FAMILY-INDEPENDENCE FOR TOPOLOGICAL AND MEASURABLE DYNAMICS

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Abstract. For a family $\mathcal{F}$ (a collection of subsets of $\mathbb{Z}_+$), the notion of $\mathcal{F}$-independence is defined both for topological dynamics (t.d.s.) and measurable dynamics (m.d.s.). It is shown that there is no non-trivial {syndetic}-independent m.d.s.; a m.d.s. is {positive-density}-independent if and only if it has completely positive entropy; and a m.d.s. is weakly mixing if and only if it is {IP}-independent. For a t.d.s. it is proved that there is no non-trivial minimal {syndetic}-independent system; a t.d.s. is weakly mixing if and only if it is {IP}-independent.

Moreover, a non-trivial proximal topological K system is constructed, and a topological proof of the fact that minimal topological K implies strong mixing is presented.

1. Introduction

By a *topological dynamical system* (t.d.s.) (X,T) we mean a compact metrizable space X together with a surjective continuous map T from X to itself. For a t.d.s. (X,T) and nonempty open subsets U and V of X let $N(U,V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}$, where $\mathbb{Z}_+$ denotes the set of non-negative integers. It turns out that many recurrence properties of t.d.s. can be described using the return times sets $N(U,V)$, see [1, 12, 15, 29, 30]. For example, for a t.d.s. (X,T) it is known that T is (topologically) *strongly mixing* iff $N(U,V)$ is cofinite, T is (topologically) *weakly mixing* iff $N(U,V)$ is thick [12] and T is (topologically) *mildly mixing* iff $N(U,V)$ is an $(\mathrm{IP}-\mathrm{IP})^*$ set [30, 21], for each pair of nonempty open subsets U and V. Huang and Ye [30] showed that a minimal system (X,T) is weakly mixing iff the lower Banach density of $N(U,V)$ is 1, and (X,T) is mildly mixing iff $N(U,V)$ is an IP^* set, for each pair of nonempty open sets U and V.

By a *measurable dynamical system* (m.d.s.) we mean a quadruple $(X,\mathcal{B},\mu,T)$, where $(X,\mathcal{B},\mu)$ is a Lebesgue space (i.e., X is a set, $\mathcal{B}$ is the σ-algebra of Borel subsets on X for some Polish topology on X, and μ is a probability measure on $\mathcal{B}$) and $T: X \to X$ is measurable and measure-preserving, that is: $\mu(B) = \mu(T^{-1}B)$ for each $B \in \mathcal{B}$. For a t.d.s (X,T), there are always invariant Borel probability measures on X and thus for each such measure μ, $(X,\mathcal{B}_X,\mu,T)$, with $\mathcal{B}_X$ the Borel σ-algebra on X, is a m.d.s.. For a m.d.s. $(X,\mathcal{B},\mu,T)$, let $\mathcal{B}^+ = \{B \in \mathcal{B} : \mu(B) > 0\}$ and $N(A,B) = \{n \in \mathbb{Z}_+ : \mu(A \cap T^{-n}B) > 0\}$ for $A, B \in \mathcal{B}^+$. It is known that T is ergodic iff $N(A,B) \neq \emptyset$ iff $N(A,B)$ is syndetic; T is weakly mixing iff the lower Banach density of $N(A,B)$ is 1 iff $N(A,B)$ is thick; and T is mildly mixing iff

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$N(A,B)$ is an IP^* set iff $N(A,B)$ is an $(\mathrm{IP}-\mathrm{IP})^*$ set for all $A,B\in\mathcal{B}^+$ iff for each IP set F and $A\in\mathcal{B}^+$, $\mu(\bigcup_{n\in F}T^{-n}A)=1$. Finally, it is known that T is intermixing iff $N(A,B)$ is cofinite for all $A,B\in\mathcal{B}^+$, see [37, 38] and references therein.

In ergodic theory there exists a rich and powerful entropy theory. The analogous notion of topological entropy was introduced soon after the measure theoretical one, and was widely studied and applied. Notwithstanding, the level of development of topological entropy theory lagged behind. In recent years however this situation is rapidly changing. A turning point occurred with F. Blanchard's pioneering papers [4, 5] in the 1990's.

In recent years a local entropy theory has been developed, see [22] for a survey. More precisely, in [4] Blanchard introduced the notions of completely positive entropy (c.p.e.) and uniformly positive entropy (u.p.e.) as topological analogues of the K-property in ergodic theory. In [5] he defined the notion of entropy pairs and used it to show that a u.p.e. system is disjoint from all minimal zero entropy systems. The notion of entropy pairs can also be used to show the existence of the maximal zero entropy factor for any t.d.s., namely the topological Pinsker factor [8]. Blanchard et al. [7] also introduced the notion of entropy pairs for an invariant Borel probability measure. Glasner and Weiss [19] introduced the notion of entropy tuples. In order to gain a better understanding of the topological version of a K-system, Huang and Ye [32] introduced the notion of entropy tuples for an invariant Borel probability measure. They showed that if (X,T) is a t.d.s. and $k\geq 2$, then a non-diagonal tuple $(x_1,\dots,x_k)$ in X^k is an entropy tuple iff for every choice of neighborhoods U_i of x_i there is a subset F of $\mathbb{Z}_+$ with positive density such that $\bigcap_{i\in F}T^{-i}U_{s(i)}\neq\emptyset$ for each $s\in\{1,\dots,k\}^F$. We mention that at the same time a theory on sequence entropy tuples and tame systems were developed [16, 26, 17]. It is Kerr and Li who captured the idea behind the results on entropy tuples, sequence entropy tuples and tame systems and treated them systematically using a notion called independence in [35, 36], which first appeared in Rosenthal's proof of his groundbreaking ℓ_1 theorem [45, 46].

Let (X,T) be a t.d.s.. For a tuple $\mathbf{A}=(A_1,\dots,A_k)$ of subsets of X, we say a subset $F\subseteq\mathbb{Z}_+$ is an *independence set* for $\mathbf{A}$ if for any nonempty finite subset $J\subseteq F$, we have

$$\bigcap_{j\in J}T^{-j}A_{s(j)}\neq\emptyset$$

for any $s\in\{1,\dots,k\}^J$. We call a tuple $\mathbf{x}=(x_1,\dots,x_k)\in X^k$ (1) an *IE-tuple* if for every product neighborhood $U_1\times\cdots\times U_k$ of $\mathbf{x}$ the tuple $(U_1,\dots,U_k)$ has an independence set of positive density; (2) an *IT-tuple* if for every product neighborhood $U_1\times\cdots\times U_k$ of $\mathbf{x}$ the tuple $(U_1,\dots,U_k)$ has an infinite independence set; (3) an *IN-tuple* if for every product neighborhood $U_1\times\cdots\times U_k$ of $\mathbf{x}$ the tuple $(U_1,\dots,U_k)$ has arbitrarily long finite independence sets. Kerr and Li [35] showed that (1) entropy tuples are exactly non-diagonal IE-tuples; (2) sequence entropy tuples are exactly non-diagonal IN-tuples, and in particular a t.d.s. (X,T) is null iff it has no non-diagonal IN-pairs; (3) a t.d.s. (X,T) is tame iff it has no non-diagonal IT-pairs. For similar results concerning m.d.s. see [36].

Thus the notion of independence is very useful to describe dynamical properties. For a family $\mathcal{F}$, the notion of $\mathcal{F}$-independence can be defined both for topological dynamics (t.d.s.) and measurable dynamics (m.d.s.). So a natural question is: for a given family $\mathcal{F}$ which dynamical property is equivalent to $\mathcal{F}$-independence? In this paper we try to answer this question.

It is shown that there is no non-trivial {syndetic}-independent m.d.s.; a m.d.s. is {positive-density}-independent iff it has completely positive entropy; and a m.d.s. is weakly mixing iff it is {infinite}-independent iff it is {IP}-independent. For a t.d.s. it is proved that there is no non-trivial minimal {syndetic}-independent system; a t.d.s. is weakly mixing iff it is {infinite}-independent iff it is {IP}-independent.

Moreover, a non-trivial proximal topological K system is constructed, and a topological proof (using independence) of the fact that minimal topological K implies strong mixing is presented. In a forthcoming paper [27] we will deal with the problem of how to localize the notion of $\mathcal{F}$-independence.

In [5] Blanchard raised the question whether there exists any non-trivial minimal uniformly positive entropy (equivalently, {positive-density}-independent of order 2 in our terminology) t.d.s.. This was answered affirmatively by Glasner and Weiss in [18]. Later Huang and Ye showed there are non-trivial minimal {positive-density}-independent t.d.s. [32]. However, the constructions in [18] and [32] are based on showing that any minimal topological model of a K-system is such an example and then using the Jewett-Krieger theorem to obtain such a topological model. So far there is no explicit topological construction of such examples. Since the family of syndetic sets is just slightly smaller than the family of positive upper Banach density sets, our result of the non-existence of non-trivial minimal {syndetic}-independent t.d.s. explains why it is so difficult to construct examples for Blanchard's question.

The paper is organized as follows. In Section 2 we investigate the relationship between a given family $\mathcal{F}$ and the associated block family $b\mathcal{F}$. In Section 3, the basic properties of $\mathcal{F}$-independence for a t.d.s. are discussed. Particularly we show that $\mathcal{F}$ and $b\mathcal{F}$ define the same notion of independence. In Section 4, the basic properties of $\mathcal{F}$-independence for a m.d.s. are discussed. In Section 5, we investigate classes of $\mathcal{F}$-independent systems for t.d.s. and show that there is no non-trivial minimal {syndetic}-independent t.d.s.. Moreover, a non-trivial proximal topological K system is constructed. In Section 6, we investigate classes of $\mathcal{F}$-independent systems for m.d.s. and show that a m.d.s. is {positive-density}-independent iff it has completely positive entropy. We also show that there is no non-trivial {syndetic}-independent m.d.s.. In Section 7, we give a topological proof of the fact that minimal topological K implies strong mixing. An interesting combinatorial result, which is needed for the proof of non-existence of no-trivial minimal {syndetic}-independent t.d.s., is established in the Appendix.

Throughout this paper, we use $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the sets of nonnegative integers and positive integers respectively. For a subset F of $\mathbb{Z}$ and $m \in \mathbb{Z}$ we denote $\{j+m : j \in F\}$ by $F+m$. For a subshift X of $\{0,1,\dots,k\}^{\mathbb{Z}_+}$ or $\{0,1,\dots,k\}^{\mathbb{Z}}$ and $a \in \{0,1,\dots,k\}^{\{1,\dots,m\}}$ for some $m \in \mathbb{N}$, we denote $\{x \in X : (x(0), x(1), \dots, x(m-1)) = a\}$ by $[a]_X$. For a t.d.s. (X,T) and subsets $U, V \subseteq X$, we denote by $N(U,V)$ the set $\{n \in \mathbb{Z}_+ : U \bigcap T^{-n}V \neq \emptyset\}$; for $x \in X$ we shall write $N(x,U)$ for $N(\{x\},U)$. For a m.d.s. $(X,\mathcal{B},\mu,T)$ and $A, B \in \mathcal{B}$, we denote by $N(A,B)$ the set $\{n \in \mathbb{Z}_+ : \mu(A \bigcap T^{-n}B) > 0\}$.

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2. Preliminary

The idea of using families to describe dynamical properties goes back at least to Gottschalk and Hedlund [23]. It was developed further by Furstenberg [12, 13]. For a systematic study and recent results, see [1, 15, 29, 30].

Let us recall some notations related to a family (for details see [1]). Let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upward. That is, $F_1 \subseteq F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ generates a family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supseteq A \text{ for some } A \in \mathcal{A}\}$. If a proper family $\mathcal{F}$ is closed under taking finite intersection, then $\mathcal{F}$ is called a *filter*. For a family $\mathcal{F}$, the *dual family* is

$$\mathcal{F}^* = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}.$$

$\mathcal{F}^*$ is a family, proper if $\mathcal{F}$ is. Clearly,

$$(\mathcal{F}^*)^* = \mathcal{F} \text{ and } \mathcal{F}_1 \subseteq \mathcal{F}_2 \Longrightarrow \mathcal{F}_2^* \subseteq \mathcal{F}_1^*.$$

There is an important property being well studied: the Ramsey property. We say that a family $\mathcal{F}$ has the *Ramsey property* if whenever $F_1 \cup F_2 \in \mathcal{F}$, one has either $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$. One can show that a proper family $\mathcal{F}$ has the Ramsey property if and only if $\mathcal{F}^*$ is a filter [1, page 26].

Denote by $\mathcal{F}_{\text{inf}}$ the family of all infinite subsets of $\mathbb{Z}_+$ and by $\mathcal{F}_{\text{c}}$ the dual family $\mathcal{F}_{\text{inf}}^*$. Note that $\mathcal{F}_{\text{c}}$ is the collection of all cofinite subsets of $\mathbb{Z}_+$. **All the families considered in this paper are assumed to be proper and contained in $\mathcal{F}_{\text{inf}}$.**

Let F be a subset of $\mathbb{Z}_+$. The *lower density* and *upper density* of F are defined by

$$\underline{d}(F) = \liminf_{n \to +\infty} \frac{1}{n}|F \cap \{0, 1, \ldots, n-1\}| \text{ and } \overline{d}(F) = \limsup_{n \to +\infty} \frac{1}{n}|F \cap \{0, 1, \ldots, n-1\}|.$$

If $\underline{d}(F) = \overline{d}(F) = d(F)$, we then say that the *density* of F is $d(F)$. The *upper Banach density* of F is defined by

$$BD^*(F) = \limsup_{|I| \to +\infty} \frac{|S \cap I|}{|I|},$$

where I is taken over all nonempty finite intervals of $\mathbb{Z}_+$.

We denote by $\mathcal{F}_{\text{pd}}$ the family generated by sets with positive density, by $\mathcal{F}_{\text{pud}}$ the family of sets with positive upper density, and by $\mathcal{F}_{\text{pubd}}$ the family of sets with positive upper Banach density.

Note that a subset F of $\mathbb{Z}_+$ is said to be *thick* if for any $n \in \mathbb{N}$ there exists some $m \in \mathbb{Z}_+$ such that $\{m, m+n, \ldots, m+n\} \subseteq F$. An infinite subset $F = \{s_1 < s_2 < \cdots\}$ of $\mathbb{Z}_+$ is said to be *syndetic* if $\{s_{n+1} - s_n : n \in \mathbb{N}\}$ is bounded. A subset of $\mathbb{Z}_+$ is called *piecewise syndetic* if it is the intersection of a thick set and a syndetic set. We denote by $\mathcal{F}_{\text{t}}$, $\mathcal{F}_{\text{s}}$ and $\mathcal{F}_{\text{ps}}$ the families of thick sets, syndetic sets and piecewise syndetic sets respectively.

A subset F of $\mathbb{Z}_+$ is called a *central set* if there exists a t.d.s. (X,T), a point $x \in X$, a minimal point $y \in X$ which is proximal to x and a neighborhood U_y of y such that $F \supseteq N(x, U_y)$ [13, Section 8.3]. Here y is proximal to x means that for a compatible metric d of X, one has $\inf_{n\in\mathbb{Z}_+} d(T^n x, T^n y) = 0$. We denote by $\mathcal{F}_{\rm cen}$ the family of all central sets.

A subset F of $\mathbb{Z}_+$ is called an IP-set if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that F consists of $a_{n_1} + a_{n_2} + \cdots + a_{n_k}$ for all $k \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_k$. We denote by $\mathcal{F}_{\rm ip}$ the family generated by all IP-sets.

Definition 2.1. Let $\mathcal{F}$ be a family. The *block family* of $\mathcal{F}$, denoted by $b\mathcal{F}$, is the family consisting of sets $S \subseteq \mathbb{Z}_+$ for which there exists some $F \in \mathcal{F}$ such that for every finite subset W of F one has $m + W \subseteq S$ for some $m \in \mathbb{Z}$.

Clearly $\mathcal{F} \subseteq b\mathcal{F}$ and $b(b\mathcal{F}) = b\mathcal{F}$. It is also clear that $b\mathcal{F}_{\rm inf} = \mathcal{F}_{\rm inf}$ and $b\mathcal{F}_{\rm c} = \mathcal{F}_{\rm t}$.

Example 2.2. It is clear that $b\mathcal{F}_{\rm pd} \subseteq b\mathcal{F}_{\rm pud} \subseteq \mathcal{F}_{\rm pubd}$. It is a result of Ellis that $\mathcal{F}_{\rm pubd} \subseteq b\mathcal{F}_{\rm pd}$ [13, Theorem 3.20] (one can also give a topological proof for it, using an argument similar to that in the proof of Lemma 4.5). Thus one has $b\mathcal{F}_{\rm pd} = b\mathcal{F}_{\rm pud} = \mathcal{F}_{\rm pubd}$.

Example 2.3. It is clear that $b\mathcal{F}_{\rm s} \subseteq \mathcal{F}_{\rm ps}$. Let $S_1 \in \mathcal{F}_{\rm t}$ and $S_2 \in \mathcal{F}_{\rm s}$. Then for each $n \in \mathbb{N}$ we can find some $a_n \in \mathbb{Z}_+$ with $[a_n, a_n + n] \subseteq S_1$. Some subsequence of the sequence $\{1_{([a_n,a_n+n]\cap S_2)-a_n}\}_{n\in\mathbb{N}}$ converges in $\{0,1\}^{\mathbb{Z}_+}$ to 1_F for some subset F of $\mathbb{Z}_+$. It is easy to see that F is syndetic and that for every finite subset W of F one has $m + W \subseteq S_1 \cap S_2$ for some $m \in \mathbb{Z}_+$. Therefore $b\mathcal{F}_{\rm s} \supseteq \mathcal{F}_{\rm ps}$, and hence $b\mathcal{F}_{\rm s} = \mathcal{F}_{\rm ps}$.

Example 2.4. It is clear that $\mathcal{F}_{\rm cen} \subseteq \mathcal{F}_{\rm ps}$ and hence $b\mathcal{F}_{\rm cen} \subseteq b\mathcal{F}_{\rm ps} = \mathcal{F}_{\rm ps}$. Let $S \in \mathcal{F}_{\rm ps}$. Denote by X the smallest closed shift-invariant subset of $\{0,1\}^{\mathbb{Z}}$ containing 1_S. Note that $S = N(1_S, [1]_X)$. By [9, Theorem 6] there is a minimal point x of X contained in $[1]_X$. Say, $x = 1_{S'}$. Set $F = S' \cap \mathbb{Z}_+$. Then $F = N(x, [1]_X)$ is central. Since x is in X, it is easy to see that for every finite subset W of F one has $m + W \subseteq S$ for some $m \in \mathbb{Z}$. This means that $S \in b\mathcal{F}_{\rm cen}$. Therefore $b\mathcal{F}_{\rm cen} \supseteq \mathcal{F}_{\rm ps}$, and hence $b\mathcal{F}_{\rm cen} = \mathcal{F}_{\rm ps}$.

The following result shows the relation between the block family and the broken family introduced in [9, Defintion 2].

