $n_1 = n_2$ or $n_1 + n_2 = 1$.*

Proof. From the Pimsner-Voiculescu exact sequence [26] one knows that $K_0(A_{\theta_j})$ is a free abelian group of rank 1 or 2^{n_j-1} depending on whether $n_j = 0$ or $n_j > 0$.

We prove first the "only if" part. Comparing the ranks of the K_0-groups we see that either $n_1 = n_2$ or $n_1 + n_2 = 1$ (i.e., one of n_1 and n_2 is 0 and the other is 1). Every unital C^*-algebra admitting an ergodic action of a compact abelian group is nuclear [23, Lemma 6.2], [12, Proposition 3.1]. Since A_{θ_j} admits an ergodic action of $\mathbb{T}^{n_j}$, it is nuclear and hence is exact. For a unital C^*-algebra A, a state on the scaled ordered K_0-group $(K_0(A)_+, K_0(A), [1_A])$ is a positive unital homomorphism from $(K_0(A)_+, K_0(A), [1_A])$ to $(\mathbb{R}_+, \mathbb{R}, 1)$. When A is exact, every state on $(K_0(A)_+, K_0(A), [1_A])$ comes from a tracial state on A [3, Corollary 3.4], [16, Theorem 9.2]. Therefore, $(K_0(A_{\theta_j})_+, K_0(A_{\theta_j}), [1_{A_{\theta_j}}])$ has a unique state, which is exactly ω_{θ_j}. Take an order isomorphism ψ from $(K_0(A_{\theta_1})_+, K_0(A_{\theta_1}))$ onto $(K_0(A_{\theta_2})_+, K_0(A_{\theta_2}))$. Then $\omega_{\theta_2} \circ \psi$ is a nontrivial positive homomorphism from $(K_0(A_{\theta_1})_+, K_0(A_{\theta_1}))$ to $(\mathbb{R}_+, \mathbb{R})$. Set μ equal to the value of $[1_{A_{\theta_1}}]$ under $\omega_{\theta_2} \circ \psi$. Then $\mu > 0$ and $\frac{1}{\mu}(\omega_{\theta_2} \circ \psi)$ is a state on $(K_0(A_{\theta_1})_+, K_0(A_{\theta_1}), [1_{A_{\theta_1}}])$. Consequently, $\frac{1}{\mu}(\omega_{\theta_2} \circ \psi) = \omega_{\theta_1}$. Evaluating both sides on $K_0(A_{\theta_1})$ we get $\omega_{\theta_2}(K_0(A_{\theta_2})) = \mu\omega_{\theta_1}(K_0(A_{\theta_1}))$.

Next we prove the "if" part. Note that $\omega_{\theta_j}(K_0(A_{\theta_j}))$ is a torsion-free finitely generated abelian group, and hence is a free abelian group. Taking a lifting of $\omega_{\theta_j}(K_0(A_{\theta_j}))$ in $K_0(A_{\theta_j})$ and identifying this lifting with $\omega_{\theta_j}(K_0(A_{\theta_j}))$, we may assume that $K_0(A_{\theta_j}) = \ker(\omega_{\theta_j}) \oplus \omega_{\theta_j}(K_0(A_{\theta_j}))$ and that ω_{θ_j} is exactly the projection onto the second summand. Now we need to distinguish the cases θ_j is rational or nonrational. Suppose that both θ_1 and θ_2 are nonrational. Then $n_1 = n_2$. By [31, Theorem 6.1], $(K_0(A_{\theta_j}))_+$ consists of exactly the elements of $K_0(A_{\theta_j})$ on which ω_{θ_j} is strictly positive, together with 0. The multiplication by μ is an order isomorphism from $\omega_{\theta_1}(K_0(A_{\theta_1}))$ onto $\omega_{\theta_2}(K_0(A_{\theta_2}))$. Then $\ker(\omega_{\theta_1})$ and $\ker(\omega_{\theta_2})$ have the same rank. Taking any isomorphism from $\ker(\omega_{\theta_1})$ onto $\ker(\omega_{\theta_2})$, we get an order isomorphism from $(K_0(A_{\theta_1})_+, K_0(A_{\theta_1}))$ onto $(K_0(A_{\theta_2})_+, K_0(A_{\theta_2}))$, as desired.

