COMPACT QUANTUM METRIC SPACES AND ERGODIC ACTIONS OF COMPACT QUANTUM GROUPS

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Abstract. We show that for any co-amenable compact quantum group $A = C(\mathcal{G})$ there exists a unique compact Hausdorff topology on the set $\mathrm{EA}(\mathcal{G})$ of isomorphism classes of ergodic actions of $\mathcal{G}$ such that the following holds: for any continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T the map $T \to \mathrm{EA}(\mathcal{G})$ sending each t in T to the isomorphism class of the fibre at t is continuous if and only if the function counting the multiplicity of γ in each fibre is continuous over T for every equivalence class γ of irreducible unitary representations of $\mathcal{G}$. Generalizations for arbitrary compact quantum groups are also obtained. In the case $\mathcal{G}$ is a compact group, the restriction of this topology on the subset of isomorphism classes of ergodic actions of full multiplicity coincides with the topology coming from the work of Landstad and Wassermann. Podleś spheres are shown to be continuous in the natural parameter as ergodic actions of the quantum SU(2) group. We also introduce a notion of regularity for quantum metrics on $\mathcal{G}$, and show how to construct a quantum metric from any ergodic action of $\mathcal{G}$, starting from a regular quantum metric on $\mathcal{G}$. Furthermore, we introduce a quantum Gromov-Hausdorff distance between ergodic actions of $\mathcal{G}$ when $\mathcal{G}$ is separable and show that it induces the above topology.

1. Introduction

An ergodic action of a compact group G on a unital C^*-algebra B is a strongly continuous action of G on B such that the fixed point algebra consists only of scalars. For an irreducible representation of G on a Hilbert space H, the conjugate action of G on the algebra $B(H)$ is ergodic. On the other hand, ergodic actions of G on commutative unital C^*-algebras correspond exactly to translations on homogeneous spaces of G. Thus the theory of ergodic actions of G connects both the representation theory and the study of homogeneous spaces. See [14, 20, 26, 42, 43, 44] and references therein.

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Olesen, Pedersen, and Takesaki classified faithful ergodic actions of an abelian compact group as skew-symmetric bicharacters on the dual group [26]. Landstad and Wassermann generalized their result independently to show that ergodic actions of full multiplicity of an arbitrary compact group G are classified by equivalence classes of dual cocycles [20, 43]. However, the general case is quite difficult–so far there is no classification of (faithful) ergodic actions of compact groups, not to mention compact quantum groups. In this paper we are concerned with topological properties of the whole set $\mathrm{EA}(G)$ of isomorphism classes of ergodic actions of a compact group G, and more generally, the set $\mathrm{EA}(\mathcal{G})$ of isomorphism classes of ergodic actions of a compact quantum group $A = C(\mathcal{G})$.

As a consequence of their classification, Olesen, Pedersen, and Takesaki showed that the set of isomorphism classes of faithful ergodic actions of an abelian compact group has a natural abelian compact group structure. From the work of Landstad and Wassermann, the set $\mathrm{EA}(G)_{\mathrm{fm}}$ of ergodic actions of full multiplicity of an arbitrary compact group G also carries a natural compact Hausdorff topology.

There are many ergodic actions not of full multiplicity, such as conjugation actions associated to irreducible representations and actions corresponding to translations on homogeneous spaces (unless G is finite or the homogeneous space is G itself). In the physics literature concerning string theory and quantum field theory, people talk about fuzzy spheres, the matrix algebras $M_n(\mathbb{C})$, converging to the two-sphere S^2 (see the introduction of [35] and references therein). One important feature of this convergence is that each term carries an ergodic action of $\mathrm{SU}(2)$, which is used in the construction of this approximation of S^2 by fuzzy spheres. Thus if one wants to give a concrete mathematical foundation for this convergence, it is desirable to include the $\mathrm{SU}(2)$ symmetry. However, none of these actions involved are of full multiplicity, and hence the topology of Landstad and Wassermann does not apply here.