Proposition 2.5. Let $\mathcal{F}$ be a family. Let $S \subseteq \mathbb{Z}_+$. Then $S \in b\mathcal{F}$ if and only if there exist an $F = \{p_1 < p_2 < \dots\} \in \mathcal{F}$ and a (not necessarily strictly) increasing sequence $\{b_j\}_{j=1}^\infty$ of integers such that $S \supseteq \bigcup_{j=1}^\infty \{b_j + \{p_1, p_2, \dots, p_j\}\}$.

Proof. The "if" part is trivial.

Suppose that $S \in b\mathcal{F}$. Let $F = \{p_1 < p_2 < \dots\} \in \mathcal{F}$ witnessing this. Then for each $j \in \mathbb{N}$ we find some $b_j \in \mathbb{Z}$ with $b_j + \{p_1, \dots, p_j\} \subseteq S$. Note that $b_j + p_1 \geq 0$ for every $j \in \mathbb{N}$. Thus we can find an increasing subsequence $\{b_{j_k}\}_{k=1}^\infty$ of $\{b_j\}_{j=1}^\infty$. Then for each $k \in \mathbb{N}$ we have $b_{j_k} + \{p_1, \dots, p_k\} \subseteq b_{j_k} + \{p_1, \dots, p_{j_k}\} \subseteq S$. Thus $S \supseteq \bigcup_{k=1}^\infty \{b_{j_k} + \{p_1, p_2, \cdots, p_k\}\}$. This proves the "only if" part. □

The next result follows from Proposition 3.7 and Lemma 3.9, which we shall prove in the next section.

Proposition 2.6. If $\mathcal{F}$ has the Ramsey property, then so does $b\mathcal{F}$.

We remark that if $b\mathcal{F}$ has the Ramsey property, it is not necessarily true that $\mathcal{F}$ has the Ramsey property. For example, $\mathcal{F}_{\mathrm{pud}}$ and $\mathcal{F}_{\mathrm{pubd}}$ have the Ramsey property, while $\mathcal{F}_{\mathrm{pd}}$ does not.

3. Independence: topological case

In this section, for a given family $\mathcal{F}$, we define $\mathcal{F}$-independence for t.d.s., and discuss 1-independence for various families. Recall first the notion of independence set introduced in [35, Definition 2.1].

Definition 3.1. Lt (X,T) be a t.d.s.. For a tuple $\mathbf{A}=(A_1,\dots,A_k)$ of subsets of X, we say that a subset $F\subseteq\mathbb{Z}_+$ is an *independence set* for $\mathbf{A}$ if for any nonempty finite subset $J\subseteq F$, we have

$$\bigcap_{j\in J} T^{-j}A_{s(j)}\neq\emptyset$$

for any $s\in\{1,\dots,k\}^J$.

We shall denote the collection of all independence sets for $\mathbf{A}$ by $\mathrm{Ind}(A_1,\dots,A_k)$ or $\mathrm{Ind}\mathbf{A}$. The basic properties of independence sets are listed below.

Lemma 3.2. The following hold:

1. If $F\in\mathrm{Ind}(A_1,\dots,A_k)$ and $F_1\subseteq F$, then $F_1\in\mathrm{Ind}(A_1,\dots,A_k)$.
2. $F=\{a_1,a_2,\dots\}$ is in $\mathrm{Ind}(A_1,\dots,A_k)$ if and only if $\{a_1,\dots,a_n\}$ is in $\mathrm{Ind}(A_1,\dots,A_k)$ for each $n\in\mathbb{N}$.
3. If $m\in\mathbb{Z}$ and $F, m+F\subseteq\mathbb{Z}_+$, then F is in $\mathrm{Ind}(A_1,\dots,A_k)$ if and only if $m+F$ is so.
4. Let $F\subseteq\mathbb{Z}_+$ and X be the subshift of $\{0,1\}^{\mathbb{Z}}$ generated by $\{1_E: E\subseteq F\}$. Then $F\in\mathrm{Ind}([0]_X,[1]_X)$.

Definition 3.3. Let $\mathcal{F}$ be a family. We say that $\mathcal{F}$ has the *dynamical Ramsey property*, if for any t.d.s. (X,T), any $k\in\mathbb{N}$ and closed subsets $A_1,A_2,\dots,A_k,A_{1,1},A_{1,2}$ of X with $A_1=A_{1,1}\cup A_{1,2}$, whenever $\mathrm{Ind}(A_1,A_2,\dots,A_k)\cap\mathcal{F}\neq\emptyset$, one has either $\mathrm{Ind}(A_{1,1},A_2,\dots,A_k)\cap\mathcal{F}\neq\emptyset$ or $\mathrm{Ind}(A_{1,2},A_2,\dots,A_k)\cap\mathcal{F}\neq\emptyset$.

It was shown in [35, Lemmas 3.8 and 6.3] that the families $\mathcal{F}_{\mathrm{pd}}$ and $\mathcal{F}_{\mathrm{inf}}$ have the dynamical Ramsey property.

Similar to the definition of u.p.e. of order n (see [32]), we have

Definition 3.4. Let $\mathcal{F}$ be a family, $k\in\mathbb{N}$ and (X,T) be a t.d.s.. A tuple $(x_1,\dots,x_k)\in X^k$ is called an *$\mathcal{F}$-independent tuple* if for any neighborhoods $U_1,\dots,U_k$ of $x_1,\dots,x_k$ respectively, one has $\mathrm{Ind}(U_1,\dots,U_k)\cap\mathcal{F}\neq\emptyset$. A t.d.s. is said to be *$\mathcal{F}$-independent of order* k, if for each tuple of nonempty open subsets $U_1,\dots,U_k$, $\mathrm{Ind}(U_1,\dots,U_k)\cap\mathcal{F}\neq\emptyset$, and a t.d.s. is said to be *$\mathcal{F}$-independent*, if it is *$\mathcal{F}$-independent* of order k for each $k\in\mathbb{N}$.

Standard arguments as in [5] show the following:

Proposition 3.5. Let $\mathcal{F}$ be a family with the dynamical Ramsey property, and let (X,T) be a t.d.s.. The following are true:

1. If $\mathbf{A}=(A_1,\dots,A_k)$ is a tuple of closed subsets of X with $\mathrm{Ind}\mathbf{A}\cap\mathcal{F}\neq\emptyset$, then there exists $x_j\in A_j$ for each $1\le j\le k$ such that $(x_1,\dots,x_k)$ is an $\mathcal{F}$-independent tuple.

(2) Let $k \in \mathbb{N}$. Then the set of $\mathcal{F}$-independent k-tuples of X is a closed $T \times \cdots \times T$-invariant subset of X^k.
(3) Let (Y, S) be a t.d.s. and $\pi : X \to Y$ be a factor map, i.e., π is continuous surjective and equivariant. Let $k \in \mathbb{N}$. Then $\pi \times \cdots \times \pi$ maps the set of $\mathcal{F}$-independent k-tuples of X onto the set of $\mathcal{F}$-independent k-tuples of Y.

Recall that two t.d.s. (X, T) and (Y, S) are said to be *disjoint* [12] if $X \times Y$ is the only nonempty closed subset Z of $X \times Y$ satisfying $(T \times S)(Z) = Z$ and projecting surjectively to X and Y under the natural projections $X \times Y \to X$ and $X \times Y \to Y$ respectively. Following the arguments in the proofs of [5, Proposition 6] and [8, Theorem 2.1] we have

Theorem 3.6. Let $\mathcal{F}$ be a family with the dynamical Ramsey property. The following are true:

(1) Each t.d.s. being $\mathcal{F}$-independent of order 2 is disjoint from any minimal system without non-diagonal $\mathcal{F}$-independent pair.
(2) Each t.d.s. admits a maximal factor with no non-diagonal $\mathcal{F}$-independent pairs.

Different families might lead to the same notion of independence. In fact, it follows from Lemma 3.2(2)(3) that $\mathrm{Ind}(A_1, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$ if and only if $\mathrm{Ind}(A_1, \ldots, A_k) \cap b\mathcal{F} \neq \emptyset$. Thus we have:

Proposition 3.7. Let $\mathcal{F}$ be a family. Then:

(1) The families $\mathcal{F}$ and $b\mathcal{F}$ define the same notion of independence.
(2) $\mathcal{F}$ has the dynamical Ramsey property if and only if $b\mathcal{F}$ does.

Theorem 3.8. Let $\mathcal{F}_1, \mathcal{F}_2$ be two families having the dynamical Ramsey property. Then each $\mathcal{F}_1$-independent pair is an $\mathcal{F}_2$-independent pair and viceversa if and only if $b\mathcal{F}_1 = b\mathcal{F}_2$.

Proof. The "if" part follows from Proposition 3.7.

Now assume that each $\mathcal{F}_1$-independent pair is an $\mathcal{F}_2$-independent pair. We are going to show that $b\mathcal{F}_1 \subseteq b\mathcal{F}_2$.

Let $F \in \mathcal{F}_1$. Denote by X the smallest closed shift-invariant subset of $\{0, 1\}^{\mathbb{Z}}$ containing $\{1_E : E \subseteq F\}$. Then $F \in \mathrm{Ind}([0]_X, [1]_X)$ and

$$X = \overline{\{T^i 1_E : i \in \mathbb{Z}, E \subseteq F\}},$$

where T denotes the shift. Since $\mathcal{F}_1$ has the dynamical Ramsey property, there exists $(x, y) \in [0]_X \times [1]_X$ which is $\mathcal{F}_1$-independent. As each $\mathcal{F}_1$-independent pair is an $\mathcal{F}_2$-independent pair, we get that $\mathrm{Ind}([0]_X, [1]_X) \cap \mathcal{F}_2 \neq \emptyset$. Let $F' \in \mathrm{Ind}([0]_X, [1]_X) \cap \mathcal{F}_2$. For any finite subset W of F', there exists $x_W \in \bigcap_{k \in W} T^{-k}([1]_X)$. Then $x_W(k) = 1$ for every $k \in W$. Since $x_W \in X$, it follows that there exists some $m \in \mathbb{Z}$ with $m + W \subseteq F$. Therefore $F \in b\mathcal{F}_2$. Thus $\mathcal{F}_1 \subseteq b\mathcal{F}_2$, and hence $b\mathcal{F}_1 \subseteq b(b\mathcal{F}_2) = b\mathcal{F}_2$. This proves the "only if" part. □

From Theorem 3.8 one sees that if a family $b\mathcal{F}$ has the dynamical Ramsey property, then among the families which has the dynamical Ramsey property and defines the same independence as $\mathcal{F}$ does, $b\mathcal{F}$ is the largest one.

Lemma 3.9. Let $\mathcal{F}$ be a family. If $\mathcal{F}$ has the Ramsey property, then for any t.d.s. (X,T) and closed subsets Y, Y_1, Y_2 of X with $Y = Y_1 \cup Y_2$ and $\mathrm{Ind}(Y) \cap \mathcal{F} \neq \emptyset$, one has either $\mathrm{Ind}(Y_1) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}(Y_2) \cap \mathcal{F} \neq \emptyset$. The converse holds if furthermore $\mathcal{F} = b\mathcal{F}$.

Proof. Suppose that $\mathcal{F}$ has the Ramsey property. Consider a t.d.s. (X,T) and closed subsets Y, Y_1, Y_2 of X with $Y = Y_1 \cup Y_2$ and $\mathrm{Ind}(Y) \cap \mathcal{F} \neq \emptyset$. Take $F \in \mathrm{Ind}(Y) \cap \mathcal{F}$. Then $\cap_{n\in F} T^{-n}(Y) \neq \emptyset$. Say, $x \in \cap_{n\in F} T^{-n}Y$. Set $F_j = \{n \in F : T^n x \in Y_j\}$ for $j = 1, 2$. Then $F = F_1 \cup F_2$, and hence either $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$. Thus either $\mathrm{Ind}(Y_1) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}(Y_2) \cap \mathcal{F} \neq \emptyset$.

Now suppose that $\mathcal{F} = b\mathcal{F}$, and for any t.d.s. (X,T) and closed subsets Y, Y_1, Y_2 of X with $Y = Y_1 \cup Y_2$ and $\mathrm{Ind}(Y) \cap \mathcal{F} \neq \emptyset$, one has either $\mathrm{Ind}(Y_1) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}(Y_2) \cap \mathcal{F} \neq \emptyset$. Let $F \in \mathcal{F}$ and $F = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$. Denote by X the smallest closed shift-invariant subset of $\{0,1,2\}^{\mathbb{Z}}$ containing $1_{F_1} + 2 \cdot 1_{F_2}$. Then $F \in \mathrm{Ind}([1]_X \cup [2]_X)$. By assumption we have either $\mathrm{Ind}([1]_X) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}([2]_X) \cap \mathcal{F} \neq \emptyset$. Without loss of generality let us assume that $\mathrm{Ind}([1]_X) \cap \mathcal{F} \neq \emptyset$. Say, $F' \in \mathrm{Ind}([1]_X) \cap \mathcal{F}$. Since X is the orbit closure of $1_{F_1} + 2 \cdot 1_{F_2}$, it follows that for any finite subset W of F' there exists some $m \in \mathbb{Z}$ with $m + W \subseteq F_1$. Thus $F_1 \in b\mathcal{F} = \mathcal{F}$. Therefore $\mathcal{F}$ has the Ramsey property. □

From Proposition 3.7 and Lemma 3.9 we get:

Proposition 3.10. Let $\mathcal{F}$ be a family. If $\mathcal{F}$ has the dynamical Ramsey property, then $b\mathcal{F}$ has the Ramsey property.

We remark that if $\mathcal{F}$ has the dynamical Ramsey property, it is not necessarily true that $\mathcal{F}$ has the Ramsey property. For example, $\mathcal{F}_{\mathrm{pd}}$ has the dynamical Ramsey property, but not the Ramsey property.

It is easy to see that $\mathcal{F}_{\mathrm{ps}}$ has the Ramsey property. It is also known that $\mathcal{F}_{\mathrm{cen}}$ has the Ramsey property [3, Corollary 2.16]. The celebrated Hindman theorem [25] says that $\mathcal{F}_{\mathrm{ip}}$ has the Ramsey property. This leads to the following questions:

Question 3.11. Is there any family which has the Ramsey property but not the dynamical Ramsey property?

Question 3.12. Do the families $\mathcal{F}_{\mathrm{ps}}$ and $\mathcal{F}_{\mathrm{ip}}$ have the dynamical Ramsey property?

To end the section we shall discuss 1-independence for various families. Denote by $\mathcal{F}_{\mathrm{rs}}$ the family generated by $\{n\mathbb{Z}_+ : n \in \mathbb{N}\}$. The following notion was introduced in [31]. Let (X,T) be a t.d.s.. We say that (X,T) has *dense small periodic sets*, if for any nonempty open subset U of X there exist a nonempty closed $A \subseteq U$ and $k \in \mathbb{N}$ such that $T^k A \subseteq A$. To state our result we need a local version of this notion. That is, for a point x in a t.d.s. (X,T), x is called *quasi regular* if for each neighborhood U of x, there exist a nonempty closed $A \subseteq U$ and $k \in \mathbb{N}$ such that $T^k A \subseteq A$. The closed set of quasi regular points of T is denoted by $\mathrm{QR}(T)$.

Theorem 3.13. Let (X,T) be a t.d.s.. Then

(1) $x \in X$ is $\mathcal{F}_{\mathrm{ip}}$-independent iff $x \in \overline{\mathrm{Rec}(T)}$, where $\mathrm{Rec}(T)$ denotes the set of recurrent points of T. Thus, (X,T) is $\mathcal{F}_{\mathrm{ip}}$-independent of order 1 iff $\overline{\mathrm{Rec}(T)} = X$.

(2) $x \in X$ is $\mathcal{F}_{\text{inf}}$-independent iff $x \in \overline{\Lambda(T)}$, where $\Lambda(T) = \cup_{x\in X}\omega(x,T)$ and $w(x,T) = \cap_{k\geq 0}\overline{\cup_{n\geq k}\{T^n x\}}$. Thus, (X,T) is $\mathcal{F}_{\text{inf}}$-independent of order 1 iff $\overline{\Lambda(T)} = X$, iff $\overline{\text{Rec}(T)} = X$.

(3) $x \in X$ is $\mathcal{F}_{\text{pubd}}$-independent iff $x \in \overline{\text{M}(T)}$, where $\text{M}(T) = \cup_{\mu\in \text{M}(X,T)}\text{supp}(\mu)$ and $\text{M}(X,T)$ denotes the set of all invariant Borel probability measures on X. Thus, (X,T) is $\mathcal{F}_{\text{pubd}}$-independent of order 1 iff $\overline{\text{M}(T)} = X$, iff there exists a $\mu \in \text{M}(X,T)$ with full support.

(4) $x \in X$ is $\mathcal{F}_{\text{ps}}$-independent if $x \in \overline{\text{AP}(T)}$, where $\text{AP}(T)$ denotes the set of minimal points of T. Thus, (X,T) is $\mathcal{F}_{\text{ps}}$-independent of order 1 iff $\overline{\text{AP}(T)} = X$.

(5) $x \in X$ is $\mathcal{F}_{\text{rs}}$-independent iff $x \in \text{QR}(T)$. Thus, (X,T) is $\mathcal{F}_{\text{rs}}$-independent of order 1 iff $\text{QR}(T) = X$.

Proof. (1). Assume that $x \in X$ is $\mathcal{F}_{\text{ip}}$-independent and U is a closed neighborhood of x. Then $\text{Ind}(U) \cap \mathcal{F}_{\text{ip}} \neq \emptyset$, and hence there are an IP-set F and $y \in X$ such that $T^i y \in U$ for each $i \in F$. By [9, Theorem 5], $U \cap \text{Rec}(T) \neq \emptyset$, i.e. $x \in \overline{\text{Rec}(T)}$.

Conversely, assume that $x \in \overline{\text{Rec}(T)}$ and U is an open neighborhood of x. Then there exists a $y \in \text{Rec}(T) \cap U$. By [13, Theorem 2.17], the set $N(y,U)$ contains an IP-set. Thus $\text{Ind}(U) \cap \mathcal{F}_{\text{ip}} \neq \emptyset$.

(2). The first statement follows easily from the definition. The statement that the $\mathcal{F}_{\text{inf}}$-independence of order 1 for (X,T) implies $\overline{\text{Rec}(T)} = X$ follows from the fact that if (X,T) is *non-wandering* in the sense that $\mathbb{N} \cap N(U,U) \neq \emptyset$ for every nonempty open subset U of X, then $\overline{\text{Rec}(T)} = X$ [13, Theorem 1.27].

(3). This was proved in [35, Proposition 3.12].

(4). Assume that $x \in X$ is $\mathcal{F}_{\text{ps}}$-independent and U is a closed neighborhood of x. Then $\text{Ind}(U) \cap \mathcal{F}_{\text{ps}} \neq \emptyset$, and hence there are a piecewise syndetic set F and $y \in X$ such that $T^i y \in U$ for each $i \in F$. By [9, Theorem 6], $U \cap \text{AP}(T) \neq \emptyset$, i.e. $x \in \overline{\text{AP}(T)}$.

Conversely, assume that $x \in \overline{\text{AP}(T)}$ and U is an open neighborhood of x. Then there is $y \in \text{AP}(T) \cap U$. By a well-known result of Gottschalk, $N(y,U)$ contains a syndetic set. Thus $\text{Ind}(U) \cap \mathcal{F}_{\text{ps}} \neq \emptyset$.