Now assume that at least one of θ_1 and θ_2 is rational. Comparing the ranks of $\omega_{\theta_1}(K_0(A_{\theta_1}))$ and $\omega_{\theta_2}(K_0(A_{\theta_2}))$ we see that both θ_1 and θ_2 are rational. Note that the partial action of $\mathrm{SO}(n, n|\mathbb{Z})$ on $\mathcal{T}_n$ preserves rationality. Thus, if we take $\theta = \theta_j$ and $n = n_j$ in Proposition 3.3, then $\tilde{\theta}$ given in Proposition 3.3 must be rational and hence k given there must be 0. By [21, Theorem 1.1], $A^\infty_{\theta_j}$ is completely Morita equivalent to $A^\infty_{0_{n\times n}}$, where $0_{n\times n}$ is the zero $n \times n$ matrix. Consequently, A_{θ_1} and A_{θ_2} are Morita equivalent and hence have isomorphic ordered K_0-groups. This finishes the proof of Proposition 4.1. □

The "only if" part of Proposition 4.1 and the proof of the "if" part show

Corollary 4.2. *Let $\theta \in \mathcal{T}_n$. Then the following statements are equivalent:*

(1) *θ is rational;*
(2) *A^∞_θ is completely Morita equivalent to $A^\infty_{0_{n\times n}}$, where $0_{n\times n}$ is the zero $n \times n$ matrix;*
(3) *A_θ is Morita equivalent to $A_{0_{n\times n}} = C(\mathbb{T}^n)$, where $C(\mathbb{T}^n)$ is the algebra of continuous functions on $\mathbb{T}^n$;*
(4) *$\omega_\theta(K_0(A_\theta))$ has rank 1.*

Lemma 4.3. *Let* θ_1 *and* θ_2 *in* $\mathcal{T}_n$ *be such that*

$$\theta_1 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta}_1 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta}_2 \end{pmatrix},$$

with $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{T}_k$*. If the ordered* K_0*-groups of* A_{θ_1} *and* A_{θ_2} *are isomorphic, then so also are those of* $A_{\tilde{\theta}_1}$ *and* $A_{\tilde{\theta}_2}$*.*

Proof. Note that $A_{\theta_j} = A_{\tilde{\theta}_j} \otimes C(\mathbb{T}^{n-k})$. Taking the evaluation at any point of $\mathbb{T}^{n-k}$ we get a unital $*$-homomorphism φ_j from A_{θ_j} to $A_{\tilde{\theta}_j}$. Denote by $(\varphi_j)_*$ the induced homomorphism from $K_0(A_{\theta_j})$ to $K_0(A_{\tilde{\theta}_j})$. Then $\omega_{\tilde{\theta}_j} \circ (\varphi_j)_*$ is exactly ω_{θ_j}. Using the embedding $A_{\tilde{\theta}_j} \hookrightarrow A_{\tilde{\theta}_j} \otimes C(\mathbb{T}^{n-k}) = A_{\theta_j}$ one sees that $(\varphi_j)_*$ is surjective. Thus,

$$\omega_{\theta_j}(K_0(A_{\theta_j})) = (\omega_{\tilde{\theta}_j} \circ (\varphi_j)_*)(K_0(A_{\theta_j})) = \omega_{\tilde{\theta}_j}(K_0(A_{\tilde{\theta}_j})).$$

Now Lemma 4.3 follows from Proposition 4.1. □

As we mentioned in Subsection 2.2, the center of A_θ is isomorphic to the algebra of continuous functions on $\widehat{H_{\sigma_\theta}}$, and hence depends only on the rank of H_{σ_θ}, which can be calculated from θ arithmetically.

We are ready to prove Theorem 1.1 in the case of noncommutative tori.