For compact quantum groups there are even more interesting examples of ergodic actions [41]. Podleś introduced a family of quantum spheres S^2_{qt}, parameterized by a compact subset T_q of the real line, as ergodic actions of the quantum $\mathrm{SU}(2)$ group $\mathrm{SU}_q(2)$ satisfying certain spectral conditions [29]. These quantum spheres carry interesting noncommutative differential geometry [8, 9]. One also expects that Podleś quantum spheres are continuous in the natural parameter t as ergodic actions of $\mathrm{SU}_q(2)$.

Continuous change of C^*-algebras is usually described qualitatively as continuous fields of C^*-algebras over locally compact Hausdorff spaces [7, Chapter 10]. There is no difficulty to formulate the equivariant version–continuous fields of actions of compact groups [32] or even compact quantum groups (see Section 5 below). Thus if there is any natural topology on $\mathrm{EA}(\mathcal{G})$, the relation with continuous fields of ergodic actions should be clarified.

One distinct feature of the theory of compact quantum groups is that there is a full compact quantum group and a reduced compact group associated to each compact quantum group $\mathcal{G}$, which may not be the same. A compact quantum group $\mathcal{G}$ is called co-amenable if the full and reduced compact quantum groups coincide. This is the case for compact groups and $\mathrm{SU}_q(N)$ (for $0 < |q| < 1$). Our result is simplified in such case. Denote by $\hat{\mathcal{G}}$ the set of equivalence classes of irreducible unitary representations of $\mathcal{G}$. For each ergodic action of $\mathcal{G}$, one can talk about the multiplicity of each $\gamma \in \hat{\mathcal{G}}$ in this action [30], which is known to be finite for the compact group case by [14] and, for the compact quantum group case by [5].

Theorem 1.1. Let $\mathcal{G}$ be a co-amenable compact quantum group. Then $\mathrm{EA}(\mathcal{G})$ has a unique compact Hausdorff topology such that the following holds: for any continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T the map $T \to \mathrm{EA}(\mathcal{G})$ sending each $t \in T$ to the isomorphism class of the fibre at t is continuous if and only if the function counting the multiplicity of γ in each fibre is continuous over T for each γ in $\hat{\mathcal{G}}$.

In particular, fuzzy spheres converge to S^2 as ergodic actions of $\mathrm{SU}(2)$ (see [21, Example 10.12]). Podleś quantum spheres are also continuous as ergodic actions of $\mathrm{SU}_q(2)$:

Theorem 1.2. Let q be a real number with $0 < |q| < 1$, and let T_q be the parameter space of Podleś quantum spheres. The map $T_q \to \mathrm{EA}(\mathrm{SU}_q(2))$ sending t to the isomorphism class of S^2_{qt} is continuous.

When $\mathcal{G}$ is not co-amenable, the more appropriate object to study is a certain quotient space of $\mathrm{EA}(\mathcal{G})$. To each ergodic action of $\mathcal{G}$, there is an associated full ergodic action and an associated reduced ergodic action (see Section 3 below), which are always isomorphic when $\mathcal{G}$ is co-amenable. Two ergodic actions are said to be equivalent if the associated full (reduced resp.) actions are isomorphic. Denote by $\mathrm{EA}^{\sim}(\mathcal{G})$ the quotient space of $\mathrm{EA}(\mathcal{G})$ modulo this equivalence relation. We also have to deal with semi-continuous fields of ergodic actions in the general case.

Theorem 1.3. Let $\mathcal{G}$ be a compact quantum group. Then $\mathrm{EA}^{\sim}(\mathcal{G})$ has a unique compact Hausdorff topology such that the following holds: for any semi-continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T the map $T \to \mathrm{EA}^{\sim}(\mathcal{G})$ sending each $t \in T$ to the equivalence class of the fibre at t is continuous if and only if the function counting the multiplicity of γ in each fibre is continuous over T for each γ in $\hat{\mathcal{G}}$.