(5). It is clear that if $x \in \text{QR}(T)$ then x is an $\mathcal{F}_{\text{rs}}$-independent point. Assume now that x is an $\mathcal{F}_{\text{rs}}$-independent point. Let U be a closed neighborhood of x. Then there is exists a $k \in \mathbb{N}$ such that $k\mathbb{Z}_+$ is in $\text{Ind}(U)$. Take $z \in \cap_{n\in\mathbb{Z}_+} T^{-kn}U$. Then $T^{kn}z \in U$ for all $n \in \mathbb{Z}_+$. Thus $A := \overline{\{T^{kn}z : n \in \mathbb{Z}_+\}}$ is contained in U. It is clear that $T^k A \subseteq A$. □

Remark 3.14. The family $b\mathcal{F}_{\text{rs}}$ does not have the Ramsey property.

Proof. Let (X,T) be a non-trivial *totally minimal* t.d.s., i.e., X is minimal under T^k for every $k \in \mathbb{N}$. For example, any minimal (X,T) with X being a connected topological space is totally minimal [50, II(9.6)8]. Let U be a nonempty open subset of X with $\overline{U} \neq X$. Then $X = X_1 \cup X_2$ with $X_1 = \overline{U}$ and $X_2 = X \setminus U$. Let $y \in X$. We claim that $N(y,X_i) \notin b\mathcal{F}_{\text{rs}}$ for each $i = 1,2$. Assume the contrary that $N(y,X_1) \in b\mathcal{F}_{\text{rs}}$. This means that there are $d \in \mathbb{N}$ and a sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{Z}_+$ such that for each i, $T^{n_i+dj}(y) \in X_1$ for each $0 \leq j \leq i$. Replacing $\{n_i\}_{i\in\mathbb{N}}$ by a subsequence if necessary, we may assume that $T^{n_i}(y)$ converges to some $z \in X$.

Then $z \in X_1$ and $T^{dj}(z) \in X_1$ for each $j \in \mathbb{N}$, contradicting the assumption that (X, T) is totally minimal. The same argument shows that $N(y, X_2) \notin b\mathcal{F}_{\rm rs}$. Since $\mathbb{Z}_+ = N(y, X) = N(y, X_1) \cup N(y, X_2)$, we conclude that $b\mathcal{F}_{\rm rs}$ does not have the Ramsey property. □

4. Independence: measurable case

In this section, for a given family $\mathcal{F}$, we define $\mathcal{F}$-independence for m.d.s., and discuss 1-independence for various families. First we define independence sets for m.d.s., similar to that for t.d.s. in Definition 3.1.

Definition 4.1. Lt $(X, \mathcal{B}, \mu, T)$ be a m.d.s.. For a tuple $\mathbf{A} = (A_1, \ldots, A_k)$ of sets in $\mathcal{B}$, we say that a subset $F \subseteq \mathbb{Z}_+$ is an *independence set* for $\mathbf{A}$ if for any nonempty finite subset $J \subseteq F$, we have

$$\mu(\bigcap_{j \in J} T^{-j} A_{s(j)}) > 0$$

for any $s \in \{1, \ldots, k\}^J$.

We shall still denote the collection of all independence sets for $\mathbf{A}$ by $\mathrm{Ind}(A_1, \ldots, A_k)$ or $\mathrm{Ind}\mathbf{A}$. Note that Lemma 3.2.(1)-(3) holds also for m.d.s..

Proposition 4.2. Let $\mathcal{F}$ be a family with the dynamical Ramsey property. For any m.d.s. $(X, \mathcal{B}, \mu, T)$, any $k \in \mathbb{N}$ and $A_1, A_2, \ldots, A_k, A_{1,1}, A_{1,2} \in \mathcal{B}$ with $A_1 = A_{1,1} \cup A_{1,2}$, if $\mathrm{Ind}(A_1, A_2, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$, then either $\mathrm{Ind}(A_{1,1}, A_2, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}(A_{1,2}, A_2, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$.

Proof. Set $B_{k+1} = X$, $B_0 = A_{1,1}$, $B_1 = A_{1,2}$, and $B_i = A_i$ for $2 \leq i \leq k$. Denote by Y the set of elements s in $\Sigma_{k+2} := \{0, 1, \ldots, k+1\}^{\mathbb{Z}_+}$ satisfying that for any nonempty finite subset J of $\mathbb{Z}_+$, $\mu(\bigcap_{j \in J} T^{-j} A_{s(j)}) > 0$. Then Y is a closed subset of Σ_{k+2}, and contains the constant function $k+1$. It is also easily checked that $\sigma(Y) = Y$, where σ denotes the shift map. Thus (Y, σ) is a t.d.s.. Note that $\mathrm{Ind}(A_1, A_2, \ldots, A_k) = \mathrm{Ind}([0]_Y \cup [1]_Y, [2]_Y, \ldots, [k]_Y)$, $\mathrm{Ind}(A_{1,1}, A_2, \ldots, A_k) = \mathrm{Ind}([0]_Y, [2]_Y, \ldots, [k]_Y)$, and $\mathrm{Ind}(A_{1,2}, A_2, \ldots, A_k) = \mathrm{Ind}([1]_Y, [2]_Y, \ldots, [k]_Y)$. Thus $\mathrm{Ind}([0]_Y \cup [1]_Y, [2]_Y, \ldots, [k]_Y) \cap \mathcal{F} \neq \emptyset$. Since $\mathcal{F}$ has the dynamical Ramsey property, either $\mathrm{Ind}([0]_Y, [2]_Y, \ldots, [k]_Y) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}([1]_Y, [2]_Y, \ldots, [k]_Y) \cap \mathcal{F} \neq \emptyset$. That is, either $\mathrm{Ind}(A_{1,1}, A_2, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$ or $\mathrm{Ind}(A_{1,2}, A_2, \ldots, A_k) \cap \mathcal{F} \neq \emptyset$. □

Next we define $\mathcal{F}$-independence for m.d.s., similar to that for t.d.s. in Definition 3.4.

Definition 4.3. Let $\mathcal{F}$ be a family and $k \in \mathbb{N}$. We say that a m.d.s. $(X, \mathcal{B}, \mu, T)$ is *$\mathcal{F}$-independent of order k* if for each tuple $(A_1, \ldots, A_k)$ of sets in $\mathcal{B}$ with positive measures, $\mathrm{Ind}(A_1, \ldots, A_n) \cap \mathcal{F} \neq \emptyset$. It is said to be *$\mathcal{F}$-independent*, if it is $\mathcal{F}$-independent of order k for each $k \in \mathbb{N}$.

Note that Proposition 3.7.(1) holds also for m.d.s..

Remark 4.4. Given a probability space $(X, \mathcal{B}, \mu)$, one may consider the equivalence relation defined on $\mathcal{B}$ by $A \sim B$ exactly when $\mu(A \Delta B) = 0$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B. The set of equivalence classes in $\mathcal{B}$, denoted by $\tilde{\mathcal{B}}$, has the induced operation of taking complement and countable union. Furthermore, μ descends to a function $\tilde{\mu}$ on $\tilde{\mathcal{B}}$. The pair $(\tilde{B}, \tilde{\mu})$ is called a *measure algebra* [13, Section 5.1] [14, Section 2.1]. Given a measurable and measure-preserving map $T : X \to X$, one also gets an induced map $\widetilde{T^{-1}} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ preserving $\tilde{\mu}$, complement and countable union. For any family $\mathcal{F}$ and $k \in \mathbb{N}$, it is clear that whether a m.d.s. $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}$-independent of order k or not depends only on the triple $(\tilde{\mathcal{B}}, \tilde{\mu}, \widetilde{T^{-1}})$.

Consider a m.d.s. $(X, \mathcal{B}, \mu, T)$ or a t.d.s. (X, T). Let $\mathbf{A} = (A_1, \dots, A_n)$ be a tuple of subsets of X (in $\mathcal{B}$ for m.d.s.). For each $k \in \mathbb{N}$ set $a_k = \max_{F \in \mathrm{Ind}\mathbf{A}} |F \cap [0, k-1]|$. Then the function $k \mapsto a_k$ on $\mathbb{N}$ is *subadditive* in the sense that $a_{k+j} \leq a_k + a_j$ for all $k, j \in \mathbb{N}$. Thus the limit $\lim_{k \to +\infty} \frac{a_k}{k}$ exists and is equal to $\inf_{k \in \mathbb{N}} \frac{a_k}{k}$ (see for example [51, Theorem 4.9]). We call this limit the *independence density* of $\mathbf{A}$ and denote it by $I(\mathbf{A})$ (see the discussion before Proposition 3.23 in [35] for the case of actions of discrete amenable groups). The following lemma was proved by Glasner and Weiss in the second paragraph of the proof of Theorem 3.2 in [20], using Birkhoff's ergodic theorem. We give a topological proof here.

Lemma 4.5. There exists $F \in \mathrm{Ind}\mathbf{A}$ with $d(F) = I(\mathbf{A})$.

Proof. For each $k \in \mathbb{N}$ we claim that there exists $F_k \in \mathrm{Ind}\mathbf{A}$ such that $F_k \subseteq [0, k-1]$ and $|F_k \cap [0, j-1]| \geq j(I(\mathbf{A}) - \frac{1}{k})$ for all $1 \leq j \leq k$. Suppose that this is not true. Then $I(\mathbf{A}) - \frac{1}{k} > 0$. Furthermore, for any $F \in \mathrm{Ind}\mathbf{A}$ we can find a strictly increasing sequence $\{b_i\}_{i \in \mathbb{N}}$ in $\mathbb{Z}_+$ such that $b_1 = 0$, and $b_{i+1} - b_i \leq k$ and $|F \cap [b_i, b_{i+1})| < (b_{i+1} - b_i)(I(\mathbf{A}) - \frac{1}{k})$ for all $i \in \mathbb{N}$. Set $m = k^2 + 1$. Take $F \in \mathrm{Ind}\mathbf{A}$ with $|F \cap [0, m-1]| = a_m$, and let $\{b_i\}_{i \in \mathbb{N}}$ be as above. Then $b_s < m \leq b_{s+1}$ for some $s \in \mathbb{N}$. Thus

$$\begin{aligned} m \cdot I(\mathbf{A}) &\leq a_m = |F \cap [0, m-1]| = |F \cap [b_s, m-1]| + \sum_{i=1}^{s-1} |F \cap [b_i, b_{i+1})| \\ &\leq k + \sum_{i=1}^{s-1} (b_{i+1} - b_i)(I(\mathbf{A}) - \frac{1}{k}) \leq k + m(I(\mathbf{A}) - \frac{1}{k}), \end{aligned}$$

which contradicts $m = k^2 + 1$. This proves our claim.

Now some subsequence of $\{1_{F_k}\}_{k \in \mathbb{N}}$ converges in $\{0, 1\}^{\mathbb{Z}_+}$ to 1_F for some $F \in \mathbb{Z}_+$. Clearly $F \in \mathrm{Ind}\mathbf{A}$ and $|F \cap [0, k-1]| \geq k \cdot I(\mathbf{A})$ for every $k \in \mathbb{N}$. We also have $\limsup_{k \to +\infty} \frac{|F \cap [0, k-1]|}{k} \leq \lim_{k \to +\infty} \frac{a_k}{k} = I(\mathbf{A})$. Therefore F has density $I(\mathbf{A})$. □

We now discuss 1-independence for various families. Using Birkhoff's ergodic theorem, Bergelsen proved part (1) of the following theorem [2, Theorem 1.2]. Here we give a different proof.

Theorem 4.6. Let $(X, \mathcal{B}, \mu, T)$ be a m.d.s.. The following hold:

(1) For any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $F \in \mathcal{F}_{\mathrm{pd}} \cap \mathrm{Ind}(A)$ with density at least $\mu(A)$. In particular, $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\mathrm{pd}}$-independent of order 1.

(2) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\mathrm{s}}$-independent of order 1 iff T is a.e. periodic, iff $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\mathrm{rs}}$-independent of order 1, iff for each $A \in \mathcal{B}$, a.e. every point of A returns to A syndetically, iff for each $A \in \mathcal{B}$, a.e. every point of A returns to A along $n\mathbb{Z}_+$ for some $n \in \mathbb{N}$.

(3) Let $(X,\mathcal{B},\mu,T)$ be a m.d.s. If A is in $\mathcal{B}$ with $\mu(\bigcup_{i=0}^{n-1}T^{-i}A)=1$ for some $n\in\mathbb{N}$, then $\mathrm{Ind}(A)\cap\mathcal{F}_{\mathrm{s}}\neq\emptyset$.

Proof. (1). For each $k\in\mathbb{N}$ let a_k be defined as before Lemma 4.5 for $\mathbf{A}=(A)$. Then $\sum_{j=0}^{k-1}1_{T^{-j}A}\leq a_k$ a.e. on X. Thus $k\mu(A)=\int_X\sum_{j=0}^{k-1}1_{T^{-j}A}\,d\mu\leq a_k$. It follows that $I(\mathbf{A})\geq\mu(A)$. By Lemma 4.5 we can find $F\in\mathrm{Ind}(A)$ with $d(F)=I(\mathbf{A})\geq\mu(A)$.

(2). By Theorem 6.8 the first condition implies the second one. Clearly the second condition implies the third one and the fifth one, the third one implies the first one, and the fifth one implies the fourth one. Thus it suffices to show that the fourth condition implies the first one.

Let $A\in\mathcal{B}$ with $\mu(A)>0$ and assume that a.e. every point of X returns to A syndetically. For each $n\in\mathbb{N}$ set $A_n=\bigcap_{j\in\mathbb{Z}_+}\bigcup_{i=0}^{n-1}T^{-j-i}A$. Then $\mu(X\setminus\bigcup_{n\in\mathbb{N}}A_n)=0$ and thus there exists $n\in\mathbb{N}$ with $\mu(A_n)>0$. Denote by N the union of the measure zero ones among $\bigcap_{j\in J}T^{-j}A$ for J running over nonempty finite subsets of $\mathbb{Z}_+$. Then $\mu(N)=0$, and hence $\mu(A_n\setminus N)=\mu(A_n)>0$. Take $x\in A_n\setminus N$. Then there exists $F\in\mathcal{F}_{\mathrm{s}}$ such that for each $j\in F$, $T^jx\in A$. For each nonempty finite subset J of F, $x\in(\bigcap_{j\in J}T^{-j}A)\setminus N$, thus $\mu(\bigcap_{j\in J}T^{-j}A)>0$. That is, $F\in\mathrm{Ind}(A)$.

(3). The condition implies that $\mu(A_n)=1$. Thus the conclusion follows from the last paragraph. □

5. Classes of topological $\mathcal{F}$-independence

5.1. **General discussion.** In this subsection we characterize $\mathcal{F}_{\mathrm{inf}}$ (resp. $\mathcal{F}_{\mathrm{ip}}$) independent t.d.s. in Theorem 5.1, construct a nontrivial topological K system with a unique minimal point in Example 5.7, and discuss $\mathcal{F}_{\mathrm{rs}}$-independence at the end.

A t.d.s. (X,T) is said to be *(topologically) transitive* if for any nonempty open subsets U and V of X, $N(U,V)$ is nonempty; it is called *weakly mixing* if $(X\times X,T\times T)$ is transitive. The equivalence of the conditions (1), (2) and (3) in the following theorem was proved in [35, Theorem 8.6]. Here we strengthen it by adding the conditions (4) and (5).

Theorem 5.1. For a t.d.s. (X,T) the following are equivalent:

(1) (X,T) is weakly mixing.
(2) (X,T) is $\mathcal{F}_{\mathrm{inf}}$-independent of order 2.
(3) (X,T) is $\mathcal{F}_{\mathrm{inf}}$-independent.
(4) (X,T) is $\mathcal{F}_{\mathrm{ip}}$-independent of order 2.
(5) (X,T) is $\mathcal{F}_{\mathrm{ip}}$-independent.

Proof. It is clear that (5)⇒(4)⇒(2) and (5)⇒(3)⇒(2). The implication (2)⇒(1) follows from the fact that if for any nonempty open subsets U and V of X one has $N(U,U)\cap N(U,V)\neq\emptyset$, then (X,T) is weakly mixing [40, Lemma]. ([40, Lemma] was proved only for invertible t.d.s., but it is easy to modify the proof to make it work for any t.d.s..) Thus it suffices to show that (1)⇒(5).

Now assume that (X,T) is weakly mixing. Then each $(X\times\cdots\times X,T\times\cdots\times T)$ is transitive [12, Proposition II.3]. Thus, for any $n\in\mathbb{N}$, if $U_1,\dots,U_n$ and $V_1,\dots,V_n$ are nonempty open subsets of X, then $\mathbb{N}\cap(\bigcap_{i=1}^nN(U_i,V_i))\neq\emptyset$. For any given nonempty open subsets $U_1,\dots,U_n$ of X, we are going to find an IP-set F in $\mathrm{Ind}(U_1,\dots,U_n)$.

First there exists a $t_1 \in \mathbb{N}$ such that $t_1 \in \bigcap_{(i_1,i_2)\in\{1,\ldots,n\}^2} N(U_{i_1}, U_{i_2})$. Assume that $t_1, \ldots, t_k$ in $\mathbb{N}$ are defined such that $\{a_1, \ldots, a_j\} := \{0\} \cup \{t_{i_1} + \cdots + t_{i_l} : 1 \le i_1 < \cdots < i_l \le k\}$ is in $\mathrm{Ind}(U_1, \ldots, U_n)$. Pick $t_{k+1} \in \mathbb{N}$ such that

$$t_{k+1} \in \bigcap_{1\le i_m \le n, 1\le m\le 2j} N(T^{-a_1}U_{i_1} \cap \ldots \cap T^{-a_j}U_{i_j}, T^{-a_1}U_{i_{j+1}} \cap \ldots \cap T^{-a_j}U_{i_{2j}}).$$

Then $\{0\} \cup \{t_{i_1} + \ldots + t_{i_l} : 1 \le i_1 < \ldots < i_l \le k+1\} = \{a_1, \ldots, a_j\} \cup \{a_1 + t_{k+1}, \ldots, a_j + t_{k+1}\}$ is in $\mathrm{Ind}(U_1, \ldots, U_n)$. This implies that the IP-set generated by the sequence $\{t_i\}_{i\in\mathbb{N}}$ is in $\mathrm{Ind}(U_1, \ldots, U_n)$. □

Petersen [41] showed there exists a t.d.s. which is strictly ergodic, strongly mixing, and has zero topological entropy. Thus in such a system every tuple is $\mathcal{F}_{\rm ip}$-independent, while no non-diagonal tuple is $\mathcal{F}_{\rm pd}$-independent.

A t.d.s. is called an *E-system* if it is transitive and has an invariant Borel probability measure with full support; it is called an *M-system* if it is transitive and the set of minimal points is dense; it is called *totally transitive* if (X, T^k) is transitive for every $k \in \mathbb{N}$. By Theorems 3.13 and 5.1 we have

Corollary 5.2. Let (X, T) be a t.d.s.. The following hold:

(1) If (X, T) is $\mathcal{F}_{\rm pd}$-independent of order 2, then it is an E-system.
(2) If (X, T) is $\mathcal{F}_{\rm ps}$-independent of order 2, then it is an M-system.
(3) If (X, T) is $\mathcal{F}_{\rm rs}$-independent of order 1, then it has dense small periodic sets. If it is $\mathcal{F}_{\rm rs}$-independent of order 2, then it is totally transitive, and hence is disjoint from all minimal systems by [31, Theorem 3.4].