Proof of Theorem 1.1 in the case of noncommutative tori. The "only if" part follows from the fact that Morita equivalence between unital algebras (or rings) preserves both the ordered K_0-group and the center [1, Proposition 21.10]. Consider the "if" part. Suppose that A_{θ_1} and A_{θ_2} have isomorphic ordered K_0-groups and centers. By the "only if" part and Theorem 3.1, we may assume that

$$\theta_j = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta}_j \end{pmatrix}$$

for some nondegenerate $\tilde{\theta}_j \in \mathcal{T}_{k_j}$. Say that θ_j is in $\mathcal{T}_{n_j}$. Then $A_{\theta_j} \cong C(\mathbb{T}^{n_j-k_j}) \otimes A_{\tilde{\theta}_j}$. From the Pimsner-Voiculescu exact sequence [26] one knows (as recalled above) that $K_0(A_{\theta_j})$ is a free abelian group of rank 1 or 2^{n_j-1}, depending on whether $n_j = 0$ or $n_j > 0$. It follows that either $n_1 = n_2$, or $n_1 + n_2 = 1$. Note that $A_\theta = \mathbb{C}$ if $n_j = 0$ and $A_\theta = C(\mathbb{T})$ if $n_j = 1$. Therefore we must have $n_1 = n_2$. Also note that the center of A_{θ_j} is isomorphic to $C(\mathbb{T}^{n_j-k_j})$. Thus $k_1 = k_2$. By Lemma 4.3, $A_{\tilde{\theta}_1}$ and $A_{\tilde{\theta}_2}$ have isomorphic ordered K_0-groups. Since $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are nondegenerate, both $A_{\tilde{\theta}_1}$ and $A_{\tilde{\theta}_2}$ are simple. Phillips has shown that simple noncommutative tori are classified up to strong Morita equivalence by their ordered K_0-groups [25]. Therefore, $A_{\tilde{\theta}_1}$

and $A_{\tilde{\theta}_2}$ are strongly Morita equivalent. Consequently, A_{θ_1} and A_{θ_2} are strongly Morita equivalent. This finishes the proof of Theorem 1.1 in the case of noncommutative tori. □

Remark 4.4. In the case of simple noncommutative tori, note that the result of Phillips in [25] refers only to the ordered K_0-group, not the center. Theorem 1.1 in fact also does this as the center is the scalars in this case. If we consider only noncommutative tori of dimension 2 or 3, then Theorem 1.1 holds without mentioning the centers, since in these cases the dimension of the center is determined by the ordered K_0-group. The reason in the case of 2-dimensional noncommutative tori is that in this case the dimension of the center of A_θ is either 2 or 0, depending as $\omega_\theta(K_0(A_\theta))$ has rank 1 or 2, as is easily seen from the arithmetical description of $\omega_\theta(K_0(A_\theta))$ given above. The reason in the case of 3-dimensional noncommutative tori is that in this case the center of A_θ has dimension 3, 1, or 0, depending as $\omega_\theta(K_0(A_\theta))$ has rank 1, 2, or at least 3. However, Example 4.5 below shows that if we consider n-dimensional noncommutative tori for a fixed $n \geq 4$, then Theorem 1.1 does not hold any longer without keeping track of the centers.

Example 4.5. Let γ be a real algebraic integer of degree 2 (for example, $\sqrt{2}$). Set

$$\theta_1 = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & -\gamma & 0 \end{pmatrix}.$$

Then by the arithmetical description of the range of the trace on K_0 given above, $\omega_{\theta_1}(K_0(A_{\theta_1})) = \omega_{\theta_2}(K_0(A_{\theta_2}))$, and hence A_{θ_1} and A_{θ_2} have isomorphic ordered K_0-groups by Proposition 4.1. But the center of A_{θ_1} has dimension 2, while that of A_{θ_2} has dimension 0.

5. Twisted group algebras of finitely generated abelian groups

In this section we extend the results of Sections 3 and 4 to the twisted group algebras of arbitrary finitely generated abelian groups.

Denote by G_{tor} the torsion subgroup of a finitely generated abelian group G. The rank of G is the dimension of the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall that a 2-cocycle on G is nondegenerate if the subgroup H_σ defined in Subsection 2 is $\{0\}$. The following result is a generalization of Theorem 3.1.

Theorem 5.1. *Let σ be a 2-cocycle on a finitely generated abelian group G. Then there exist a finitely generated abelian group $G' = G'_1 \oplus G'_2$ and a skew-symmetric bicharacter σ' on G' such that* $\mathrm{rank}(G) = \mathrm{rank}(G')$, *$G'_2$ is torsion-free, $\sigma'(G'_1, G') = 1$, the restriction of σ' to G'_2 is nondegenerate, and $S(G'; \sigma')$ is completely Morita equivalent to $S(G; \sigma)$.*