Motivated partly by the need to give a mathematical foundation for various approximations in the string theory literature, such as the approximation of S^2 by fuzzy spheres in above, Rieffel initiated the theory of compact quantum metric spaces and quantum Gromov-Hausdorff distances [34, 37]. As the information of the metric on a compact metric space X is encoded in the Lipschitz seminorm on $C(X)$, a quantum metric on (the non-commutative space corresponding to) a unital C^*-algebra B is a (possibly $+\infty$-valued) seminorm on B satisfying suitable conditions. The seminorm is called a Lip-norm. Given a length function on a compact group G, Rieffel showed how to induce a quantum metric on (the C^*-algebra carrying) any ergodic action of G [33]. We find that the right generalizations of length functions for a compact quantum group $A = C(\mathcal{G})$ are Lip-norms on A being finite on the algebra $\mathcal{A}$ of regular functions, which we call regular Lip-norms. Every separable co-amenable A has a bi-invariant regular Lip-norm (Corollary 8.10). Then we have the following generalization of Rieffel's construction (see Section 2 below for more detail on the notation), answering a question Rieffel raised at the end of Section 3 in [37].

Theorem 1.4. Suppose that $\mathcal{G}$ is a co-amenable compact quantum group and L_A is a regular Lip-norm on A. Let $\sigma : B \to B \otimes A$ be an ergodic action of $\mathcal{G}$ on a unital C^*-algebra B. Define a (possibly $+\infty$-valued) seminorm on B via

$$L_B(b) = \sup_{\varphi \in S(B)} L_A(b * \varphi) \tag{1}$$

for all $b \in B$, where $S(B)$ denotes the state space of B and $b * \varphi = (\varphi \otimes \mathrm{id})(\sigma(b))$. Then L_B is finite on the algebra $\mathcal{B}$ of regular functions and is a Lip-norm on B with $r_B \leq 2r_A$, where r_B and r_A are the radii of B and A respectively.

As an important step towards establishing a mathematical foundation for various convergence in the string theory literature, such as the convergence of fuzzy spheres to S^2, Rieffel introduced a quantum Gromov-Hausdorff distance $\mathrm{dist_q}$ between compact quantum metric spaces and showed, among many properties of $\mathrm{dist_q}$, that the fuzzy spheres converge to S^2 under $\mathrm{dist_q}$ when they are all endowed with the quantum metrics induced from the ergodic actions of SU(2) for a fixed length function on SU(2) [35]. Two generalizations of $\mathrm{dist_q}$ are introduced in [15] and [22] in order to distinguish the algebra structures (see also [16]). However, none of these quantum distances distinguishes the group symmetries. That is, there exist non-isomorphic ergodic actions of a compact group such that quantum distances between the compact quantum metric spaces induced by these ergodic actions are zero (see Example 9.1 below). One of the features of our quantum distances in [22, 21] is that they can be adapted easily to take care of other algebraic structures. Along the lines in [22, 21], we introduce a quantum distance $\mathrm{dist_e}$ (see Definition 9.3 below) between the compact quantum metric spaces coming from ergodic action of $\mathcal{G}$ as in Theorem 1.4. This distance distinguishes the ergodic actions:

Theorem 1.5. Let $\mathcal{G}$ be a co-amenable compact quantum group with a fixed left-invariant regular Lip-norm L_A on A. Then $\mathrm{dist_e}$ is a metric on $\mathrm{EA}(\mathcal{G})$ inducing the topology in Theorem 1.1.

The organization of this paper is as follows. In section 2 we recall some basic definitions and facts about compact quantum groups, their actions, and compact quantum metric spaces. Associated full and reduced actions are discussed in Section 3. The topologies on $\mathrm{EA}(\mathcal{G})$ and $\mathrm{EA}^{\sim}(\mathcal{G})$ are introduced in Section 4. We also prove that $\mathrm{EA}^{\sim}(\mathcal{G})$ is compact Hausdorff there. In Section 5 we clarify the relation between semi-continuous fields of ergodic actions and the topology introduced in Section 4. This completes the proofs of Theorems 1.1 and 1.3. The continuity of Podleś quantum spheres is discussed in Section 6. In Section 7 we show that the topology of Landstad and Wassermann on $\mathrm{EA}(G)_{\mathrm{fm}}$ for a compact group G is simply the relative topology of $\mathrm{EA}(G)_{\mathrm{fm}}$ in $\mathrm{EA}(G)$. Theorems 1.4 and 1.5 are proved in Sections 8 and 9 respectively.