Definition 5.3. We say that a t.d.s. is *topological K* if it is $\mathcal{F}_{\rm pubd}$-independent.

By [32, Theorem 8.3] and [35, Theorem 3.16] a t.d.s. is topological K if and only if its every finite cover by non-dense open subsets has positive topological entropy.

Next we show that there is an invertible topological K system with only one minimal point. Recall that a t.d.s. (X, T) is said to be *proximal* if the orbit closure of every point in $(X \times X, T \times T)$ has nonempty intersection with the diagonal. Following [35] we shall call $\mathcal{F}_{\rm pubd}$-independent tuples of a t.d.s. as IE-tuples. To construct the example we need

Lemma 5.4. Let (X, T) be a t.d.s.. We have:

(1) Suppose that (X, T) has a transitive point x. Then T is topologically K if and only if for each $j \in \mathbb{N}$, $(x, Tx, \ldots, T^{j-1}x)$ is an IE-tuple.
(2) (X, T) has only one minimal point if and only if (X, T) is proximal.

Proof. (1). This follows from the fact that the set of IE j-tuples is closed in X^j for each $j \in \mathbb{N}$.

(2). The "only if" part is trivial. Assume that (X, T) is proximal. Take $x \in X$. Say, (y, y) is in the intersection of the diagonal and the orbit closure of (x, Tx). Then $Ty = y$. Let $z \in X$. Then the orbit closures of y and z have nonempty intersection, which of course has to be $\{y\}$. It follows that if z is minimal, then $z = y$. □

For a t.d.s. (X, T), recall its *natural extension* $(\tilde{X}, \tilde{T})$ defined as follows. $\tilde{X}$ is the closed subspace of $\prod_{n\in\mathbb{N}} X$ consisting of $(x_1, x_2, \ldots)$ with $T(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$, and $\tilde{T}$ is defined as $\tilde{T}(x_1, x_2, \ldots) = (T(x_1), x_1, x_2, \ldots)$. Note that $\tilde{T}$ is a homeomorphism and the projection $\pi : \tilde{X} \to X$ sending $(x_1, x_2, \dots)$ to x_1 is a factor map. It is well known that (X, T) and $(\tilde{X}, \tilde{T})$ share many dynamical properties. Here we need a special case.

Lemma 5.5. Let (X, T) be a t.d.s.. The following are true:

1. Let $\mathcal{F}$ be a family and $k \in \mathbb{N}$. Then (X, T) is $\mathcal{F}$-independent of order k if and only if $(\tilde{X}, \tilde{T})$ is so.
2. (X, T) is proximal if and only if $(\tilde{X}, \tilde{T})$ is so.

Proof. (1). The "if" part follows from the fact that if a t.d.s. is $\mathcal{F}$-independent of order k, then so is every factor. Suppose that (X, T) is $\mathcal{F}$-independent of order k. Let $U_1, \dots, U_k$ be nonempty open subsets of $\tilde{X}$. Then there exist nonempty open subsets $V_1, \dots, V_k$ of X and $m \in \mathbb{N}$ such that if $(x_1, x_2, \dots)$ is in $\tilde{X}$ and $x_m \in V_j$ for some $1 \le j \le k$, then $(x_1, x_2, \dots)$ is in U_j.

We claim that $\mathrm{Ind}(V_1, \dots, V_k) \subseteq \mathrm{Ind}(U_1, \dots, U_k)$. Let $F \in \mathrm{Ind}(V_1, \dots, V_k)$, J be a nonempty finite subset of F, and $s \in \{1, \dots, k\}^J$. Then $\bigcap_{j \in J} T^{-j} V_{s(j)} \neq \emptyset$. Take $y \in \bigcap_{j \in J} T^{-j} V_{s(j)}$. We can find $\tilde{x} = (x_1, x_2, \dots) \in \tilde{X}$ such that $x_m = y$. Then $\tilde{x} \in \bigcap_{j \in J} \tilde{T}^{-j} U_{s(j)}$. Thus $F \in \mathrm{Ind}(U_1, \dots, U_k)$. This proves out claim.

Since $\mathcal{F} \cap \mathrm{Ind}(V_1, \dots, V_k) \neq \emptyset$, we get $\mathcal{F} \cap \mathrm{Ind}(U_1, \dots, U_k) \neq \emptyset$. Therefore $(\tilde{X}, \tilde{T})$ is also $\mathcal{F}$-independent of order k. This proves the "only if" part.

(2). This is trivial. □

For $p \ge 2$ let $\Lambda_p = \{0, 1, \dots, p-1\}$ with the discrete topology, $\Sigma_p = \Lambda_p^{\mathbb{Z}_+}$ with the product topology and $\sigma : \Sigma_p \to \Sigma_p$ be the shift. For $n \in \mathbb{N}$ and $a = (a(1), a(2), \dots, a(n)) \in \Lambda_p^n$ (a block of length n), let $|a| = n$, $\sigma(a) = (a(2), \dots, a(n))$. We say that a *appears* in $x = (x(1), x(2), \dots) \in \Sigma_p$ or $x \in \Lambda_p^m$ with $m \ge n$ if there is $j \in \mathbb{N}$ with $a = (x(j), x(j+1), \dots, x(j+n-1))$ (write $a < x$ for short) and we use t^i to denote $t \dots t$ (i times). For $b = (b(1), \dots, b(m)) \in \Lambda_p^m$, denote $(a(1), \dots, a(n), b(1), \dots, b(m)) \in \Lambda_p^{n+m}$ by ab. Denote $(ii\dots)$ by $\mathbf{i}$, $0 \le i \le p-1$. We also need the following lemma. In view of [5, Proposition 2] and [32, Theorem 7.3] or [35, Theorem 3.16], it is equivalent to [28, Lemma 4.1]. One can also prove it directly using IE-pairs instead of entropy pairs in the proof of [28, Lemma 4.1].

Lemma 5.6. There is an E-system (Y, σ) contained in (Σ_3, σ) with a unique minimal point $\mathbf{0}$ such that Y has an IE-pair (x_1, x_2) in $[1]_Y \times [2]_Y$.

Example 5.7. There exists a non-trivial invertible t.d.s. which is topological K and has a unique minimal point.

Proof. By Lemma 5.5 it suffices to show that there exists a non-trivial t.d.s. which is topological K and proximal. We use the idea in the proof of Theorem 4.2 in [28]. The main idea is to construct a recurrent point $x \in \Sigma_2$ with the following two properties:

(I) for any $j \in \mathbb{N}$, $(x, \sigma(x), \dots, \sigma^{j-1}(x))$ is an IE-tuple of (X, σ), where X is the orbit closure of x, and

(II) for each $n \in \mathbb{N}$, 0^n appears in x syndetically.

By Lemma 5.4 it is clear that (X, σ) is topological K with a unique minimal point $\mathbf{0}$. First we give the detailed construction of the recurrent point x.

Let (Y, σ) be the system constructed in Lemma 5.6 and let y be a transitive point of Y. By [35, Theorem 3.18] for each $m \in \mathbb{N}$ we can find an IE-tuple $(z_{m,1}, \dots, z_{m,m})$ of Y with $z_{m,1}, \dots, z_{m,m}$ being pairwise distinct and all in $[1]_Y \cup [2]_Y$. Then we can find $t_m \in \mathbb{N}$ such that $z_{m,1}[0, t_m], \dots, z_{m,m}[0, t_m]$ are pairwise distinct, where $z[0, t]$ denotes $(z(0), \dots, z(t))$. Define a map $f_m : \Lambda_3^{t_m+1} \to \Lambda_{m+1}$ by $f_m(a) = j$ if $a = z_{m,j}[0, t_m]$ for some $1 \leq j \leq m$ and $f_m(a) = 0$ otherwise.

Take $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(k) \leq k$ for each $k \in \mathbb{N}$ and for each $m \in \mathbb{N}$, $\phi^{-1}(m)$ is infinite.

Set $A_1 = (10)$, $n_1 = |A_1| = 2$. Set

$$C_{1,0} = 0^{2n_1},\ C_{1,1} = A_1 0^{n_1}.$$

Suppose that $A_1, \dots, A_k$, $C_{m,i}$ for $0 < m \leq k$ and $0 \leq i \leq m$, and $n_1, \dots, n_k$ are defined. We define inductively A_{k+1}, $C_{k+1,i}$ for $i = 0, 1, \dots, k+1$, and n_{k+1}.

Say, $m = \phi(k)$. Since (Y, σ) has a unique minimal point $\mathbf{0}$, there exists $\ell_k \in \mathbb{N}$ with $\ell_k \geq t_m$ such that 0^{n_k} appears in y with gaps bounded above by ℓ_k. Set $b_k = 2\ell_k n_k$, and set

$$A_{k+1} = A_k 0^{n_k} C_{m,f_m(y[0,t_m])} C_{m,f_m(y[1,t_m+1])} \dots C_{m,f_m(y[b_k - t_m, b_k])} 0^{2n_k},\ n_{k+1} = |A_{k+1}|, \quad \text{and}$$
$$C_{k+1,0} = 0^{2n_{k+1}}, C_{k+1,i} = \sigma^{i-1}(A_{k+1}) 0^{i-1} 0^{n_{k+1}} \text{ for } i = 1, 2, \dots, k+1.$$

It is clear that $x := \lim_{k \to +\infty} A_k$ is a recurrent point of σ in Σ_2. Denote by X the orbit closure of x in Σ_2. We claim that x satisfies (I) and (II).

(I). Given $j \in \mathbb{N}$, we show that $(x, \sigma(x), \dots, \sigma^{j-1}(x))$ is an IE-tuple of (X, σ). Suppose that $V_0', V_1', \dots, V_{j-1}'$ are neighborhoods of $x, \sigma(x), \dots, \sigma^{j-1}(x)$ respectively. Then there is some $m \in \mathbb{N}$ with $m > j$ such that $V_i \subseteq V_i'$ for all $0 \leq i \leq j-1$, where $V_i := [\sigma^i(A_m)0^i]_X$ for all $0 \leq i \leq m-1$.

Since $(z_{m,1}, \dots, z_{m,m})$ is an IE-tuple of Y, there exists some $d > 0$ such that for any $n \in \mathbb{N}$ we can find a finite subset $J \subseteq \mathbb{Z}_+$ with $|J| \geq n$ contained in an interval with length at most $d|J|$ such that for any $s \in \{1, 2, \dots, m\}^J$ one has $\bigcap_{i \in J} \sigma^{-i} U_{s(i)} \neq \emptyset$, where $U_j = (z_{m,j}[0, t_m])_Y$ for $1 \leq j \leq m$. Since y is a transitive point of Y, we have $\sigma^N(y) \in \bigcap_{i \in J} \sigma^{-i} U_{s(i)}$ for some $N \in \mathbb{Z}_+$. Then $y_{[N+i, N+i+t_m]} = z_{m,s(i)}[0, t_m]$ for all $i \in J$. Take $k \geq N + \max J + t_m$ with $\phi(k) = m$. Then $b_k \geq k \geq N + i + t_m$ for all $i \in J$, and

$$A_{k+1} = A_k 0^{n_k} C_{m,f_m(y[0,t_m])} C_{m,f_m(y[1,t_m+1])} \dots C_{m,f_m(y[b_k - t_m, b_k])} 0^{2n_k}.$$

Note that $f_m(y[N+i, N+i+t_m]) = s(i)$ for all $i \in J$. Thus $\sigma^{2n_k + 2(N+i)n_m}(x) \in [C_{m,s(i)}]_X \subseteq V_{s(i)-1}$ for all $i \in J$. It follows that $\sigma^{2n_k + 2Nn_m}(x) \in \bigcap_{i \in 2n_m J} \sigma^{-i} V_{\psi(i)}$ for the map $\psi \in \{0, 1, \dots, m-1\}^{2n_m J}$ defined by $\psi(2n_m i) = s(i) - 1$ for all $i \in J$. Therefore $2n_m J$ is an independence set for $(V_0, \dots, V_{m-1})$. Clearly $2n_m J$ is contained in an interval with length at most $2n_m d|J| = 2n_m d|2n_m J|$. Thus by Lemma 4.5 $(x, \sigma(x), \dots, \sigma^{j-1}(x))$ is an IE-tuple of (X, σ).

(II). We now show that for each $n \in \mathbb{N}$, 0^n appears in x syndetically. It suffices to prove that for each $k \in \mathbb{N}$, 0^{n_k} appears in x syndetically with gaps bounded above by $2b_k$.

Fix $k \in \mathbb{N}$. Say, $\phi(k) = m$. By the construction

$$A_{k+1} = A_k 0^{n_k} C_{m,f_m(y[0,t_m])} C_{m,f_m(y[1,t_m+1])} \dots C_{m,f_m(y[b_k - t_m, b_k])} 0^{2n_k}.$$

Note that $f_m(a) = 0$ for every $a \in \Lambda_3^{t_m+1}$ with $a(0) = 0$. As 0^{n_k} appears in y with gaps bounded above by ℓ_k, 0^{n_k} appears in $C_{m,f_m(y[0,t_m])}C_{m,f_m(y[1,t_m+1])}\dots C_{m,f_m(y[b_k-t_m,b_k])}$ with gaps bounded above by $2n_m\ell_k \le 2n_k\ell_k = b_k$. Thus 0^{n_k} appears in A_{k+1} with gaps bounded above by $b_k + n_k \le 2b_k$.

Assume that 0^{n_k} appears in A_ℓ with gaps bounded above by $2b_k$, where $\ell \ge k+1$. Now we are going to prove that this is also true for $\ell+1$. Set $m' = \phi(\ell)$. First note that

$$A_{\ell+1} = A_\ell 0^{n_\ell} C_{m',f_{m'}(y[0,t_{m'}])}C_{m',f_{m'}(y[1,t_{m'}+1])}\dots C_{m',f_{m'}(y[b_\ell-t_{m'},b_\ell])}0^{2n_\ell}.$$

If $m' \ge k+1$, then by the induction assumption and the construction of $C_{m',i}$ we know that 0^{n_k} appears in $A_{\ell+1}$ with gaps bounded above by $2b_k$. If $m' \le k$, then by the induction assumption and the discussion similar to the case of A_{k+1}, we know that 0^{n_k} appears in $A_{\ell+1}$ with gaps bounded above by $2b_k$. Hence 0^{n_k} appears in x syndetically with gaps bounded above by $2b_k$, as $x = \lim_{\ell\to+\infty} A_\ell$. □

Definition 5.8. We say that a t.d.s (X, T) is *Bernoulli* if it is conjugate to $(A^{\mathbb{Z}_+}, \sigma)$, where A is a compact metrizable space with $|A| \ge 2$ and σ is the shift.

Theorem 5.9. A Bernoulli system is $\mathcal{F}_{\rm rs}$-independent.

Proof. Let (X, T) be a Bernoulli system. Without loss of generality we may assume that $(X, T) = (A^{\mathbb{Z}_+}, \sigma)$ as above. Let $U_1, \dots, U_n$ be nonempty open subsets of X for some $n \in \mathbb{N}$. Then there exist some $k \in \mathbb{N}$ and nonempty subsets $A_{i,j} \subseteq A$ for $1 \le i \le n$ and $0 \le j < k$ such that $U_i \supseteq A_{i,0} \times \dots \times A_{i,k-1} \times \prod_{\ell \ge k} A$ for all $1 \le i \le n$. It follows that $k\mathbb{N} \subseteq \mathrm{Ind}(U_1, \dots, U_n)$. Thus (X, T) is $\mathcal{F}_{\rm rs}$-independent. □

Recall that a t.d.s. (X, T) is called *strongly mixing* if for any nonempty open subsets U and V of X, $N(U, V)$ is a cofinite subset of $\mathbb{Z}_+$. In [4, Example 5] Blanchard constructed examples of invertible t.d.s. which are $\mathcal{F}_{\rm rs}$-independent of order 2 and are not strongly mixing. In fact, the Property P defined in [4] is exactly the same as $\mathcal{F}_{\rm rs}$-independence of order 2. It is easily checked that the condition in [4, Proposition 4] actually implies $\mathcal{F}_{\rm rs}$-independence. Thus Blanchard's examples are actually $\mathcal{F}_{\rm rs}$-independent. Thus $\mathcal{F}_{\rm rs}$-independence does not imply strong mixing and hence does not imply Bernoulli.

A factor map $\pi : (X, T) \to (Y, S)$ between t.d.s. is said to be an *almost one-to-one extension* if the set $\{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is dense in X.

For a sequence $K = \{k_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ with k_{n+1} being divisible by k_n for each $n \in \mathbb{N}$, the *adding machine* (X_K, T_K) associated to K is defined as follows. X_K is the projective limit of $\varprojlim_{n\to+\infty} \mathbb{Z}/k_n\mathbb{Z}$, as a metrizable compact abelian group, and T_K is the addition by 1.

For a t.d.s. (X, T), recall that $x \in X$ is called a *regular minimal point* [23, Definition 3.38] if for each neighborhood U of x, there exists $k \in \mathbb{N}$ such that $N(x, U) \supseteq k\mathbb{Z}_+$. It is known that if x is a regular minimal point, then its orbit closure is an almost one-to-one extension of some adding machine, see for instance [31, Proposition 3.5]. Now we show

Proposition 5.10. Let (X, T) be a minimal t.d.s.. The following are equivalent:

(1) (X, T) has dense small periodic sets.
(2) (X, T) is an almost one-to-one extension of some adding machine.

(3) X has a regular minimal point.

Proof. By [31, Proposition 3.5] (2) and (3) are equivalent. (3)⇒(1) is trivial.

(1)⇒(2). For any nonempty open subset U of X, let B be a nonempty closed subset of U with $T^k B \subseteq B$ for some $k \in \mathbb{N}$. Take $x \in B$. Then the argument in the proof of [31, Proposition 3.5] shows that the orbit closure A of x under T^k is a nonempty clopen subset of U and there exists some $\ell \in \mathbb{N}$ such that $\{A, TA, \dots, T^{\ell-1}A\}$ is a clopen partition of X and $T^\ell A = A$.

Fix a compatible metric on X. Starting with some nonempty open subset U of X with $\mathrm{diam}(U) < 1$, we obtain A and ℓ as above, and set $A_1 = A$ and $\ell_1 = \ell$. Inductively, assuming that we have found subsets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k$ and positive integers $\ell_1, \ell_2, \dots, \ell_k$ such that $\mathrm{diam}(A_j) < 1/j$, $\{A_j, TA_j, \dots, T^{\ell_j-1}A_j\}$ is a clopen partition of X, and $T^{\ell_j}A_j = A_j$ for all $1 \le j \le k$. We shall find A_{k+1} and ℓ_{k+1} with the same property. Let U be a nonempty open subset of A_k with $\mathrm{diam}(U) < 1/(k+1)$. We obtain A and ℓ as above, and set $A_{k+1} = A$ and $\ell_{k+1} = \ell$.