Proof. Since G is finitely generated and abelian, we can find a non-negative integer n and a surjective homomorphism ψ from $\mathbb{Z}^n$ to G. Then the pull-back $\psi^*(\sigma)$ of σ under ψ is a 2-cocycle on $\mathbb{Z}^n$. In Subsection 2.2 we noticed that there exists an element θ of $\mathcal{T}_n$ such that $\psi^*(\sigma)$ is cohomologous to σ_θ via a 1-cochain λ on $\mathbb{Z}^n$. By Theorem 3.1, A^∞_θ is completely Morita equivalent to $A^\infty_{\theta'}$ for some $\theta' \in \mathcal{T}_n$ of the form

$$\theta' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \end{pmatrix},$$

where $\tilde{\theta} \in \mathcal{T}_k$ for some $0 \leq k \leq n$ and is nondegenerate. Let ${}_{A^\infty_{\theta'}}E_{A^\infty_\theta}$ be a complete Morita equivalence $A^\infty_{\theta'}$-A^∞_θ-bimodule with constant-curvature connections on $E_{A^\infty_\theta}$ and ${}_{A^\infty_{\theta'}}E$ and a Lie algebra isomorphism $\phi : \mathrm{Lie}(\widehat{\mathbb{Z}^n}) \to \mathrm{Lie}(\widehat{\mathbb{Z}^n})$ as in Subsection 2.1.

In Subsection 2.1 we mentioned that complete Morita equivalence passes to quotient algebras (with the actions of certain Lie subalgebras). We shall find an ideal J of A^∞_θ such that $A^\infty_\theta/J \cong S(G;\sigma)$ and identify the corresponding ideal $K(J)$ of $A^\infty_{\theta'}$. Then $A^\infty_{\theta'}/K(J)$ is completely Morita equivalent to $S(G;\sigma)$.

Note that there is a $*$-homomorphism ψ_* from $A_\theta = C^*(\mathbb{Z}^n; \sigma_\theta)$ onto $C^*(G;\sigma)$ sending u_g to $\lambda_g u_{\psi(g)}$ for all $g \in \mathbb{Z}^n$. From the universal property of $C^*(G;\sigma)$ one sees that $\ker(\psi_*)$ is the closed ideal of $C^*(\mathbb{Z}^n;\sigma_\theta)$ generated by $u_{g_j} - \lambda_{g_j}\lambda_0^{-1}, 1 \leq j \leq m$ for any basis $g_1, \cdots, g_m$ for $\ker(\psi)$. Denote $u_{g_j} - \lambda_{g_j}\lambda_0^{-1}$ by v_j. Observing that v_j is in the center of $C^*(\mathbb{Z}^n;\sigma_\theta)$, we see that $\ker(\psi_*)$ is the closure of $\sum_{j=1}^m v_j C^*(\mathbb{Z}^n;\sigma_\theta)$ in $C^*(\mathbb{Z}^n;\sigma_\theta)$. Denote by J the intersection of $S(\mathbb{Z}^n;\sigma_\theta)$ and $\ker(\psi_*)$. Then J is a closed two-sided ideal of $S(\mathbb{Z}^n;\sigma_\theta)$ and is the closure of $\sum_{j=1}^m v_j S(\mathbb{Z}^n;\sigma_\theta)$ in $S(\mathbb{Z}^n;\sigma_\theta)$. Using ψ we may identify $\hat{G}$ with a closed subgroup of $\widehat{\mathbb{Z}^n}$, and hence identify $\mathrm{Lie}(\hat{G})$ with a Lie subalgebra of $\mathrm{Lie}(\widehat{\mathbb{Z}^n})$. Clearly, ψ_* sends $A^\infty_\theta = S(\mathbb{Z}^n;\sigma_\theta)$ into $S(G;\sigma)$. It follows that $\mathrm{Lie}(\hat{G})$ preserves J. It is easily checked that elements of $S(G;\sigma)$ are of the form $\sum_{g\in G} c_g u_g$ for the coefficient function $G \ni g \mapsto c_g$ belonging to the Schwarz space $\mathcal{S}(G)$ (cf. the proof of the corollary on page 468 of [22]). Thus ψ_* sends $S(\mathbb{Z}^n;\sigma_\theta)$ onto $S(G;\sigma)$. Consequently, A^∞_θ/J and $S(G;\sigma)$ are isomorphic as pre-C^*-algebras, in a way compatible with the actions of $\mathrm{Lie}(\hat{G})$.