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2. Preliminaries

In this section we collect some definitions and facts about compact quantum groups and compact quantum metric spaces.

Throughout this paper we use $\otimes$ for the spatial tensor product of C^*-algebras, and $\odot$ for the algebraic tensor product of vector spaces. $A = C(\mathcal{G})$ will be a compact quantum group.

2.1. **Compact quantum groups and actions.** We recall first some definitions and facts about compact quantum groups. See [23, 47, 48] for more detail.

A *compact quantum group* (A, Φ) is a unital C^*-algebra A and a unital $*$-homomorphism $\Phi : A \to A \otimes A$ such that $(\mathrm{id} \otimes \Phi)\Phi = (\Phi \otimes \mathrm{id})\Phi$ and that both $\Phi(A)(1_A \otimes A)$ and $\Phi(A)(A \otimes 1_A)$ are dense in $A \otimes A$, where $\Phi(A)(1_A \otimes A)$ ($\Phi(A)(A \otimes 1_A)$ resp.) denotes the linear span of $\Phi(a)(1_A \otimes a')$ ($\Phi(a)(a' \otimes 1_A)$ resp.) for $a, a' \in A$. We shall write A as $C(\mathcal{G})$ and say that $\mathcal{G}$ is the compact quantum group. The Haar measure is the unique state h of A such that $(\mathrm{id} \otimes h)\Phi = (h \otimes \mathrm{id})\Phi = h$.

A unitary representation of $\mathcal{G}$ on a Hilbert space H is a unitary $u \in M(K(H) \otimes A)$ such that $(\mathrm{id} \otimes \Phi)(u) = u_{12}u_{13}$, where $K(H)$ is the algebra of compact operators, $M(K(H) \otimes A)$ is the multiplier algebra of $K(H) \otimes A$, and we use the leg numbering notation [31, page 385]. When H is finite dimensional, u^c denotes the contragradient representation acting on the conjugate Hilbert space of H. For unitary representations v and w of $\mathcal{G}$, the tensor product representation $v \textcircled{\scriptsize T} w$ is defined as $v_{13}w_{23}$ in the leg numbering notation. Denote by $\hat{\mathcal{G}}$ the set of equivalence classes of irreducible unitary representations of $\mathcal{G}$. For each $\gamma \in \hat{\mathcal{G}}$ fix $u^\gamma \in \gamma$ acting on H_γ and an orthonormal basis in H_γ. Each H_γ is finite dimensional. Denote by d_γ the dimension of H_γ. Then we may identify $B(H_\gamma)$ with $M_{d_\gamma}(\mathbb{C})$, and hence $u^\gamma \in M_{d_\gamma}(A)$. Denote by A_γ the linear span of $(u^\gamma_{ij})_{ij}$. Then

$$A^*_\gamma = A_{\gamma^c}, \quad \Phi(A_\gamma) \subseteq A_\gamma \odot A_\gamma, \tag{2}$$

and $\mathcal{A} := \oplus_{\gamma \in \hat{\mathcal{G}}} A_\gamma$ is the algebra of *regular functions* in A. For any $1 \leq i, j \leq d_\gamma$ denote by ρ^γ_{ij} the unique element in A' such that

$$\rho^\gamma_{ij}(u^\beta_{sk}) = \delta_{\gamma\beta}\delta_{is}\delta_{jk} \tag{3}$$

(the existence of such ρ^γ_{ij} is guaranteed by [47, Theorem 5.7]). Moreover ρ^γ_{ij} is of the form $h(\cdot a)$ for some $a \in A_{\gamma^c}$. Denote $\sum_{1 \leq i \leq d_\gamma} \rho^\gamma_{ii}$ by ρ^γ. Denote the class of the trivial representations of $\mathcal{G}$ by γ_0.