Now the argument in the proof of [31, Proposition 3.5] shows that (X, T) is an almost one-to-one extension of some adding machine. □

5.2. **Non-existence of non-trivial minimal $\mathcal{F}_s$-independent t.d.s.** It was shown in [32, Theorem 3.4] that there exist non-trivial minimal topological K systems (the existence of nontrivial minimal u.p.e. systems was proved earlier by Glasner and Weiss [18], answering a question of Blanchard [5]). As a contrast, we have

Theorem 5.11. There is no non-trivial minimal t.d.s. which is $\mathcal{F}_s$-independent of order 2.

To prove Theorem 5.11, we need some preparation. Crucial to the proof of Theorem 5.11 is the following combinatorial result, which is also of independent interest. We postpone its proof to the Appendix. Recall the notion introduced before Lemma 5.6.

Theorem 5.12. Let $p, \ell \in \mathbb{N}$ with $p \ge 2$. For any integer $m \ge 4\ell + 2$, given any sequence $\{A_n\}_{n \in \mathbb{Z}_+}$ of subsets of Λ_p^m with $|A_n| \le \ell$ for each $n \in \mathbb{Z}_+$, there exists $x \in \Sigma_p$ such that $x[n, n+m-1] \notin A_n$ for every $n \in \mathbb{Z}_+$.

We remark that under the conditions of Theorem 5.12, the set $\{x \in \Sigma_p : x[n, n+m-1] \notin A_n \text{ for all } n \in \mathbb{Z}_+\}$ is small in both topological and measure-theoretical sense: it is a closed subset of Σ_p with empty interior and has measure 0 for the product measure on Σ_p associated to any probability vector $(t_0, \dots, t_{p-1})$ with $\sum_{j=0}^{p-1} t_j = 1$ and $t_j > 0$ for all $0 \le j \le k-1$. The following lemma is important for the proof of Theorem 5.11 and also can be applied to show that an $\mathcal{F}_s$-independent t.d.s. is disjoint from all minimal t.d.s. [11].

Lemma 5.13. For every minimal subshift $X \subseteq \Sigma_2$, $\mathrm{Ind}([0]_X, [1]_X)$ does not contain any syndetic set.

Proof. We argue by contradiction. Assume that $X \subseteq \Sigma_2$ is a minimal subshift and $\mathrm{Ind}([0]_X, [1]_X)$ contains a syndetic set F. Say, $F = \{n_0 < n_1 < \dots\}$ with $\ell = \max_{j \in \mathbb{Z}_+}(n_{j+1} - n_j)$. Let m be as in Theorem 5.12 for $p = 2$ and ℓ. Take $a \in \Lambda_2^{m\ell}$ such that a appears in some element of X. For each $j \in \mathbb{Z}_+$, set A_j to be the subset of Λ_2^m consisting of elements of the form $(a(k), a(k + n_{j+1} - n_j), a(k +$

$n_{j+2}-n_j),\dots,a(k+n_{j+m-1}-n_j))$ for $1\le k\le \ell$. Then $|A_j|\le \ell$ for all $j\in\mathbb{Z}_+$. By Theorem 5.12 we can find $x\in\Sigma_2$ such that $x[j,j+m-1]\notin A_j$ for every $j\in\mathbb{Z}_+$. Since $F\in \mathrm{Ind}([0]_X,[1]_X)$, we can find $y\in X$ with $y(n_j)=x(j)$ for all $j\in\mathbb{Z}_+$. As X is minimal, there exists some $i\ge n_1$ such that $y[i,i+m\ell-1]=a$. Say, $n_{j-1}<i\le n_j$. Set $k=n_j-i+1$. Then $x(s)=y(n_s)=a(k+n_s-n_j)$ for all $j\le s\le j+m-1$, which contradicts that $x[j,j+m-1]\notin A_j$. □

We are ready to prove Theorem 5.11.

Proof of Theorem 5.11. We shall show that if (Y,S) is a minimal t.d.s., and V_0,V_1 are disjoint closed subsets of X with nonempty interior, then $\mathrm{Ind}(V_0,V_1)\cap\mathcal{F}_{\mathrm{s}}$ does not contain any syndetic set.

It is well known that we can find a minimal t.d.s. (X_1,T_1) and a factor map $\pi:(X_1,T_1)\to(Y,S)$ such that X_1 is a closed subset of a Cantor set (see for eample [6, page 34]). It is easy to see that $\mathrm{Ind}(V_0,V_1)=\mathrm{Ind}(\pi^{-1}(V_0),\pi^{-1}(V_1))$. Write X as the disjoint union of clopen subsets U_0 and U_1 such that $U_j\supseteq\pi^{-1}(V_j)$ for $j=0,1$. Then $\mathrm{Ind}(V_0,V_1)\subseteq\mathrm{Ind}(U_0,U_1)$.

Define a coding $\phi:X_1\to\Sigma_2$ such that for each $x\in X_1$, $\phi(x)=(x_0,x_1,\dots)$, where $x_i=j$ if $T_1^i(x)\in U_j$ for all $i\in\mathbb{Z}_+$. Then $X=\phi(X_1)$ is a minimal subshift contained in Σ_2 and $\phi:X_1\to X$ is a factor map. It is easy to verify that $\mathrm{Ind}(U_0,U_1)\subseteq\mathrm{Ind}([0]_X,[1]_X)$.

By Lemma 5.13 we know that $\mathrm{Ind}([0]_X,[1]_X)$ does not contain any syndetic set. Then $\mathrm{Ind}(V_0,V_1)$ does not contain any syndetic set either. □

5.3. **Finite product.** In this subsection we investigate the question for which family $\mathcal{F}$, the product of finitely many $\mathcal{F}$-independent t.d.s. remains $\mathcal{F}$-independent.

It is known that if $\mathcal{F}=\mathcal{F}_{\mathrm{pd}}$ the question has a positive answer [32, Theorem 8.1] [35, Theorem 3.15]. We now show that the question has a positive answer for $\mathcal{F}=\mathcal{F}_{\mathrm{rs}},\mathcal{F}_{\mathrm{ps}}$. It is clear that

$$\mathcal{F}_{\mathrm{rs}}\subseteq\mathcal{F}_{\mathrm{cen}}\subseteq\mathcal{F}_{\mathrm{ps}}\subseteq\mathcal{F}_{\mathrm{pubd}}.$$

We need the following lemma. It is also needed for the proof of Theorem 7.1 later.

Lemma 5.14. For any $d>0$, $k\in\mathbb{N}$, and finite subset $F\subseteq\mathbb{Z}_+$ with $d|F|>k$, there exists $N=N(d,k,F)\in\mathbb{N}$ such that for any nonempty finite interval $I\subseteq\mathbb{Z}_+$ and $S\subseteq I$ with $\frac{|S|}{|I|}\ge d$ and $|I|\ge N$ one has $|S\cap(F+p)|\ge k$ for some $p\in\mathbb{Z}$.

Proof. Take $N\in\mathbb{N}$ such that $\frac{d|F|}{1+(\max F)/N}\ge k$. For each $j\in F$ the set $S-j$ is contained in $[\min I-\max F,\max I]\subseteq\mathbb{Z}$. Then we can find some $p\in[\min I-\max F,\max I]$ such that p is contained in $S-j$ for at least $\frac{|S|\cdot|F|}{|[\min I-\max F,\max I]|}$ j's in F. Set $W=\{j\in F:p\in S-j\}$. Then $(W+p)\subseteq S\cap(F+p)$ and

$$|W|\ge\frac{|S|\cdot|F|}{|[\min I-\max F,\max I]|}=\frac{|S|\cdot|F|}{|I|+\max F}\ge\frac{d|F|}{1+(\max F)/|I|}\ge k.$$

□

We need the following simple lemma. For a subset K of $\mathbb{Z}_+$, denote by X_K the set of limit points of the sequence $\{\sigma^n 1_K\}_{n\in\mathbb{Z}_+}$ in $\{0,1\}^{\mathbb{Z}_+}$, where σ denotes the shift map on $\{0,1\}^{\mathbb{Z}_+}$. Note that (X_K,σ) is a t.d.s..

Lemma 5.15. The following statements hold:

(1) Let $S_1, S_2 \in \mathcal{F}_{\text{pubd}}$. Then there are two subsets K_1, K_2 of $\mathbb{Z}_+$ such that $1_{K_i} \in X_{S_i}$, $i = 1, 2$, and $K_1 \cap K_2 \in \mathcal{F}_{\text{pud}}$.
(2) Let $S_1, S_2 \in \mathcal{F}_{\text{ps}}$. Then there are two subsets K_1, K_2 of $\mathbb{Z}_+$ such that $1_{K_i} \in X_{S_i}$, $i = 1, 2$, and $K_1 \cap K_2 \in \mathcal{F}_{\text{s}} \cap \mathcal{F}_{\text{cen}}$.
(3) Let $S_1, S_2 \in \mathcal{F}_{\text{rs}}$. Then there are two subsets K_1, K_2 of $\mathbb{Z}_+$ such that $1_{K_i} \in X_{S_i}$, $i = 1, 2$, and $K_1 \cap K_2 \in \mathcal{F}_{\text{rs}}$.

Proof. (1). Set $X_i = X_{S_i}$. Recall the independence density defined before Lemma 4.5. We have $I([1]_{X_i}) = BD^*(S_i) > 0$ for $i = 1, 2$. For each $k \in \mathbb{N}$, take a finite interval J_1 in $\mathbb{Z}_+$ with $|J_1| = k$ and a set $F_1 \in \text{Ind}([1]_{X_1})$ with $F_1 \subseteq J_1$ and $|F_1| \geq |J_1| I([1]_{X_1})$. Note that we can find arbitrarily long finite interval J_2 in $\mathbb{Z}_+$ and a set $F_2 \in \text{Ind}([1]_{X_1})$ with $F_2 \subseteq J_2$ and $|F_2| \geq |J_2| I([1]_{X_2})$. By Lemma 5.14, when $|I_2|$ is large enough, we have $|F_1 \cap (F_2 + p)| \geq I([1]_{X_2})|F_1| - 1 \geq |J_1| I([1]_{X_1}) I([1]_{X_2}) - 1$ for some $p \in \mathbb{Z}$. Consider the t.d.s. $(X_1 \times X_2, \sigma \times \sigma)$. Note that $F_1 \cap (F_2 + p) \in \text{Ind}([1]_{X_1} \times [1]_{X_2})$. It follows that $I([1]_{X_1} \times [1]_{X_2}) \geq I([1]_{X_1}) I([1]_{X_2}) > 0$. By Lemma 4.5 we can find $F \in \text{Ind}([1]_{X_1} \times [1]_{X_2})$ with density $I([1]_{X_1} \times [1]_{X_2})$. Take $x \in \bigcap_{n \in F} (\sigma \times \sigma)^{-n}([1]_{X_1} \times [1]_{X_2})$. Say, $x = (1_{K_1}, 1_{K_2})$ for some $K_1, K_2 \in \mathbb{Z}_+$. Then $1_{K_i} \in X_i$, $i = 1, 2$, and $K_1 \cap K_2 \supseteq F$. It follows that $K_1 \cap K_2 \in \mathcal{F}_{\text{pud}}$.

(2). Set $X_i = X_{S_i}$. Since S_i is in $\mathcal{F}_{\text{ps}}$, there exists $1_{F_i} \in X_i$ with $F_i \in \mathcal{F}_{\text{s}}$. By Lemma 5.4.(2), for each $i = 1, 2$, there is a minimal set $M_i \neq \{(0, 0, \dots)\}$ contained in X_i. Consider the t.d.s. $(M_1 \times M_2, \sigma \times \sigma)$. Let M be a minimal set contained in $M_1 \times M_2$ and take $x \in M$. Say, $x = (1_{K_1}, 1_{K_2})$ for some $K_1, K_2 \subseteq \mathbb{Z}_+$. Then K_1 and K_2 are nonempty. For any $j, k \in \mathbb{Z}_+$, since $\sigma^j \times \sigma^k$ is a factor map from M to a minimal set in $M_1 \times M_2$, $\sigma^j \times \sigma^k(x) = (\sigma^j 1_{K_1}, \sigma^k 1_{K_2})$ is also a minimal point. Replacing x by $\sigma^{\min K_1} \times \sigma^{\min K_2}(x)$ if necessary, we may assume that $\min K_1 = \min K_2 = 0$. Then $K_1 \cap K_2 = N(x, [1]_{X_1} \times [1]_{X_2})$ is syndetic and central.

(3). This is trivial. □

Theorem 5.16. The product of finitely many $\mathcal{F}_{\text{s}}$-(resp. $\mathcal{F}_{\text{rs}}, \mathcal{F}_{\text{pd}}$) independent t.d.s. is $\mathcal{F}_{\text{s}}$-(resp. $\mathcal{F}_{\text{rs}}, \mathcal{F}_{\text{pd}}$) independent.

Proof. We shall prove the case $\mathcal{F} = \mathcal{F}_{\text{s}}$, and the proof for the other cases is similar. Let (X_i, T_i) be an $\mathcal{F}_{\text{s}}$-independent t.d.s. for $i = 1, 2$. Let $U_1, \dots, U_n$ and $V_1, \dots, V_n$ be nonempty open subsets of X_1 and X_2 respectively. Then there are syndetic sets $S_1 \in \text{Ind}(U_1, \dots, U_n)$ and $S_2 \in \text{Ind}(V_1, \dots, V_n)$. By Lemma 5.15 there are two subsets K_1, K_2 of $\mathbb{Z}_+$ such that $1_{K_i} \in X_{S_i}$, $i = 1, 2$, and $K_1 \cap K_2$ is syndetic. It is clear that $K_1 \in \text{Ind}(U_1, \dots, U_n)$ and $K_2 \in \text{Ind}(V_1, \dots, V_n)$. Thus, $K_1 \cap K_2 \in \text{Ind}(U_1 \times V_1, \dots, U_n \times V_n)$. This implies that $(X_1 \times X_2, T_1 \times T_2)$ is $\mathcal{F}_{\text{s}}$-independent. The theorem follows by induction. □

Since a family $\mathcal{F}$ has the Ramsey property if and only if its dual family $\mathcal{F}^*$ has the finite intersection property, we have

Theorem 5.17. Let $\mathcal{F}$ be a family with the Ramsey property. Then the product of finitely many $\mathcal{F}^*$-independent t.d.s. remains $\mathcal{F}^*$-independent.

In [52, page 278] Weiss constructed two weakly mixing t.d.s. whose product is not transitive. (Weiss's example was only stated to be $\mathbb{Z}$-weakly mixing, but is easily checked to be $\mathbb{Z}_+$-weakly mixing.) In view of Theorem 5.1, this implies that the product of $\mathcal{F}_{\rm inf}$-independent ($\mathcal{F}_{\rm ip}$-independent resp.) t.d.s. may fail to be $\mathcal{F}_{\rm inf}$-independent ($\mathcal{F}_{\rm ip}$-independent resp.).

6. Classes of measurable $\mathcal{F}$-independence

6.1. **General discussion.** In this subsection we characterize $\mathcal{F}_{\rm inf}$-(resp. $\mathcal{F}_{\rm ip}$, $\mathcal{F}_{\rm pubd}$) independent m.d.s. in Theorems 6.1 and 6.2.

Recall that a m.d.s (X, T) is said to be *ergodic* if for any $A, B \in \mathcal{B}$ with positive measures, $N(A, B)$ is nonempty; it is called *weakly mixing* if $T \times T$ is ergodic. Similar to the topological case (Theorem 5.1) we have

Theorem 6.1. For a m.d.s. $(X, \mathcal{B}, \mu, T)$ the following are equivalent:

(1) $(X, \mathcal{B}, \mu, T)$ is weakly mixing.
(2) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\rm inf}$-independent of order 2.
(3) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\rm inf}$-independent.
(4) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\rm ip}$-independent of order 2.
(5) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\rm ip}$-independent.

Proof. It is clear that (5)⇒(4)⇒(2) and (5)⇒(3)⇒(2). The implication (2)⇒(1) follows from the fact that if for any $A, B \in \mathcal{B}$ with positive measures one has $N(A, B) \cap N(A, A) \neq \emptyset$, then $(X, \mathcal{B}, \mu, T)$ is weakly mixing [13, Theorem 4.31]. ([13, Theorem 4.31] was proved only for invertible m.d.s., as it depended on [13, Theorem 4.30] which in turn was only proved for invertible m.d.s.; [39, Theorem 3.4] proved [13, Theorem 4.30] for any m.d.s., thus [13, Theorem 4.31] also holds for any m.d.s..)

When T is weakly mixing, so is $T \times \cdots \times T$ [51, Theorem 1.24]. Thus the proof of (1)⇒(5) in Theorem 5.1 also applies here. □

It was proved in [32, Theorem 8.3] and [35, Theorem 3.16] that a t.d.s. is topological K if and only if its every finite cover by non-dense open subsets has positive entropy. Moreover, it is shown in [32, Theorem 9.4] that there exists t.d.s. which is $\mathcal{F}_{\rm pubd}$-independent of order 2 but is not $\mathcal{F}_{\rm pubd}$-independent of order 3. Now we show that in the measurable setup the situation is different.

We refer the reader to [39, Chapter 4] for the basics of the entropy theory. A m.d.s. $(X, \mathcal{B}, \mu, T)$ is said to have *completely positive entropy* if for every non-trivial countable measurable partition α of X with $0 < H(\alpha) < \infty$ one has $h_\mu(T, \alpha) > 0$. The Rohlin-Sinai theorem says that an invertible m.d.s. has completely positive entropy if and only if it is an K-automorphism [43] [39, Theorem 4.12].

For $a \geq 2$ let $\Omega_a = \{0, 1, \dots, a-1\}^{\mathbb{Z}}$ and $Y \subseteq \Omega_a$. A subset $I \subseteq \mathbb{Z}$ is called an *interpolating set for* Y if $Y|_I = \Omega_a|_I$. Now suppose that $(X, \mathcal{B}, \mu, T)$ is an invertible m.d.s. and that $\mathcal{P} = \{P_0, P_1, \dots, P_{a-1}\}$ is a finite measurable partition of X. Construct a set $Y_{\mathcal{P}} \subseteq \Omega_a$ as follows:

$$Y_{\mathcal{P}} = \{\omega \in \Omega_a : \text{for all nonempty finite subsets } J \subseteq \mathbb{Z},\ \mu(\bigcap_{j \in J} T^{-j} P_{\omega_j}) > 0\}.$$

Glasner and Weiss showed that an invertible m.d.s. $(X, \mathcal{B}, \mu, T)$ has completely positive entropy if and only if for every finite measurable partition $\mathcal{P} = \{P_0, P_1, \dots, P_{a-1}\}$ of X with $\min_{0 \le j \le a-1} \mu(P_j) > 0$ the set $Y_{\mathcal{P}}$ has interpolating sets of positive density. In our terminology, clearly interpolating sets of $\mathcal{P}$ are exactly the independence sets of the tuple $(P_0, P_1, \dots, P_{a-1})$. Now we extend the result of Glasner and Weiss to general m.d.s..

Theorem 6.2. Let $(X, \mathcal{B}, \mu, T)$ be a m.d.s.. Then the following are equivalent:

(1) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\text{pubd}}$-independent.
(2) $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\text{pubd}}$-independent of order 2.
(3) $(X, \mathcal{B}, \mu, T)$ has completely positive entropy.