Now we need to find the corresponding ideal $K(J)$ of $A^\infty_{\theta'}$. There is a $*$-isomorphism φ from the center of A^∞_θ onto that of $A^\infty_{\theta'}$ determined by $\varphi(a)x = xa$ for all $a \in A^\infty_\theta$ and $x \in E$. Clearly, the corresponding closed $A^\infty_{\theta'}$-A^∞_θ-submodule Y of E is the closure of $\sum_{j=1}^m Ev_j$ in E, and $K(J)$ is the closed two-sided ideal of $A^\infty_{\theta'}$ generated by $\varphi(v_j), 1 \le j \le m$. Recall that the center of $A^\infty_\theta = S(G;\sigma_\theta)$ is the closed linear span of u_g, $g \in H_{\sigma_\theta}$, in $S(G;\sigma_\theta)$ (see Subsection 2.2). Moreover, the eigenvectors of the restriction of the action of $\mathrm{Lie}(\widehat{\mathbb{Z}^n})$ on $S(\mathbb{Z}^n;\sigma_\theta)$ to the center are exactly the scalar multiples of the u_g's for $g \in H_{\sigma_\theta}$. Therefore, for any $g \in H_{\sigma_\theta}$, up to a scalar multiple, $\varphi(u_g)$ is equal to $u_{g'}$ for some $g' \in H_{\sigma_{\theta'}}$. The map η from H_{σ_θ} to $H_{\sigma_{\theta'}}$ sending g to g' is easily seen to be an isomorphism. Consequently, $K(J)$ is the closed two-sided ideal of $A^\infty_{\theta'}$ generated by $u_{\varphi(g_j)} - \gamma_j, 1 \le j \le m$, where γ_j is a certain element of $\mathbb{T}$. Note that $\ker(\psi) \subseteq H_{\sigma_\theta}$ and $\mathbb{Z}^{n-k} \oplus 0^k = H_{\sigma_{\theta'}}$. Set $G' = \mathbb{Z}^n/\eta(\ker(\psi))$. Then $G' = G'_1 \oplus G'_2$ for $G'_1 = H_{\sigma_{\theta'}}/\eta(\ker(\psi))$ and $G'_2 = \mathbb{Z}^k$. Set σ' equal to the skew-symmetric bicharacter on G' such that $\sigma'(G'_1, G') = 1$ and the restrictions of σ' and $\sigma_{\theta'}$ to G'_2 coincide. Then the pull-back of σ' under the quotient map ψ' from $\mathbb{Z}^n$ to G' is exactly $\sigma_{\theta'}$, and the restriction of σ' to $\mathbb{Z}^k$ is nondegenerate. Note that for any basis $h_1, \cdots, h_{n-k}$ for $\mathbb{Z}^{n-k}$ and any $\mu_1, \cdots, \mu_{n-k} \in \mathbb{T}$ there is a $*$-homomorphism from $C^*(\mathbb{Z}^n;\sigma_{\theta'}) = A_{\theta'}$ onto $C^*(G';\sigma')$ sending u_{h_j} to $\mu_j u_{\psi(h_j)}$ for all $1 \le j \le k$ and sending u_h to $u_{\psi(h)}$ for all $h \in \mathbb{Z}^k$. Using the elementary divisor theorem [20, Theorem III.7.8], we can choose suitable $\mu_1, \cdots, \mu_{n-k}$ such that the kernel of the above $*$-homomorphism is exactly the closed two-sided ideal of $C^*(\mathbb{Z}^n;\sigma_{\theta'})$ generated by $u_{\varphi(g_j)} - \gamma_j, 1 \le j \le m$. Then $K(J)$ is the intersection of this ideal and $S(\mathbb{Z}^n;\sigma_{\theta'}) = A^\infty_{\theta'}$. As in the last paragraph, we may identify $\mathrm{Lie}(\widehat{G'})$ with a Lie subalgebra of $\mathrm{Lie}(\mathbb{Z}^n)$, and $K(J)$ is invariant under the action of $\mathrm{Lie}(\widehat{G'})$. Furthermore, $A^\infty_{\theta'}/K(J)$ and $S(G';\sigma')$ are isomorphic as pre-C^*-algebras, in a way compatible with the actions of $\mathrm{Lie}(\widehat{G'})$. Observing that $\mathrm{Lie}(\hat{G})$ (resp. $\mathrm{Lie}(\widehat{G'})$) consists of exactly those elements of $\mathrm{Lie}(\widehat{\mathbb{Z}^n})$ (resp. $\mathrm{Lie}(\widehat{\mathbb{Z}^n})$) acting trivially on the center of A^∞_θ (resp. $A^\infty_{\theta'}$), we see that $\mathrm{Lie}(\hat{G})$ is sent onto $\mathrm{Lie}(\widehat{G'})$ under the Lie algebra isomorphism $\phi : \mathrm{Lie}(\widehat{\mathbb{Z}^n}) \to \mathrm{Lie}(\widehat{\mathbb{Z}^n})$. Therefore, $S(G';\sigma')$ and $S(G;\sigma)$ are completely Morita equivalent. From $\mathrm{rank}(G) + \mathrm{rank}(\ker(\psi)) = \mathrm{rank}(\mathbb{Z}^n)$ and