There exist a full compact quantum group (A_u, Φ_u) and a reduced compact group (A_r, Φ_r) whose algebras of regular functions and restrictions of comultiplications are the same as $(\mathcal{A}, \Phi|_{\mathcal{A}})$. The quantum group $\mathcal{G}$ is said to be *co-amenable* if the canonical surjective homomorphism $A_u \to A_r$ is an isomorphism [2, Definition 6.1] [3, Theorem 3.6]. There is a unique $*$-homomorphism $e : \mathcal{A} \to \mathbb{C}$ such that

$(e \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes e)\Phi = \mathrm{id}$ on $\mathcal{A}$, which is called the *counit*. The quantum group $\mathcal{G}$ is co-amenable exactly if it has bounded counit and faithful Haar measure [3, Theorem 2.2].

Next we recall some facts about actions of $\mathcal{G}$. See [5, 30] for detail.

Definition 2.1. [30, Definition 1.4] A (left) *action* of $\mathcal{G}$ on a unital C^*-algebra B is a unital $*$-homomorphism $\sigma : B \to B \otimes A$ such that

(1) $(\mathrm{id} \otimes \Phi)\sigma = (\sigma \otimes \mathrm{id})\sigma$,
(2) $\sigma(B)(1_B \otimes A)$ is dense in $B \otimes A$.

The *fixed point algebra* of σ is $B^\sigma = \{b \in B : \sigma(b) = b \otimes 1_A\}$. The action σ is *ergodic* if $B^\sigma = \mathbb{C}1_B$.

Remark 2.2. When A has bounded counit, the proof of [12, Lemma 1.4.(a)] shows that $(\mathrm{id} \otimes e)\sigma = \mathrm{id}$ on B and that σ is injective.

Let $\sigma : B \to B \otimes A$ be an action of $\mathcal{G}$ on a unital C^*-algebra B. For any $b \in B, \varphi \in B'$ and $\psi \in A'$ set

$$b * \varphi = (\varphi \otimes \mathrm{id})(\sigma(b)), \quad \psi * b = (\mathrm{id} \otimes \psi)(\sigma(b)). \tag{4}$$

Also set $E^\gamma_{ij}, E^\gamma : B \to B$ by

$$E^\gamma_{ij} = (\mathrm{id} \otimes \rho^\gamma_{ij})\sigma, \quad E^\gamma = (\mathrm{id} \otimes \rho^\gamma)\sigma. \tag{5}$$

Then [5, page 98]

$$E^\beta E^\gamma = \delta_{\beta\gamma} E^\gamma. \tag{6}$$

Set

$$B_\gamma = E^\gamma(B), \quad \mathcal{B} = \oplus_{\gamma \in \hat{\mathcal{G}}} B_\gamma. \tag{7}$$

Then $\mathcal{B}$ is a dense $*$-subalgebra of B [30, Theorem 1.5] [5, Lemma 11, Proposition 14] (the ergodicity condition in [5] is not used in Lemma 11 and Proposition 14 therein), which we shall call the algebra of *regular functions* for σ. Moreover,

$$B^*_\gamma = B_{\gamma^c}, \quad B_\alpha B_\beta \subseteq \sum_{\gamma \preceq \alpha \textcircled{\scriptsize T} \beta} B_\gamma, \quad \sigma(B_\gamma) \subseteq B_\gamma \odot A_\gamma. \tag{8}$$

There exist a set J_γ and a linear basis $\mathscr{S}_\gamma = \{e_{\gamma k i} : k \in J_\gamma, 1 \le i \le d_\gamma\}$ of B_γ [30, Theorem 1.5] such that

$$\sigma(e_{\gamma k i}) = \sum_{1 \le j \le d_\gamma} e_{\gamma k j} \otimes u^\gamma_{ji}. \tag{9}$$