To prove Theorem 6.2, we need some preparation. For a Lebesgue space $(X, \mathcal{B}, \mu)$ and a measurable partition α of X, we denote by $\hat{\alpha}$ the σ-algebra generated by the items of α; for a family $\{\mathcal{B}_j\}_{j\in J}$ of sub-σ-algebras of $\mathcal{B}$, we denote by $\bigvee_{j\in J} \mathcal{B}_j$ the sub-σ-algebra of $\mathcal{B}$ generated by $\bigcup_{j\in J} \mathcal{B}_j$. For a m.d.s. $(X, \mathcal{B}, \mu, T)$, a measurable partition α of X, and $0 \le n \le m \le \infty$, we denote $\bigvee_{j=n}^m T^{-j}\alpha$ and $\bigvee_{j=n}^m T^{-j}\hat{\alpha}$ by α_n^m and $\hat{\alpha}_n^m$ respectively. The following lemma is [39, Lemma 4.6] for non-invertible m.d.s..

Lemma 6.3. Let $(X, \mathcal{B}, \mu, T)$ be a m.d.s., and let α and β be countable measurable partitions of X with $H(\alpha), H(\beta) < \infty$. If $\beta \le \alpha$ or $\alpha \le \beta$, then $\lim_{n\to+\infty} \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) = H(\alpha|\hat{\alpha}_1^\infty)$.

Proof. We follow the proof of [39, Lemma 4.6]. Consider first the case $\beta \le \alpha$. The sequence of σ-algebras $\{\hat{\alpha}_1^n \vee \hat{\beta}_{n+1}^\infty\}_{n\in\mathbb{N}}$ is increasing and their union generates the σ-algebra $\hat{\alpha}_1^\infty$. By the increasing Martingale theorem [39, Theorem 2.6] one has

$$\lim_{n\to+\infty} H(\alpha|\hat{\alpha}_1^n \vee \hat{\beta}_{n+1}^\infty) = H(\alpha|\hat{\alpha}_1^\infty).$$

Since

$$H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) = \sum_{i=0}^{n-1} H(T^{-i}\alpha|\hat{\alpha}_{i+1}^{n-1} \vee \hat{\beta}_n^\infty) = \sum_{i=0}^{n-1} H(\alpha|\hat{\alpha}_1^{n-1-i} \vee \hat{\beta}_{n-i}^\infty) = \sum_{i=0}^{n-1} H(\alpha|\hat{\alpha}_1^i \vee \hat{\beta}_{i+1}^\infty),$$

we conclude that $\lim_{n\to+\infty} \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) = H(\alpha|\hat{\alpha}_1^\infty)$.

Next we consider the case $\alpha \le \beta$. One has

$$\limsup_{n\to+\infty} \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) \le \lim_{n\to+\infty} \frac{1}{n} H(\alpha_0^{n-1}|\hat{\alpha}_n^\infty) = H(\alpha|\hat{\alpha}_1^\infty),$$

where the second equality comes from the above paragraph. One also has

$$\frac{1}{n} H(\beta_0^{n-1}|\hat{\beta}_n^\infty) = \frac{1}{n} H(\beta_0^{n-1} \vee \alpha_0^{n-1}|\hat{\beta}_n^\infty) = \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) + \frac{1}{n} H(\beta_0^{n-1}|\hat{\alpha}_0^{n-1} \vee \hat{\beta}_n^\infty),$$

and

$$\frac{1}{n} H(\beta_0^{n-1}|\hat{\alpha}_n^\infty) = \frac{1}{n} H(\beta_0^{n-1} \vee \alpha_0^{n-1}|\hat{\alpha}_n^\infty) = \frac{1}{n} H(\alpha_0^{n-1}|\hat{\alpha}_n^\infty) + \frac{1}{n} H(\beta_0^{n-1}|\hat{\alpha}_0^\infty).$$

Since $H(\beta_0^{n-1}|\hat{\alpha}_0^{n-1} \vee \hat{\beta}_n^\infty) \le H(\beta_0^{n-1}|\hat{\alpha}_0^\infty)$, we get

$$\frac{1}{n} H(\beta_0^{n-1}|\hat{\beta}_n^\infty) - \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) \le \frac{1}{n} H(\beta_0^{n-1}|\hat{\alpha}_n^\infty) - \frac{1}{n} H(\alpha_0^{n-1}|\hat{\alpha}_n^\infty).$$

Taking $\limsup$ on both sides, by the above paragraph we get

$$H(\beta|\hat{\beta}_1^\infty) - \liminf_{n\to+\infty} \frac{1}{n} H(\alpha_0^{n-1}|\hat{\beta}_n^\infty) \le H(\beta|\hat{\beta}_1^\infty) - H(\alpha|\hat{\alpha}_1^\infty).$$

That is, $\liminf_{n\to+\infty}\frac{1}{n}H(\alpha_0^{n-1}|\hat{\beta}_n^\infty)\geq H(\alpha|\hat{\alpha}_1^\infty)$. Therefore $\lim_{n\to+\infty}\frac{1}{n}H(\alpha_0^{n-1}|\hat{\beta}_n^\infty)=H(\alpha|\hat{\alpha}_1^\infty)$ as desired. □

For a m.d.s. $(X,\mathcal{B},\mu,T)$, denote by $\mathcal{P}(T)$ the *Pinsker σ-algebra* of T [51, page 113], consisting of $A\in\mathcal{B}$ such that $h_\mu(T,\{A,X\setminus A\})=0$. For a Lebesgue space $(X,\mathcal{B},\mu)$ and sub-σ-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$, we write $\mathcal{B}_1\subseteq_\mu\mathcal{B}_2$ if for every $A_1\in\mathcal{B}_1$ we can find $A_2\in\mathcal{B}_2$ with $\mu(A_1\Delta A_2)=0$; we write $\mathcal{B}_1=_\mu\mathcal{B}_2$ if $\mathcal{B}_1\subseteq_\mu\mathcal{B}_2$ and $\mathcal{B}_2\subseteq_\mu\mathcal{B}_1$. The next theorem appeared in [42, 12.3]. For the convenience of the reader, we give a proof here.

Theorem 6.4. Let $(X,\mathcal{B},\mu,T)$ be a m.d.s.. Then $\mathcal{P}(T)=_\mu\bigvee_\alpha\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty$ for α running over countable measurable partitions of X with $H(\alpha)<\infty$.

Proof. Let α,β be countable measurable partitions of X with $H(\alpha),H(\beta)<\infty$ and $\hat{\beta}\subseteq\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty$. For any $m\leq n$ in $\mathbb{Z}_+$, one has $T^{-m}\hat{\beta}\subseteq\hat{\alpha}_n^\infty$. Thus $\frac{1}{n}H(\beta_0^{n-1}|\hat{\alpha}_n^\infty\vee\hat{\beta}_n^\infty)=0$ for every $n\in\mathbb{N}$. Taking limit, by Lemma 6.3 we get $H(\beta|\hat{\beta}_1^\infty)=0$. That is, $h_\mu(T,\beta)=0$. Thus $\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty\subseteq\mathcal{P}(T)$.

Conversely, let $A\in\mathcal{P}(T)$. Set $\alpha=\{A,X\setminus A\}$. Then $0=h_\mu(T,\alpha)=H(\alpha|\alpha_1^\infty)$. Thus $\hat{\alpha}\subseteq_\mu\hat{\alpha}_1^\infty$ [14, Proposition 14.18.1]. It follows that $T^{-n}\hat{\alpha}\subseteq_\mu\hat{\alpha}_{n+1}^\infty$ and hence $\hat{\alpha}_n^\infty\subseteq_\mu\hat{\alpha}_{n+1}^\infty$ for every $n\in\mathbb{Z}_+$. Then for each $n\in\mathbb{N}$ we can find $A_n\in\hat{\alpha}_n^\infty$ with $\mu(A\Delta A_n)=0$. Note that $\bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n}A_m\in\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty$ and $\mu(A\Delta(\bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n}A_m))=0$. Therefore $\mathcal{P}(T)\subseteq_\mu\bigvee_\alpha\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty$ for α running over countable measurable partitions of X with $H(\alpha)<\infty$. □

The next result appeared implicitly in [42, 13.2]. For completeness, we give a proof here.

Theorem 6.5. Let $(X,\mathcal{B},\mu,T)$ be a m.d.s.. Then the following are equivalent:

(1) $(X,\mathcal{B},\mu,T)$ has completely positive entropy.
(2) For every countable measurable partition α of X with $H(\alpha)<\infty$, one has $\lim_{n\to+\infty}h_\mu(T^n,\alpha)=H(\alpha)$.

Proof. (1)⇒(2): Let α be a countable measurable partition of X with $H(\alpha)<\infty$. For each $n\in\mathbb{N}$ one has $h_\mu(T^n,\alpha)=H(\alpha|\bigvee_{j=1}^\infty T^{-jn}\hat{\alpha})\geq H(\alpha|\hat{\alpha}_n^\infty)$. By the decreasing Martingale theorem [14, Theorem 14.28] we have $\lim_{n\to+\infty}H(\alpha|\hat{\alpha}_n^\infty)=H(\alpha|\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty)$. Since T has completely positive entropy, $\mathcal{P}(T)$ is exactly the σ-algebra of measurable subsets of X with measure 0 or 1. Thus $H(\alpha|\bigcap_{n\in\mathbb{Z}_+}\hat{\alpha}_n^\infty)=H(\alpha)$ by Theorem 6.4. Therefore $\liminf_{n\to+\infty}h_\mu(T^n,\alpha)\geq H(\alpha)$. On the other hand, for each $n\in\mathbb{N}$ one has $h_\mu(T^n,\alpha)\leq H(\alpha)$. Thus $\lim_{n\to+\infty}h_\mu(T^n,\alpha)=H(\alpha)$.

(2)⇒(1): Let α be a countable measurable partition of X with $0<H(\alpha)<\infty$. For each $n\in\mathbb{N}$ one has $h_\mu(T,\alpha)\geq\frac{1}{n}h_\mu(T^n,\alpha)$. Since $\lim_{n\to+\infty}h_\mu(T^n,\alpha)=H(\alpha)>0$, we conclude that $h_\mu(T,\alpha)>0$. □

Lemma 6.6. A non-trivial m.d.s. being $\mathcal{F}_{\rm pubd}$-independent of order 2 has positive entropy.

Proof. Assume that $(X,\mathcal{B},\mu,T)$ is a non-trivial m.p.s. being $\mathcal{F}_{\rm pubd}$-independent of order 2 and has entropy 0. Clearly $(X,\mathcal{B},\mu,T)$ is ergodic. By Rosenthal's extension of the Jewett-Krieger theorem to non-invertible m.d.s. [44], there exists a t.d.s. $(\widehat{X},\widehat{T})$ with a unique invariant Borel probability measure $\widehat{\mu}$ such that $\widehat{\mu}$ has full support and the m.d.s. $(X, \mathcal{B}, \mu, T)$ and $(\widehat{X}, \mathcal{B}_{\widehat{X}}, \widehat{\mu}, \widehat{T})$ are isomorphic, where $\mathcal{B}_{\widehat{X}}$ denotes the Borel σ-algebra of $\widehat{X}$, in the sense that there are $X_0 \in \mathcal{B}$, $\widehat{X}_0 \in \mathcal{B}_{\widehat{X}}$ and a measure-preserving bijection $\phi : X_0 \to \widehat{X}_0$ with $\mu(X_0) = \widehat{\mu}(\widehat{X}_0) = 1$, $TX_0 \subseteq X_0$, $\widehat{T}\widehat{X}_0 \subseteq \widehat{X}_0$, and $\phi \circ T = \widehat{T} \circ \phi$. By the variational principle [51, Theorem 8.6], one has

$$h_{\mathrm{top}}(\widehat{T}) = h_{\widehat{\mu}}(\widehat{T}) = h_\mu(T) = 0.$$

Since $(X, \mathcal{B}, \mu, T)$ is non-trivial, $(\widehat{X}, \widehat{T})$ is a non-trivial t.d.s.. Thus we can find two disjoint closed subsets A, B of $\widehat{X}$ with $\widehat{\mu}(A) > 0$, $\widehat{\mu}(B) > 0$. Set $\mathcal{U} = \{\widehat{X} \setminus A, \widehat{X} \setminus B\}$. Then $\mathcal{U}$ is an open cover of $\widehat{X}$, and for any $F \in \mathrm{Ind}(A, B)$ we have

$$\begin{aligned}
0 = h_{\mathrm{top}}(\widehat{T}) \geq h_{\mathrm{top}}(\widehat{T}, \mathcal{U}) &= \lim_{n\to+\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} \widehat{T}^{-i}\mathcal{U}) \\
&\geq \limsup_{n\to+\infty} \frac{1}{n} \log N(\bigvee_{i\in F\cap\{0,1,\dots,n-1\}} \widehat{T}^{-i}\mathcal{U}) \\
&= \limsup_{n\to+\infty} \frac{1}{n} \log 2^{|F\cap\{0,1,\dots,n-1\}|} = \overline{d}(F) \log 2.
\end{aligned}$$

Hence $\overline{d}(F) = 0$. Thus $\mathrm{Ind}(A, B) \cap \mathcal{F}_{\mathrm{pud}} = \emptyset$. It follows from Example 2.2 and Proposition 3.7 that $\mathrm{Ind}(A, B) \cap \mathcal{F}_{\mathrm{pubd}} = \emptyset$. Thus $(\widehat{X}, \mathcal{B}_{\widehat{X}}, \widehat{\mu}, \widehat{T})$ is not $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2. Then by Remark 4.4 $(X, \mathcal{B}, \mu, T)$ is not $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2 either. ∎

We shall need the following consequence of Karpovsky and Milman's generalization of the Sauer-Perles-Shelah lemma [33, 48, 49].

Lemma 6.7. ([33]). Given $r \geq 2$ in $\mathbb{N}$ and $\lambda > \ln(r-1)$ there exists a constant $c > 0$ depending only on r and λ such that, for all $n \in \mathbb{N}$ and $S \subseteq \{1, 2, \dots, r\}^{\{0,1,\dots,n-1\}}$ satisfying $|S| \geq e^{\lambda n}$ there is an $I \subseteq \{0, 1, 2, \dots, n-1\}$ with $|I| \geq cn$ and $S|_I = \{1, 2, \dots, r\}^I$.

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. When $(X, \mathcal{B}, \mu, T)$ is a trivial system, this is obvious. So we suppose that $(X, \mathcal{B}, \mu, T)$ is non-trivial. (1)⇒(2) is obvious.

(3)⇒(1): We claim first that $(X, \mathcal{B}, \mu)$ is *non-atomic* in the sense that $\mu(\{x\}) = 0$ for every $x \in X$. In fact, since T has completely positive entropy, it is ergodic. If $\mu(\{x\}) > 0$ for some $x \in X$, then we can find some $n \in \mathbb{N}$ such that $x, Tx, \dots, T^{n-1}x$ are pairwise distinct, $T^n x = x$, and $\mu(x) = \mu(Tx) = \dots = \mu(T^{n-1}x) = \frac{1}{n}$. If $n > 1$, denoting by β the partition of X into $\{x\}$ and its complement, we have $h_\mu(T, \beta) = 0$. Thus $n = 1$, which means that $(X, \mathcal{B}, \mu, T)$ is trivial. Therefore $(X, \mathcal{B}, \mu)$ is non-atomic.

Given a tuple $(A_1, \dots, A_k)$ of sets in $\mathcal{B}$ with positive measures, we are going to show that $\mathrm{Ind}(A_1, \dots, A_k) \cap \mathcal{F}_{\mathrm{pubd}} \neq \emptyset$. Without loss of generality, we may assume that $A_1, \dots, A_k$ are pairwise disjoint.

Every non-atomic Lebesgue space is isomorphic to the closed unit interval endowed with its Borel σ-algebra and the Lebesgue measure [34, Theorem 17.41]. It follows that there exist a $r \in \mathbb{N}$ and a measurable partition $\alpha = \{B_1, \dots, B_r\}$ of X such that

$r > k$, $\mu(B_i) = \frac{1}{r}$ for $i = 1, 2, \dots, r$, and B_j is a subset of A_j for $j = 1, 2, \dots, k$. To show $\mathrm{Ind}(A_1, \dots, A_k) \cap \mathcal{F}_{\mathrm{pubd}} \neq \emptyset$, it is sufficient to show $\mathrm{Ind}(B_1, \dots, B_r) \cap \mathcal{F}_{\mathrm{pubd}} \neq \emptyset$.

By Theorem 6.5 we have $\lim_{n \to +\infty} h_\mu(T^n, \alpha) = H(\alpha) = \ln r$. Thus there exists $\ell \in \mathbb{N}$ such that $\lambda := h_\mu(T^\ell, \alpha) > \ln(r-1)$. Then $\frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-\ell i}\alpha) \geq \lambda > \ln(r-1)$ for all $n \in \mathbb{N}$. For any given finite measurable partition β of X, we define

$$|\beta|_\mu = |\{B \in \beta : \mu(\beta) > 0\}|.$$

Then $|\bigvee_{i=0}^{n-1} T^{-\ell i}\alpha|_\mu \geq e^{H(\bigvee_{i=0}^{n-1} T^{-\ell i}\alpha)} \geq e^{\lambda n}$ for all $n \in \mathbb{N}$.

Now combing this with Lemma 6.7, we see that there exists a constants $c > 0$ depending on only r and λ such that, for any $n \in \mathbb{N}$ there is an $I_n \subseteq \{0, 1, 2, \dots, n-1\}$ with $|I_n| \geq cn$ and $\mu(\bigcap_{i \in I_n} T^{-\ell i} B_{s(i)}) > 0$ for any $s \in \{1, 2, \dots, r\}^{I_n}$. This implies that $\ell I_n \in \mathrm{Ind}(B_1, \dots, B_r)$ for each $n \in \mathbb{N}$. From Lemma 4.5 we conclude that $\mathrm{Ind}(B_1, \dots, B_r) \cap \mathcal{F}_{\mathrm{pd}} \neq \emptyset$.

(2)$\Rightarrow$(3): Assume that $(X, \mathcal{B}, \mu, T)$ is $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2. Note that the definitions of independence sets and entropy apply to more general measure-theoretical dynamical systems in which the probability space does not have to be a Lebesgue space. In this sense $(X, \mathcal{P}(T), \mu, T)$ is also $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2 and has entropy 0. Since $(X, \mathcal{B}, \mu)$ is a Lebesgue space, it is easy to see that $\mathcal{B}$ is separable under the semi-metric $d(A, B) = \mu(A \Delta B)$. Then $\mathcal{P}(T)$ is also separable under this semi-metric. It follows that there is a m.d.s. $(Y, \mathcal{J}, \nu, S)$ (i.e., $(Y, \mathcal{J}, \nu)$ is a Lebesgue space) such that the measure algebra triples associated to $(X, \mathcal{P}(T), \mu, T)$ and $(Y, \mathcal{J}, \nu, S)$ in Remark 4.4 are isomorphic [13, Proposition 5.3]. Then $(Y, \mathcal{J}, \nu, S)$ is also $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2 and has entropy 0. By Lemma 6.6 $(Y, \mathcal{J}, \nu, S)$ is trivial. Thus $\mathcal{P}(T)$ consists of measurable subsets of X with measure 0 or 1. That is, $(X, \mathcal{B}, \mu, T)$ has completely positive entropy. $\square$

6.2. **Non-existence of $\mathcal{F}_{\mathrm{s}}$-independent m.d.s.** It is somewhat surprising that there is no non-trivial m.d.s. which is $\mathcal{F}_{\mathrm{s}}$-independent. In the following, we aim to show that for any non-periodic m.d.s. $(X, \mathcal{B}, \mu, T)$, there exists $A \in \mathcal{B}$ with $\mu(A) > 0$ such that $\mathrm{Ind}(A)$ does not contain a syndetic set.