$\mathrm{rank}(G') + \mathrm{rank}(\eta(\ker(\psi))) = \mathrm{rank}(\mathbb{Z}^n)$ we obtain $\mathrm{rank}(G) = \mathrm{rank}(G')$. This finishes the proof of Theorem 5.1. □

Recall the group H_σ and the skew-symmetric bicharacter σ^* defined in Subsection 2.2 for a 2-cocycle σ on a finitely generated abelian group G.

Remark 5.2. We indicate briefly another proof of Theorem 5.1. Note that σ^* induces a skew-symmetric bicharacter τ^* on $G'' := G/(H_\sigma)_{\mathrm{tor}}$. Then $\tau^* = (\sigma'')^*$ for some 2-cocycle σ'' on G''. Since $\widehat{G''}$ is a subgroup of $\hat{G}$, it also acts on $C^*(G;\sigma)$. One checks easily that $C^*(G;\sigma)$ and the direct sum of $|(H_\sigma)_{\mathrm{tor}}|$ many copies of $C^*(G'';\sigma'')$ are isomorphic in a way compatible with the actions of $\widehat{G''}$. Using the proof of [34, Proposition], by induction on $|G''_{\mathrm{tor}}|$, one can show that there is a free (abelian) subgroup G' of G'' of the same rank as G such that $C^*(G'';\sigma'')$ is isomorphic to $M_m(C^*(G';\sigma'))$ for some m and $S(G'';\sigma'')$ is completely Morita equivalent to $S(G';\sigma')$, where σ' is the restriction of σ'' to G'. Consequently, $C^*(G;\sigma)$ is isomorphic to the direct sum of $|(H_\sigma)_{\mathrm{tor}}|$ many copies of $M_m(C^*(G';\sigma'))$ and $S(G;\sigma)$ is completely Morita equivalent to the direct sum of $|(H_\sigma)_{\mathrm{tor}}|$ many copies of $S(G';\sigma')$. Then Theorem 5.1 follows from Theorem 3.1. However, the earlier detailed proof does not use induction and is more likely to be generalizable to the case of arbitrary (countable) discrete abelian groups.

Lemma 5.3. *In Theorem 5.1 one has* $|(H_\sigma)_{\mathrm{tor}}| = |(G'_1)_{\mathrm{tor}}|$ *and* $\mathrm{rank}(H_\sigma) = \mathrm{rank}(G'_1)$.

Proof. Recall that the center of $C^*(G;\sigma)$ is isomorphic to $C(\widehat{H_\sigma})$. Note that $|(H_\sigma)_{\mathrm{tor}}|$ and $\mathrm{rank}(H_\sigma)$ are the number of connected components and the dimension of $\widehat{H_\sigma}$ respectively. Since Morita equivalence preserves centers [1, Proposition 21.10], we obtain $|(H_\sigma)_{\mathrm{tor}}| = |(H_{\sigma'})_{\mathrm{tor}}| = |(G'_1)_{\mathrm{tor}}|$ and $\mathrm{rank}(H_\sigma) = \mathrm{rank}(H_{\sigma'}) = \mathrm{rank}(G'_1)$. □

From the Pimsner-Voiculescu exact sequence [26] one knows that the K_0-group (resp. K_1-group) of an n-dimensional noncommutative torus is a free abelian group of rank 2^{n-1} or 1 (resp. 2^{n-1} or 0) depending as $n > 0$ or $n = 0$. Since strong Morita equivalence between C^*-algebras preserves K-groups [7, Theorem 1.2], [15], we get