The *multiplicity* $\mathrm{mul}(B, \gamma)$ is defined as the cardinality of J_γ, which does not depend on the choice of $\mathscr{S}_\gamma$. Conversely, given a unital $*$-homomorphism $\sigma : B \to B \otimes A$ for a unital C^*-algebra B, if there exist a set J'_γ and a set of linearly independent elements $\mathscr{S}'_\gamma = \{e_{\gamma k i} :$

$k \in J'_\gamma, 1 \leq i \leq d_\gamma\}$ in B satisfying (9) for each $\gamma \in \hat{\mathcal{G}}$ such that the linear span of $\cup_{\gamma\in\hat{\mathcal{G}}}\mathscr{S}_\gamma$ is dense in B, then σ is an action of $\mathcal{G}$ on B and $\mathrm{span}\mathscr{S}'_\gamma \subseteq B_\gamma$ for each $\gamma \in \hat{\mathcal{G}}$ [30, Corollary 1.6]. In this event, if $|J'_\gamma|$ or $\mathrm{mul}(B, \gamma)$ is finite, then $B_\gamma = \mathrm{span}\mathscr{S}'_\gamma$.

We have that $B_{\gamma_0} = B^\sigma$ and that $E := E^{\gamma_0}$ is a conditional expectation from B onto B^σ [5, Lemma 4]. When σ is ergodic, $E = \omega(\cdot)1_B$ for the unique σ-invariant state ω on B.

2.2. **Compact quantum metric spaces.** In this subsection we recall some facts about compact quantum metric spaces [33, 34, 37]. Though Rieffel has set up his theory in the general framework of order-unit spaces, we shall need it only for C^*-algebras. See the discussion preceding Definition 2.1 in [34] for the reason of requiring the reality condition (10) below.

Definition 2.3. [34, Definition 2.1] By a *C^*-algebraic compact quantum metric space* we mean a pair (B, L) consisting of a unital C^*-algebra B and a (possibly $+\infty$-valued) seminorm L on B satisfying the *reality condition*

$$L(b) \;=\; L(b^*) \tag{10}$$

for all $b \in B$, such that L vanishes on $\mathbb{C}1_B$ and the metric ρ_L on the state space $S(B)$ defined by

$$\rho_L(\varphi, \psi) = \sup_{L(b)\leq 1} |\varphi(b) - \psi(b)| \tag{11}$$

induces the weak-$*$ topology. The *radius* r_B of (B, L) is defined to be the radius of $(S(B), \rho_L)$. We say that L is a *Lip-norm*.

Note that L must in fact vanish precisely on $\mathbb{C}1_B$ and take finite values on a dense subspace of B.

Let B be a unital C^*-algebra and let L be a (possibly $+\infty$-valued) seminorm on B vanishing on $\mathbb{C}1_B$. Then L and $\|\cdot\|$ induce (semi)norms $\tilde{L}$ and $\|\cdot\|^\sim$ respectively on the quotient space $\tilde{B} = B/(\mathbb{C}1_B)$.

Notation 2.4. Let

$$\mathcal{E}(B) := \{b \in B_{\mathrm{sa}} : L(b) \leq 1\}.$$

For any $r \geq 0$, let

$$\mathcal{D}_r(B) := \{b \in B_{\mathrm{sa}} : L(b) \leq 1, \|b\| \leq r\}.$$

Note that the definitions of $\mathcal{E}(B)$ and $\mathcal{D}_r(A)$ use B_{sa} instead of B. The main criterion for when a seminorm L is a Lip-norm is the following:

Proposition 2.5. [33, Proposition 1.6, Theorem 1.9] Let B be a unital C^*-algebra and let L be a (possibly $+\infty$-valued) seminorm on B satisfying the reality condition (10). Assume that L takes finite values on a dense subspace of B, and that L vanishes on $\mathbb{C}1_B$. Then L is a Lip-norm if and only if

(1) there is a constant $K \geq 0$ such that $\|\cdot\|^{\sim} \leq K\tilde{L}$ on $\tilde{B}$;

and (2) for any $r \geq 0$, the ball $\mathcal{D}_r(B)$ is totally bounded in B for $\|\cdot\|$;

or (2') for some $r > 0$, the ball $\mathcal{D}_r(B)$ is totally bounded in B for $\|\cdot\|$.