A m.d.s. $(X, \mathcal{B}, \mu, T)$ is called *non-periodic* or *free* if $\mu(\{x \in X : T^n x = x\}) = 0$ for every $n \in \mathbb{N}$. It is easy to see that an ergodic m.d.s. $(X, \mathcal{B}, \mu, T)$ is non-periodic if and only if $(X, \mathcal{B}, \mu)$ is *non-atomic* in the sense that $\mu(x) = 0$ for every $x \in X$.

Theorem 6.8. Let $(X, \mathcal{B}, \mu, T)$ be a non-periodic m.d.s.. Then for any $\varepsilon > 0$ there exists $A \in \mathcal{B}$ with $\mu(A) > 1 - \varepsilon$ such that $\mathrm{Ind}(A)$ does not contain any syndetic set.

Proof. Endow X with a Polish topology such that $\mathcal{B}$ is the corresponding Borel σ-algebra. Replacing the Polish topology on X by a finer one if necessary [34, Theorem 13.11 and Lemma 13.3], we may assume that T is continuous. Let $\varepsilon > 0$. We claim that there is a compact subset K of X such that $\mu(K) > 1 - \varepsilon$ and $A_n := \bigcap_{j \in \mathbb{Z}_+} \bigcup_{i=0}^{n-1} T^{-j-i} K$ has measure 0 for every $n \in \mathbb{N}$. Assuming this claim let us show how it implies the theorem.

Since $\mu(\bigcup_{n \in \mathbb{N}} A_n) = 0$, one has $\mu(K \setminus (\bigcup_{n \in \mathbb{N}} A_n)) = \mu(K) > 1 - \varepsilon$. By the regularity of μ [34, Theorem 17.11], we can find a compact set A contained in $K \setminus (\bigcup_{n \in \mathbb{N}} A_n)$ such that $\mu(A) > 1 - \varepsilon$. We shall show that $\mathrm{Ind}(A)$ does not contain any syndetic set.

Let $F \in \mathrm{Ind}(A)$ be nonempty. Replacing F by $F - \min F$ if necessary, we may assume that $0 \in F$. One has $\mu(\bigcap_{j\in J} T^{-j}A) > 0$ and hence $\bigcap_{j\in J} T^{-j}A \neq \emptyset$ for every nonempty finite subset J of F. Since A is compact, we conclude that $\bigcap_{j\in F} T^{-j}A$ is nonempty. Take $x \in \bigcap_{j\in F} T^{-j}A$. Then $x \in A$ and $T^j x \in A \subseteq K$ for every $j \in F$. For each $n \in \mathbb{N}$ one has $x \notin A_n$, and hence for some $j_n \in \mathbb{Z}_+$ none of $T^{j_n}x, T^{j_n+1}x, \dots, T^{j_n+n-1}x$ is in K. Then $[j_n, j_n + n - 1] \cap F = \emptyset$. Therefore F is not syndetic.

We are left to prove the above claim. Since the main idea of the proof is well illustrated in the case μ is ergodic, we consider this case first.

So assume that μ is ergodic. Since $(X, \mathcal{B}, \mu, T)$ is non-periodic, by the comment before Theorem 6.8, $(X, \mathcal{B}, \mu)$ is non-atomic. Replacing X by $\mathrm{supp}(\mu)$ if necessary, we may assume that μ has full support. Take $x \in X$ and set $W = \{T^n x : n \in \mathbb{Z}_+\}$. Then $TW \subseteq W$, and W is nonempty and countable. Since μ is non-atomic, one has $\mu(W) = 0$, and hence $\mu(X \setminus W) = 1$. By the regularity of μ, we can find a compact set K contained in $X \setminus W$ such that $\mu(K) > 1 - \varepsilon$. For any $n \in \mathbb{N}$, $\bigcup_{i=0}^n T^{-i}K$ is a closed subset of X with $\bigcup_{i=0}^n T^{-i}K \neq X$, since $W \cap (\bigcup_{i=0}^n T^{-i}K) = \emptyset$. As μ has full support, $\mu(\bigcup_{i=0}^n T^{-i}K) < 1$ for all $n \in \mathbb{N}$. Note that $A_n \in \mathcal{B}$ and $T^{-1}A_n \supseteq A_n$. Since μ is ergodic and $\mu(A_n) \leq \mu(\bigcup_{i=0}^n T^{-i}K) < 1$, we get $\mu(A_n) = 0$. This finishes the proof in the case μ is ergodic.

Now we consider the general case, using the ergodic decomposition of $(X, \mathcal{B}, \mu, T)$.

Denote by $P(X)$ the set of all probability Borel measures on X, and endow it with the σ-algebra generated by the functions $\mu' \mapsto \mu'(A)$ on $P(X)$ for all $A \in \mathcal{B}$ [34, Section 17.E].

From the ergodic decomposition of $(X, \mathcal{B}, \mu, T)$ we know that there exist a set $X' \in \mathcal{B}$ with $\mu(X') = 1$ and $TX' \subseteq X'$, a Lebesgue space $(Y, \mathcal{J}, \nu)$, a measurable map $\pi : X' \to Y$, a measurable map $y \mapsto \mu_y$ from Y to $P(X)$, and a set $Y' \in \mathcal{J}$ with $\nu(Y') = 1$ such that $\pi T = \pi$, $\pi\mu = \nu$, $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$ for all $A \in \mathcal{B}$, and $\mu_y(\pi^{-1}(y)) = 1$ and $T\mu_y = \mu_y$ and $(X, \mathcal{B}, \mu_y, T)$ is ergodic for every $y \in Y'$ [14, Theorem 3.42]. ([14, Theorem 3.42] was only proved for invertible m.d.s., but it is easy to see that the proof works for any m.d.s..)

Set $W_1 = \{x \in X : T^n x = x \text{ for some } n \in \mathbb{N}\}$. Clearly W_1 is in $\mathcal{B}$. By assumption $0 = \mu(W_1) = \int_Y \mu_y(W_1)\, d\nu(y)$. Thus $\mu_y(W_1) = 0$ for ν a.e. $y \in Y$. Replacing Y' by a smaller measurable set if necessary, we may assume that $\mu_y(W_1) = 0$ for every $y \in Y'$. Since $(X, \mathcal{B}, \mu_y, T)$ is ergodic for every $y \in Y'$, it follows that μ_y is non-atomic for every $y \in Y'$.

Endow Y with a Polish topology such that $\mathcal{J}$ is the corresponding Borel σ-algebra. Replacing the Polish topology on X by a finer one if necessary, we may assume that π is continuous.

Denote by $F(X)$ the set of all closed subsets of X, and endow it with the Effros Borel structure, i.e., the σ-algebra generated by the sets $\{Z \in F(X) : Z \cap U \neq \emptyset\}$ for all open subsets U of X. The map $\phi : P(X) \to F(X)$ sending each μ' to $\mathrm{supp}(\mu')$ is measurable [34, Exercise 17.38]. By the Kuratowski-Ryll-Nardzewski selection theorem [34, Theorem 12.13] we can find a measurable map $\psi : F(X) \to X$ such that $\psi(Z) \in Z$ for each nonempty $Z \in F(X)$.

Note that $\mathrm{supp}(\mu_y) \subseteq \pi^{-1}(y)$ for every $y \in Y'$. Thus the map $\varphi_n : Y' \to X$ sending y to $T^n(\psi(\phi(\mu_y)))$ is measurable and injective for each $n \in \mathbb{Z}_+$. Recall that a measurable space is a *standard Borel space* if the σ-algebra is the Borel σ-algebra for some Polish topology on the set. A measurable subset of a standard Borel space together with the restriction of the σ-algebra to the subset is also a standard Borel space [34, Corollary 13.4]. Thus Y' together with the restriction of $\mathcal{J}$ on Y' is a standard Borel space. The Lusin-Souslin theorem says that the image of any injective measurable map from a standard Borel space to another standard Borel space is measurable [34, Corollary 15.2]. Thus the set $W := \bigcup_{n\in\mathbb{Z}_+} \varphi_n(Y')$ is in $\mathcal{B}$. Note that $TW \subseteq W$, and $W \cap \mathrm{supp}(\mu_y)$ is nonempty and countable for every $y \in Y'$.

Since μ_y is non-atomic for every $y \in Y'$, one has $\mu_y(W) = 0$ for every $y \in Y'$. Thus $\mu(W) = \int_Y \mu_y(W)\, d\nu(y) = 0$, and hence $\mu(X \setminus W) = 1$. By the regularity of μ, we can find a compact set K contained in $X \setminus W$ such that $\mu(K) > 1 - \varepsilon$. For any $n \in \mathbb{N}$ and $y \in Y'$, $\mathrm{supp}(\mu_y) \cap (\bigcup_{i=0}^n T^{-i}K)$ is a closed subset of $\mathrm{supp}(\mu_y)$ with $\mathrm{supp}(\mu_y)\cap(\bigcup_{i=0}^n T^{-i}K) \neq \mathrm{supp}(\mu_y)$, since $W\cap(\bigcup_{i=0}^n T^{-i}K) = \emptyset$ and $W\cap\mathrm{supp}(\mu_y) \neq \emptyset$. Thus $\mu_y(\bigcup_{i=0}^n T^{-i}K) < 1$ for all $n \in \mathbb{N}$ and $y \in Y'$.

We still have $A_n \in \mathcal{B}$ and $T^{-1}A_n \supseteq A_n$. For each $y \in Y'$, since $(X, \mathcal{B}, \mu_y, T)$ is ergodic, $\mu_y(A_n)$ is equal to either 0 or 1. By the above paragraph we have $\mu_y(A_n) \leq \mu_y(\bigcup_{i=0}^{n-1} T^{-i}K) < 1$ for each $y \in Y'$. Thus $\mu_y(A_n) = 0$ for each $y \in Y'$. Therefore $\mu(A_n) = \int_Y \mu_y(A_n)\, d\nu(y) = 0$, as desired. This proves the claim and finishes the proof of the theorem. □

Now we are able to show

Theorem 6.9. There is no non-trivial m.d.s. which is $\mathcal{F}_{\mathrm{s}}$-independent of order 2.

Proof. Assume the contrary that there exists such a system $(X, \mathcal{B}, \mu, T)$. By Theorem 6.1, T is weakly mixing.

By Theorem 6.8, T is a.e. periodic. Then the set $A_n = \{x \in X : T^n x = x, T^j x \neq x \text{ for all } 1 \leq j < n\}$ has positive measure for some $n \in \mathbb{N}$. Note that $TA_n = A_n$. By [24, page 70] we can find $B \subseteq A_n$ such that $B \in \mathcal{B}$, $\mu(B) = \mu(A_n)/n$, and $B, TB, \dots, T^{n-1}B$ are pairwise disjoint. If $n \geq 2$, then $N(B, B) \cap N(B, TB) = \emptyset$, contradicts that $(X, \mathcal{B}, \mu, T)$ is weakly mixing. Thus $\mu(A_n) = 0$ for every $n \geq 2$. Then $\mu(A_1) = 1$. Since $(X, \mathcal{B}, \mu, T)$ is non-trivial, we can find some $B \subseteq A_1$ such that $B \in \mathcal{B}$ and $0 < \mu(B) < 1$. Then $N(B, X \setminus B) = \emptyset$, again contradicting that $(X, \mathcal{B}, \mu, T)$ is weakly mixing. □

Remark 6.10. Using Theorem 5.11 one can strengthen Theorem 6.9 as follows. For any nontrivial ergodic m.d.s. $(X, \mathcal{B}, \mu, T)$, Rosenthal's extension of the Jewett-Krieger theorem to non-invertible m.d.s. [44] says that there exists a t.d.s. $(\widehat{X}, \widehat{T})$ with a unique invariant Borel probability measure $\widehat{\mu}$ such that $\widehat{\mu}$ has full support and the m.d.s. $(X, \mathcal{B}, \mu, T)$ and $(\widehat{X}, \mathcal{B}_{\widehat{X}}, \widehat{\mu}, \widehat{T})$ are isomorphic, where $\mathcal{B}_{\widehat{X}}$ denotes the Borel σ-algebra of $\widehat{X}$. Then $(\widehat{X}, \widehat{T})$ is minimal (see for example [51, Theorem 6.17]). Furthermore, the proof in [44] shows that we can choose $\widehat{X}$ to be a Cantor set. For any real-valued continuous function f on X, the sequence $\{\frac{1}{n+1}\sum_{i=0}^n f \circ \widehat{T}^i\}_{n\in\mathbb{Z}_+}$ of functions on X converges to the constant function $\int_{\widehat{X}} f(x)\, d\widehat{\mu}(x)$ uniformly as $n \to +\infty$ [51, Theorem 6.19]. By Theorem 5.11 we can find disjoint nonempty clopen subsets $\widehat{V}_0$ and $\widehat{V}_1$ of $\widehat{X}$ such that $\mathrm{Ind}(\widehat{V}_0, \widehat{V}_1) \cap \mathcal{F}_{\mathrm{s}} = \emptyset$. Say, $\widehat{V}_j$ corresponds to $V_j \in \mathcal{B}$ for $j = 1, 2$. Then V_0 and V_1 are disjoint and have positive measures, and

$\mathrm{Ind}(V_0, V_1) \cap \mathcal{F}_{\mathrm{s}} = \emptyset$. Furthermore, taking f to be $1_{\widehat{V}_j}$, we see that the sequence $\{\frac{1}{n+1}\sum_{i=0}^{n} 1_{V_j} \circ T^i\}_{n\in\mathbb{Z}_+}$ converges to $\mu(V_j)$ in $L^\infty(X,\mu)$ for $j = 1, 2$.

6.3. **Finite product.** By contrast to the topological case, it is well known that the product of two weakly mixing m.d.s. is still weakly mixing [13, Proposition 4.6]. In view of Theorem 6.1, this means that the products of finitely many $\mathcal{F}_{\mathrm{inf}}$-independent ($\mathcal{F}_{\mathrm{ip}}$-independent resp.) m.d.s. are $\mathcal{F}_{\mathrm{inf}}$-independent ($\mathcal{F}_{\mathrm{ip}}$-independent resp.).

Meanwhile, it is known that the product of finitely many invertible completely positive entropy m.d.s. has completely positive entropy [39, Theorem 4.14]. As the topological case, every m.d.s. has a *natural extension* [10, Page 240], which is always invertible. The natural extension of a completely positive entropy m.d.s. has completely positive entropy [42, 13.8] (one can also deduce this from Theorem 6.5 and the fact that the natural extension of a m.d.s. is the inverse limit of a sequence of m.d.s. being identical to the original one). It follows that the product of finitely many completely positive entropy m.d.s. has completely positive entropy. In view of Theorem 6.2, this means that the product of finitely many $\mathcal{F}_{\mathrm{pd}}$-independent m.d.s. remains $\mathcal{F}_{\mathrm{pd}}$-independent. Thus we make the following conjecture.

Conjecture 6.11. For any family $\mathcal{F}$, the product of finitely many $\mathcal{F}$-independent m.d.s. remains $\mathcal{F}$-independent.

7. Topological proof of minimal topological K systems are strongly mixing

In this section we prove Theorem 7.1 and Corollary 7.3.

For a cover $\mathcal{V}$ of a compact space X by open subsets, we denote by $N(\mathcal{V})$ the minimal cardinality of subcovers of $\mathcal{V}$. Let $\mathcal{F}$ be a family. A t.d.s. (X,T) is called *$\mathcal{F}$-scattering* if for each $F = \{a_1 < a_2 < \ldots\} \in \mathcal{F}$ and each finite cover $\mathcal{U}$ of X by non-dense open subsets, one has $\lim_{n\to+\infty} N(\bigvee_{i=1}^{n} T^{-a_i}\mathcal{U}) = \infty$. It was shown in [29, Theorem 5.5] using ergodic theory that topological K systems are $\mathcal{F}_{\mathrm{inf}}$-scattering. Combining this with the fact that a minimal $\mathcal{F}_{\mathrm{inf}}$-scattering t.d.s. is strongly mixing [30, Theorem 5.6], one knows that a minimal topological K system is strongly mixing [29, Theorem 5.10]. Now we give a topological proof of the fact that a topological K system is $\mathcal{F}_{\mathrm{inf}}$-scattering.

Recall that for any $F = \{a_1 < a_2 < \ldots\} \in \mathcal{F}_{\mathrm{inf}}$ and any open cover $\mathcal{U}$ of X, the *topological sequence entropy* of T with respect to F is defined as

$$h^F_{\mathrm{top}}(T,\mathcal{U}) = \limsup_{n\to+\infty} \frac{\log N(\bigvee_{i=1}^{n} T^{-a_i}\mathcal{U})}{n}.$$

Theorem 7.1. Let (X,T) be a t.d.s., $n \geq 2$, $(x_1,\ldots,x_n)$ be an $\mathcal{F}_{\mathrm{pubd}}$-independent tuple of X with points pairwise distinct, and $U_1,\ldots,U_n$ be pairwise disjoint closed neighborhoods of $x_1,\ldots,x_n$ respectively. Set $\mathcal{U} = \{U_1^c,\ldots,U_n^c\}$. Then for any $F \in \mathcal{F}_{\mathrm{inf}}$, one has $h^F_{\mathrm{top}}(T,\mathcal{U}) > 0$. Consequently, a topological K system is $\mathcal{F}_{\mathrm{inf}}$-scattering.