Corollary 5.4. *Let σ be a 2-cocycle on a finitely generated abelian group G. Then the K_0-group (resp. K_1-group) of $C^*(G;\sigma)$ is a free abelian group with rank* $|(H_\sigma)_{\mathrm{tor}}|\cdot 2^{\mathrm{rank}(G)-1}$ *or* $|(H_\sigma)_{\mathrm{tor}}|$ *(resp.* $|(H_\sigma)_{\mathrm{tor}}|\cdot 2^{\mathrm{rank}(G)-1}$ *or* 0*) depending as* $\mathrm{rank}(G) > 0$ *or* $\mathrm{rank}(G) = 0$.

For any unital C^*-algebra A, denote by $T(A)_{K_0}$ the set of all homomorphisms from $K_0(A)$ to $\mathbb{R}$ induced by tracial states of A. Then $T(A)_{K_0}$ equipped with the topology of pointwise convergence is a compact convex set in a Hausdorff locally convex topological vector space.

Lemma 5.5. *Let σ be a 2-cocycle on a finitely generated abelian group G. Then $T(C^*(G;\sigma))_{K_0}$ is a simplex of dimension $|(H_\sigma)_{\rm tor}| - 1$. The images of $K_0(C^*(G;\sigma))$ are the same under all the vertices of $T(C^*(G;\sigma))_{K_0}$.*

Proof. Recall the skew-symmetric bicharacter σ^* on G defined in Subsection 2.2. It induces a skew-symmetric bicharacter τ^* on G/H_σ. Then $\tau^* = (\sigma')^*$ for some 2-cocycle σ' on G/H_σ. Note that σ' is nondegenerate. Thus $C^*(G/H_\sigma;\sigma')$ has a unique tracial state φ [39, Lemma 3.2]. For any closed ideal I_t of $C^*(G;\sigma)$ generated by a maximal ideal t of the center of $C^*(G;\sigma)$, one has $C^*(G;\sigma)/I_t \cong C^*(G/H_\sigma;\sigma')$.

Any extremal point of $T(C^*(G;\sigma))_{K_0}$ is induced by an extremal tracial state of $C^*(G;\sigma)$. An argument similar to that in the proof of [13, Lemma 2.2] shows that every extremal tracial state of $C^*(G;\sigma)$ factors through $C^*(G;\sigma)/I_t$ for some maximal ideal t of the center (and hence must be the pull-back φ_t of the unique tracial state of $C^*(G;\sigma)/I_t$), and also that the map from $\widehat{H_\sigma}$ to $T(C^*(G;\sigma))_{K_0}$ sending t to the homomorphism from $K_0(C^*(G;\sigma))$ to $\mathbb{R}$ induced by φ_t is locally constant. Denote by X the image of this map. Then X has cardinality at most the number of components of $\widehat{H_\sigma}$, i.e., $|(H_\sigma)_{\rm tor}|$. Also, X contains all the extremal points of $T(C^*(G;\sigma))_{K_0}$ and hence the closed convex hull of X is $T(C^*(G;\sigma))_{K_0}$ by the Krein-Milman theorem [11, Theorem V.7.4]. Evaluating elements of X at minimal projections in the center of $C^*(G;\sigma)$ we see that the closed convex hull of X is a simplex of dimension $|H_\sigma| - 1$ with vertex set X. This proves the first assertion of Lemma 5.5.

Note that if ${}_AE_B$ is a strong Morita equivalence bimodule for two unital C^*-algebras A and B, and $K(J), J, Y$ are closed two-sided ideals and a submodule as in Subsection 2.1, then the diagram

$$\begin{array}{ccc} K_0(A) & \longrightarrow & K_0(B) \\ \downarrow & & \downarrow \\ K_0(A/K(J)) & \longrightarrow & K_0(B/J) \end{array}$$

commutes, where the horizonal isomorphisms are induced by ${}_AE_B$ and ${}_{A/K(J)}E/Y_{B/J}$ respectively and the vertical homomorphisms are induced by the C^*-algebra quotient maps. Using Theorem 5.1 one sees that the homomorphism from $K_0(C^*(G;\sigma))$ to $K_0(C^*(G;\sigma)/I_t)$ induced by the C^*-algebra quotient map is surjective for every maximal ideal t of the center of $C^*(G;\sigma)$. Thus the image of $K_0(C^*(G;\sigma))$ under any element of X is the image of $K_0(C^*(G/H_\sigma;\sigma'))$ under the element of $T(C^*(G/H_\sigma;\sigma'))_{K_0}$ induced by φ. This finishes the proof of Lemma 5.5. □

Let ω_σ be any vertex of $T(C^*(G;\sigma))_{K_0}$ in Lemma 5.5. The following result is a generalization of Proposition 4.1.