In this event, r_B is exactly the minimal K such that $\|\cdot\|^{\sim} \leq K\tilde{L}$ on $(\tilde{B})_{\rm sa}$.

3. Full and reduced actions

In this section we discuss full and reduced actions associated to actions of $\mathcal{G}$. We will use the notation in subsection 2.1 freely. Throughout this section, σ will be an action of $\mathcal{G}$ on a unital C^*-algebra B.

Lemma 3.1. The conditional expectation $E = (\mathrm{id} \otimes h)\sigma$ is faithful on $\mathcal{B}$. If A is co-amenable, then E is faithful on B.

Proof. Suppose that $E(b) = 0$ for some positive b in $\mathcal{B}$. Then for any $\varphi \in S(B)$ we have $0 = \varphi(E(b)) = h(b * \varphi)$. Observe that $b * \varphi$ is in $\mathcal{A}$ and is positive. By the faithfulness of h on $\mathcal{A}$ [47, Theorem 4.2], $b * \varphi = 0$. Then $(\varphi \otimes \psi)(\sigma(b)) = \psi(b * \varphi) = 0$ for all $\varphi \in S(B)$ and $\psi \in S(A)$. Since product states separate points of $B \otimes A$ [45, Lemma T.5.9 and Proposition T.5.14], $\sigma(b) = 0$. From (6) one sees that $b \in \{\phi * b : \phi \in A'\}$. Therefore $b = 0$. The second assertion is proved similarly, in view of Remark 2.2. $\square$

For actions $\sigma_i : B_i \to B_i \otimes A$ of $\mathcal{G}$ on B_i for $i = 1, 2$, a unital $*$-homomorphism $\theta : B_1 \to B_2$ is said to be *equivariant* (with respect to σ_1 and σ_2) if $\sigma_2 \circ \theta = (\theta \otimes \mathrm{id}) \circ \sigma_1$.

Lemma 3.2. Let $\theta : B_1 \to B_2$ be a unital $*$-homomorphism equivariant with respect to actions σ_1, σ_2 of $\mathcal{G}$ on B_1 and B_2. Then

$$E^{\gamma} \circ \theta = \theta \circ E^{\gamma}, \tag{12}$$

$$\theta((B_1)_{\gamma}) \subseteq (B_2)_{\gamma} \tag{13}$$

for all $\gamma \in \hat{\mathcal{G}}$. The map θ is surjective if and only if $\theta(\mathcal{B}_1) = \mathcal{B}_2$. The map θ is injective on $\mathcal{B}_1$ if and only if θ is injective on $B_1^{\sigma_1}$.

Proof. One has

$$E^{\gamma} \circ \theta = (\mathrm{id} \otimes \rho^{\gamma}) \circ \sigma_2 \circ \theta = (\mathrm{id} \otimes \rho^{\gamma}) \circ (\theta \otimes \mathrm{id}) \circ \sigma_1$$

$$= \theta \circ (\mathrm{id} \otimes \rho^{\gamma}) \circ \sigma_1 = \theta \circ E^{\gamma},$$

which proves (12). The formula (13) follows from (12).

Since $\mathcal{B}_2$ is dense in B_2, if $\theta(\mathcal{B}_1) = \mathcal{B}_2$, then θ is surjective. Conversely, suppose that θ is surjective. Applying both sides of (12) to B_1 we get $\theta((B_1)_\gamma) = (B_2)_\gamma$ for each $\gamma \in \hat{\mathcal{G}}$. Thus $\theta(\mathcal{B}_1) = \mathcal{B}_2$.