Proof. Since $(x_1,\ldots,x_n)$ is an $\mathcal{F}_{\mathrm{pubd}}$-independent tuple, there exists an $S \in \mathrm{Ind}(U_1,\ldots,U_n)$ with positive upper Banach density d. Let $F = \{a_1 < a_2 < \ldots\}$ in $\mathcal{F}_{\mathrm{inf}}$. Then by Lemma 5.14 for any $k \in \mathbb{N}$, setting q_k to be the smallest integer no less than $2k/d$, we can find $p_k \in \mathbb{Z}$ and $W_k \subseteq \{a_1, a_2, \dots, a_{q_k}\}$ with $|W_k| = k$ and $p_k + W_k \subseteq S$. Thus, $W_k \in \mathrm{Ind}(U_1, \dots, U_n)$. This implies that

$$h^F_{\mathrm{top}}(T, \mathcal{U}) \geq \limsup_{k\to+\infty} \frac{1}{q_k} \log N(\bigvee_{j\in W_k} T^{-j}\mathcal{U}) \geq \limsup_{k\to+\infty} \frac{1}{q_k} \log(\frac{n}{n-1})^k = \frac{2}{d} \log \frac{n}{n-1} > 0.$$

Now for any finite open cover $\mathcal{V}$ of X by non-dense open subsets, we may find some $n \geq 2$, pairwise distinct $x_1, \dots, x_n$ in X and pairwise disjoint closed neighborhoods $U_1, \dots, U_n$ of $x_1, \dots, x_n$ respectively such that $\mathcal{V}$ refines $\mathcal{U} = \{U_1^c, \dots, U_n^c\}$. If (X, T) is topological K, then each tuple in X is $\mathcal{F}_{\mathrm{pubd}}$-independent. Thus for any $F = \{a_1 < a_2 < \dots\}$ in $\mathcal{F}_{\mathrm{inf}}$, by the above paragraph we have $h^F_{\mathrm{top}}(T, \mathcal{V}) \geq h^F_{\mathrm{top}}(T, \mathcal{U}) > 0$. This implies that $N(\bigvee_{i=1}^m T^{-a_i}\mathcal{V}) \to \infty$ as $m \to +\infty$, i.e., (X, T) is $\mathcal{F}_{\mathrm{inf}}$-scattering. □

Let $\mathcal{F}$ be a family. A t.d.s. (X, T) is called *$\mathcal{F}$-transitive* if for any nonempty open subsets U and V of X, one has $N(U, V) \in \mathcal{F}$; it is called *mildly mixing* if its product with any transitive t.d.s. is transitive. It was shown in [32, Theorem 7.5] that a u.p.e. system is mildly mixing. From [32, Theorem 7.3] or [35, Theorem 3.16] one knows that a t.d.s. is u.p.e. if and only if it is $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2. Denote by Δ the family in $\mathbb{Z}_+$ generated by the sets $F - F := \{a - b : a, b \in F, a - b > 0\}$ for all $F \in \mathcal{F}_{\mathrm{inf}}$. By [30, Theorem 6.6] every Δ^*-transitive system is mildly mixing. Now we strengthen the above result to show that every t.d.s. being $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2 is Δ^*-transitive. For this we need the following proposition, which appeared in [13, page 84] (see also [53, Proposition 2.3]) and also follows directly from Lemma 5.14.

Proposition 7.2. If $F \in \mathcal{F}_{\mathrm{pubd}}$, then $F - F$ is in Δ^*.

Corollary 7.3. A t.d.s. being $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2 is Δ^*-transitive.

Proof. Let (X, T) be $\mathcal{F}_{\mathrm{pubd}}$-independent of order 2. Then for any nonempty open subsets U and V of X, there exists an $F \in \mathrm{Ind}(U, V) \cap \mathcal{F}_{\mathrm{pubd}}$. Clearly $F - F \subseteq N(U, V)$. By Proposition 7.2 one has $N(U, V) \in \Delta^*$. □

To end the appendix we make the following remark. Denote by $\mathcal{F}_{\mathrm{ss}}$ the family consisting of $S \subseteq \mathbb{Z}_+$ satisfying that for each $F \in \mathcal{F}_{\mathrm{inf}}$ and each $k \in \mathbb{N}$ there exists $p_k \in \mathbb{Z}$ with $|F \cap (S + p_k)| \geq k$. It is clear that for any $S \in \mathcal{F}_{\mathrm{ss}}$ one has $S - S \in \Delta^*$.

Remark 7.4. One obvious corollary of Lemma 5.14 is that $\mathcal{F}_{\mathrm{pubd}} \subseteq \mathcal{F}_{\mathrm{ss}}$. We remark that there exists an $S \in \mathcal{F}_{\mathrm{ss}}$ containing no arithmetic progression of length 3 (and thus having zero upper Banach density by Roth's theorem [47]).

Proof. For $k \geq 3$ set

$$S_k = \{\{a_1, a_2, \dots, a_k\} \subseteq \mathbb{N} : a_j - a_i > a_i - a_s > 0 \text{ for all } 1 \leq s < i < j \leq k\}.$$

Each S_k is countable. Enumerate $\bigcup_{k\geq 3} S_k$ as $\{A_1, A_2, \dots\}$. Now let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ and set $S = \bigcup_{i\in\mathbb{N}}(A_i + t_i)$.

Now assume that F is an infinite subset of $\mathbb{Z}$. For each $k \geq 3$, inductively we can find $b_1, b_2, \dots, b_k \in F$ such that $b_j - b_i > b_i - b_s > 0$ for all $1 \leq s < i < j \leq k$. This implies that there exists $p_k \in \mathbb{Z}$ with $|F \cap (S + p_k)| \geq k$. Thus S is in $\mathcal{F}_{\mathrm{ss}}$.

If we choose t_i to grow rapidly enough, it is easy to check that S does not contain any arithmetic progression of length 3. □

We remak that the set of prime numbers is not in $\mathcal{F}_{\rm ss}$. In fact any $S = \{a_1 < a_2 < \dots\} \in \mathcal{F}_{\rm inf}$ with $a_{i+1} - a_i \to +\infty$ as $i \to +\infty$ is not in $\mathcal{F}_{\rm ss}$. Actually one can find an $F \in \mathcal{F}_{\rm inf}$ such that $|F \cap (S+p)| \le 2$ for all $p \in \mathbb{Z}$. To find such an F, start with any $b_1 < b_2$ in $\mathbb{N}$. Since $a_{i+1} - a_i \to +\infty$ as $i \to +\infty$, there are only finitely many i and j satisfying $a_j - a_i = b_2 - b_1$. From this we can find $b_3 > b_2$ in $\mathbb{N}$ such that $|\{b_1, b_2, b_3\} \cap (S+p)| \le 2$ for every $p \in \mathbb{Z}$. Inductively we find $b_3 < b_4 < b_5 < \dots$ in $\mathbb{N}$ such that $|\{b_1, b_2, \dots, b_k\} \cap (S+p)| \le 2$ for every $k \in \mathbb{N}$ and $p \in \mathbb{Z}$.

8. Appendix

In this appendix we prove Theorem 5.12.

Proof of Theorem 5.12. Take $d \in \mathbb{N}$ with $p^d > \frac{(d+p)\ell}{p}$. Let $m > d$ be large enough, which we shall determine later. It suffices to show that for every $n \in \mathbb{N}$ there exists $x_n \in \Sigma_p$ with $x_n[j, j+m-1] \notin A_j$ for all $0 \le j \le n$. Then any limit point of the sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in Σ_p satisfies the requirement.

As convention, set Λ_p^0 to be the one element set consisting of the empty word. For any $k, s, t \in \mathbb{Z}_+$ and any $y \in \Lambda_p^k$, set $|y| = k$ and denote by $\Lambda_p^s y \Lambda_p^t$ the subset of Λ_p^{s+k+t} consisting of elements of the form wyz for some $w \in \Lambda_p^s$ and $z \in \Lambda_p^t$.

For each $n \in \mathbb{Z}_+$ set B_n to be the subset of Λ_p^m consisting of elements b for which there is no $x \in \Sigma_p$ with $x[n, n+m-1] = b$ and $x[j, j+m-1] \notin A_j$ for all $0 \le j \le n$. Set C_n to be the subset of Λ_p^{m-1} consisting of elements c for which $\Lambda_p c$ in contained in B_n. Note that C_n is exactly the set of elements $c \in \Lambda_p^{m-1}$ for which there is no $x \in \Sigma_p$ with $x[n+1, n+m-1] = c$ and $x[j, j+m-1] \notin A_j$ for all $0 \le j \le n$. It follows that

$$B_{n+1} = A_{n+1} \cup \Big(\bigcup_{c \in C_n} c\Lambda_p\Big). \tag{1}$$

Then

$$p|C_{n+1}| \le |B_{n+1}| = |A_{n+1} \cup \Big(\bigcup_{c \in C_n} c\Lambda_p\Big)| \le |A_{n+1}| + p|C_n| \le \ell + p|C_n|$$

for all $n \in \mathbb{Z}_+$. Clearly $|C_0| \le \frac{|B_0|}{p} \le \frac{\ell}{p}$. Inductively one gets that $|C_n| \le \frac{(n+1)\ell}{p}$ for all $n \in \mathbb{Z}_+$.

Set D_n to be the subset of $\bigcup_{k=0}^{m-1} \Lambda_p^k$ consisting of elements y such that $y\Lambda_p^{m-1-|y|} \subseteq C_n$ but $y[1, |y|-1]\Lambda_p^{m-|y|} \not\subseteq C_n$. We put $\Lambda_p^0 \subseteq C_n$ exactly when $C_n = \Lambda_p^{m-1}$. Note that C_n is the disjoint union of $y\Lambda_p^{m-1-|y|}$ for $y \in D_n$. For each $0 \le k \le m-1$ set $D_{n,k} = \{y \in D_n : |y| = k\}$. We claim that

$$|D_{n+1,k}| \le \ell + \sum_{j=k+1}^{m-1} |D_{n,j}|$$

for all $n \in \mathbb{Z}_+$ and $0 \le k \le m-1$. This is clearly true if $k = 0$ or $D_{n,0} \ne \emptyset$. Thus assume that $1 \le k \le m-1$ and $D_{n,0} = \emptyset$. Let $y \in D_{n+1,k}$. Then $y\Lambda_p^{m-1-k}$ is contained in C_{n+1}, and hence $\Lambda_p y \Lambda_p^{m-1-k}$ is contained in $B_{n+1} = A_{n+1} \cup (\bigcup_{z \in D_n} z\Lambda_p^{m-|z|})$. If $\Lambda_p y \Lambda_p^{m-1-k}$ has nonempty intersection with $z_j \Lambda_p^{m-|z_j|}$, $j = 1, 2, \dots, p$, for some pairwise distinct $z_1, z_2, \dots, z_p \in D_n$ with $\max_{1 \le j \le p} |z_j| \le k$, then $\Lambda_p y[1, k-1]\Lambda_p^{m-k}$ is contained in $\bigcup_{j=1}^p z_j\Lambda_p^{m-|z_j|}$, and hence $y[1,k-1]\Lambda_p^{m-k}$ is contained in C_{n+1}, which contradicts the assumption $y\in D_{n+1}$. Therefore $\Lambda_p y\Lambda_p^{m-1-k}$ has nonempty intersection with $z\Lambda_p^{m-|z|}$ for at most $p-1$ elements $z\in D_n$ with $|z|\le k$. If $\Lambda_p y\Lambda_p^{m-1-k}$ does have nonempty intersection with $z\Lambda_p^{m-|z|}$ for some $z\in D_n$ with $|z|\le k$, then, since $|z|\ge 1$, one sees that $\Lambda_p y\Lambda_p^{m-1-k}\cap z\Lambda_p^{m-|z|}$ is equal to $jy\Lambda_p^{m-1-k}$ for some $j\in\Lambda_p$. Therefore

$$|\Lambda_p y\Lambda_p^{m-1-k}\cap(\bigcup_{z\in D_n,|z|\le k} z\Lambda_p^{m-|z|})|\le (p-1)|y\Lambda_p^{m-1-k}|=(p-1)p^{m-1-k},$$

and hence

$$|(\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k})\cap(\bigcup_{z\in D_n,|z|\le k} z\Lambda_p^{m-|z|})|\le |D_{n+1,k}|(p-1)p^{m-1-k}. \tag{2}$$

Note that $\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k}\subseteq\bigcup_{c\in C_{n+1}}\Lambda_p c\subseteq B_{n+1}$. Therefore

$$\begin{aligned}
|\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k}| &= |(\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k})\cap B_{n+1}|\\
&\overset{(1)}{=} |(\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k})\cap(A_{n+1}\cup(\bigcup_{z\in D_n} z\Lambda_p^{m-|z|})|\\
&\le |(\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k})\cap(\bigcup_{z\in D_n,|z|\le k} z\Lambda_p^{m-|z|})|\\
&\quad +|(\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-|y|})\cap(A_{n+1}\cup(\bigcup_{z\in D_n,|z|>k} z\Lambda_p^{m-|z|}))|\\
&\overset{(2)}{\le} |D_{n+1,k}|(p-1)p^{m-1-k}+|A_{n+1}\cup(\bigcup_{z\in D_n,|z|>k} z\Lambda_p^{m-|z|})|\\
&\le |D_{n+1,k}|(p-1)p^{m-1-k}+\ell+\sum_{z\in D_n,|z|>k} p^{m-|z|}\\
&= |D_{n+1,k}|(p-1)p^{m-1-k}+\ell+\sum_{j=k+1}^{m-1}|D_{n,j}|p^{m-j}. \qquad (3)
\end{aligned}$$

Since the sets $\Lambda_p y\Lambda_p^{m-1-k}$ for $y\in D_{n+1,k}$ are pairwise disjoint, we have

$$|\bigcup_{y\in D_{n+1,k}}\Lambda_p y\Lambda_p^{m-1-k}|=\sum_{y\in D_{n+1,k}}|\Lambda_p y\Lambda_p^{m-1-k}|=|D_{n+1,k}|p^{m-k}. \tag{4}$$

From (3) and (4) we get

$$|D_{n+1,k}|p^{m-1-k}\le \ell+\sum_{j=k+1}^{m-1}|D_{n,j}|p^{m-j},$$

and hence

$$|D_{n+1,k}|\le p^{k+1-m}\ell+\sum_{j=k+1}^{m-1}|D_{n,j}|p^{|k|+1-j}\le \ell+\sum_{j=k+1}^{m-1}|D_{n,j}|,$$

proving the claim. It follows inductively that $|D_{n,m-k}| \leq 2^{k-1}\ell$ for all $n \in \mathbb{Z}_+$ and $1 \leq k \leq m-1$.

We need to show that $B_n \neq \Lambda_p^m$ for all $n \in \mathbb{Z}_+$, equivalently, $C_n \neq \Lambda_p^{m-1}$ for all $n \in \mathbb{Z}_+$. In fact, we claim that for every $n \in \mathbb{Z}_+$, there are no $d \leq d' \leq m-1$ and $y \in \Lambda_p^{m-1-d'}$ with $y\Lambda_p^{d'} \subseteq C_n$.

We argue by contradiction. So assume that there are $n \in \mathbb{Z}_+$, $d \leq d' \leq m-1$ and $y \in \Lambda_p^{m-1-d'}$ with $y\Lambda_p^{d'} \subseteq C_n$. Let n_0 be the smallest such n, and let d' and y witness n_0. Replacing y by yw for any $w \in \Lambda_p^{d'-d}$, we may assume that $d' = d$.

Since $\frac{(n_0+1)\ell}{p} \geq |C_{n_0}| \geq p^d > \frac{(d+p)\ell}{p}$, we have $n_0 \geq d+p$. Denote $n_0 - d$ by n_1. For each $n_0 \geq n \geq n_1$, set E_n to be the subset of C_n consisting of c satisfying $c[1+n_0-n, m-1-d+n_0-n] = y$. The assumption in the above paragraph says that $E_{n_0} = y\Lambda_p^d$ and hence $|E_{n_0}| = p^d$. For each $n_0 > n \geq n_1$, we have

$$\bigcup_{c\in E_{n+1}} \Lambda_p c \subseteq \bigcup_{c\in C_{n+1}} \Lambda_p c \subseteq B_{n+1} = A_{n+1} \cup (\bigcup_{c'\in C_n} c'\Lambda_p).$$

Note that if $\Lambda_p c \cap c'\Lambda_p \neq \emptyset$ for some $c \in E_{n+1}$ and $c' \in C_n$, then c' is in E_n. Thus

$$\bigcup_{c\in E_{n+1}} \Lambda_p c \subseteq A_{n+1} \cup (\bigcup_{c'\in E_n} c'\Lambda_p),$$

and hence

$$p|E_{n+1}| = |\bigcup_{c\in E_{n+1}} \Lambda_p c| \leq |A_{n+1} \cup (\bigcup_{c'\in E_n} c'\Lambda_p)| \leq |A_{n+1}| + p|E_n| \leq \ell + p|E_n|.$$

It follows inductively that $|E_n| \geq |E_{n_0}| - \frac{(n_0-n)\ell}{p} = p^d - \frac{(n_0-n)\ell}{p}$ for all $n_0 \geq n \geq n_1$. In particular, $|E_{n_1}| \geq p^d - \frac{d\ell}{p} > \ell$.

Denote $\max(0, d+n_1-m+1)$ by n_2. For each $n_1 \geq n \geq n_2$ denote by k_n the largest number k for which there exists a subset F of C_n such that $|F| = k$ and $c[1+d+n_1-n, m-1]$ does not depend on $c \in F$. Taking F to be E_{n_1} we see that $k_{n_1} \geq |E_{n_1}| > \ell$. We claim that $pk_{n+1} \leq k_n + \ell$ for all $n_1 > n \geq n_2$. Take $F \subseteq C_{n+1}$ such that $|F| = k_{n+1}$ and $c[d+n_1-n, m-1]$ does not depend on $c \in F$. Then the set $W := \bigcup_{c\in F} \Lambda_p c$ has pk_{n+1} elements and is contained in $B_{n+1} = A_{n+1} \cup (\bigcup_{c'\in C_n} c'\Lambda_p)$. Set $F' = \{c' \in C_n : c'\Lambda_p \cap W \neq \emptyset\}$. Then $c[1+d+n_1-n, m-1]$ does not depend on $c \in F'$ and hence $|F'| \leq k_n$. Since $d+n_1-n \leq d+n_1-n_2 \leq m-1$, all the elements in W have the same right end. It follows that for any $c' \in F'$, one has $|W \cap c'\Lambda_p| = 1$. Thus

$$\begin{aligned} pk_{n+1} &= |W| = |W \cap (A_{n+1} \cup (\bigcup_{c'\in C_n} c'\Lambda_p))| = |W \cap (A_{n+1} \cup (\bigcup_{c'\in F'} c'\Lambda_p))| \\ &\leq |A_{n+1}| + \sum_{c'\in F'} |W \cap c'\Lambda_p| = |A_{n+1}| + |F'| \leq \ell + k_n. \end{aligned}$$

This proves the claim. Inductively, we get $k_n \geq p^{n_1-n} + \ell$ for all $n_1 \geq n \geq n_2$. In particular, $|C_{n_2}| \geq k_{n_2} \geq p^{n_1-n_2} + \ell$. Since $|C_0| \leq \frac{\ell}{p}$, we have $n_2 > 0$. Thus $n_2 = d+n_1-m+1$, and hence $|C_{n_2}| \geq k_{n_2} \geq p^{m-d-1} + \ell$.

Since $n_2 \le n_1 < n_0$, according to the choice of n_0, $D_{n_2,k}$ is empty for all $0 \le k \le m-1-d$. Thus

$$\begin{aligned}
|C_{n_2}| &= |\bigcup_{k=0}^{m-1}(\bigcup_{y\in D_{n_2,k}} y\Lambda_p^{m-1-k})| \\
&= |\bigcup_{k=m-d}^{m-1}(\bigcup_{y\in D_{n_2,k}} y\Lambda_p^{m-1-k})| \\
&= \sum_{k=m-d}^{m-1} |D_{n_2,k}| p^{m-1-k} \\
&\le \sum_{k=m-d}^{m-1} 2^{m-1-k}\ell\cdot p^{m-1-k} = \ell\sum_{j=0}^{d-1}(2p)^j = \frac{(2p)^d-1}{2p-1}\cdot\ell.
\end{aligned}$$

This contradicts $|C_{n_2}| \ge p^{m-d-1}+\ell$ once we take m large enough such that $\frac{(2p)^d-1}{2p-1}\cdot\ell < p^{m-1-d}+\ell$. A simple calculation shows that we may take $d = \ell+1$ and $m \ge 4\ell+2$. □

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