Proposition 5.6. *Let σ_j be a 2-cocycle on a finitely generated abelian group G_j for $j = 1, 2$. Then $C^*(G_1;\sigma_1)$ and $C^*(G_2;\sigma_2)$ have isomorphic ordered K_0-groups if and only if $\omega_{\sigma_2}(K_0(C^*(G_2;\sigma_2))) = \mu\omega_{\sigma_1}(K_0(C^*(G_1;\sigma_1)))$ for some real number $\mu > 0$, $|(H_{\sigma_1})_{\mathrm{tor}}| = |(H_{\sigma_2})_{\mathrm{tor}}|$, and* $\mathrm{rank}(G_1) = \mathrm{rank}(G_2)$ *or* $\mathrm{rank}(G_1) + \mathrm{rank}(G_2) = 1$.

Proof. Let us prove first the "only if" part. An argument similar to that in the proof of the "only if" part of Proposition 4.1 shows that $\omega_{\sigma_2}(K_0(C^*(G_2;\sigma_2))) = \mu\omega_{\sigma_1}(K_0(C^*(G_1;\sigma_1)))$ for some real number $\mu > 0$. By Lemma 5.5 the set of positive homomorphisms from $(K_0(C^*(G_j;\sigma_j))_+, K_0(C^*(G_j;\sigma_j)))$ to $(\mathbb{R}_+, \mathbb{R})$ is a cone of dimension $|(H_{\sigma_j})_{\mathrm{tor}}|$. Thus, $|(H_{\sigma_1})_{\mathrm{tor}}| = |(H_{\sigma_2})_{\mathrm{tor}}|$. Comparing the ranks of the K_0-groups, by Corollary 5.4 we obtain $\mathrm{rank}(G_1) = \mathrm{rank}(G_2)$ or $\mathrm{rank}(G_1) + \mathrm{rank}(G_2) = 1$.

For the "if" part, by Theorem 5.1 and the "only if" part we may assume that G_j and σ_j have the same properties as G' and σ' of Theorem 5.1. Then an argument similar to that in the proof of the "if" part of Proposition 4.1 completes the proof. □

An argument similar to that in the proof of Lemma 4.3 establishes the following generalization of Lemma 4.3:

Lemma 5.7. *Let σ_j be a skew-symmetric bicharacter on a finitely generated abelian group G_j for $j = 1, 2$ such that $G_j = G''_j \oplus G'_j$, $\sigma_j(G''_j, G_j) = 1$, and G'_j is torsion-free. Denote by σ'_j the restriction of σ_j to G'_j. Suppose that* $\mathrm{rank}(G'_1) = \mathrm{rank}(G'_2)$. *If the ordered K_0-groups of $C^*(G_1;\sigma_1)$ and $C^*(G_2;\sigma_2)$ are isomorphic, then so also are those of $C^*(G'_1;\sigma'_1)$ and $C^*(G'_2;\sigma'_2)$.*

Now the proof of Theorem 1.1 in the case of noncommutative tori in Section 4 extends verbatim to the general case of the twisted group C^*-algebras of arbitrary finitely generated abelian groups.

Remark 5.8. For a 2-cocycle σ on a finitely generated abelian group G, one can check easily that the maximal ideals of $C^*(G;\sigma)$ are exactly those I_t's in the proof of Lemma 5.5, and hence that the simple quotient algebras of $C^*(G;\sigma)$ are all isomorphic to $C^*(G/H_\sigma;\sigma')$ therein. The proof of Theorem 5.1 actually shows that two twisted group C^*-algebras of finitely generated abelian groups are strongly Morita equivalent if and only if they have isomorphic centers and their simple quotient algebras have isomorphic ordered K_0-groups.

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