Since $B_1^{\sigma_1} \subseteq \mathcal{B}_1$, if θ is injective on $\mathcal{B}_1$, then θ is injective on $B_1^{\sigma_1}$. Conversely, suppose that θ is injective on $B_1^{\sigma_1}$. Let $b \in \mathcal{B}_1 \cap \ker \theta$. Then $\theta(E(b^*b)) = E(\theta(b^*b)) = 0$. Thus $E(b^*b) \in \ker \theta$. By assumption we have $E(b^*b) = 0$. Then $b = 0$ by Lemma 3.1. □

Proposition 3.3. The $*$-algebra $\mathcal{B}$ has a universal C^*-algebra B_u. The canonical $*$-homomorphism $\mathcal{B} \to B_u$ is injective. Identify $\mathcal{B}$ with its canonical image in B_u. The unique $*$-homomorphism $\sigma_u : B_u \to B_u \otimes A$ extending $\mathcal{B} \overset{\sigma|_{\mathcal{B}}}{\to} \mathcal{B} \odot \mathcal{A} \hookrightarrow B_u \otimes A$ is an action of $\mathcal{G}$ on B_u. Moreover, the unique $*$-homomorphism $\pi_u : B_u \to B$ extending the embedding $\mathcal{B} \to B$ is equivariant, and the algebra of regular functions for σ_u is $\mathcal{B}$.

Proof. Let $\gamma \in \hat{\mathcal{G}}$ and let $\mathscr{S}_\gamma$ be a linear basis of B_γ satisfying (9). Set $b_k = \sum_{i=1}^{d_\gamma} e_{\gamma ki} e_{\gamma ki}^*$. Then

$$\sigma(b_k) = \sum_{1 \leq i,j,s \leq d_\gamma} e_{\gamma kj} e_{\gamma ks}^* \otimes u_{ji}^{\gamma} u_{si}^{\gamma\,*} = \sum_{1 \leq j,s \leq d_\gamma} e_{\gamma kj} e_{\gamma ks}^* \otimes \delta_{js} 1_A = b_k \otimes 1_A.$$

Thus $b_k \in B^\sigma$. Note that B^σ is a C^*-subalgebra of B. So $\|\pi(\cdot)\| \leq \|\cdot\|$ on B^σ for any $*$-representation π of $\mathcal{B}$. Consequently, $\|\pi(e_{\gamma ki})\|^2 \leq \|b_k\|$ for any $*$-representation π of $\mathcal{B}$. Thus for any $c \in \mathcal{B}$ there is some $\lambda_c \in \mathbb{R}$ such that $\|\pi(c)\| \leq \lambda_c$ for any $*$-representation π of $\mathcal{B}$. Therefore $\mathcal{B}$ has a universal C^*-algebra B_u with the canonical $*$-homomorphism $\phi : \mathcal{B} \to B_u$. Then there is a unique $*$-homomorphism $\pi_u : B_u \to B$ such that $\pi_u \circ \phi$ is the canonical embedding $\iota : \mathcal{B} \hookrightarrow B$. Since ι is injective, so is ϕ. Thus we may identify $\mathcal{B}$ with $\phi(\mathcal{B})$.

Denote by σ_u the unique $*$-homomorphism $B_u \to B_u \otimes A$ extending the $*$-homomorphism $\mathcal{B} \overset{\sigma|_{\mathcal{B}}}{\to} \mathcal{B} \odot \mathcal{A} \hookrightarrow B_u \otimes A$. According to the characterization of actions of $\mathcal{G}$ in terms of elements satisfying (9) in subsection 2.1, σ_u is an action of $\mathcal{G}$ on B_u and $\mathcal{B} \subseteq \mathcal{B}_u$. Since $\sigma \circ \pi_u$ and $(\pi_u \otimes \mathrm{id}) \circ \sigma_u$ coincide on $\mathcal{B}$, they also coincide on B_u. Thus π_u is equivariant.

Since B^σ is closed and $E(\mathcal{B}) = B^\sigma$, $B_u^{\sigma_u} = E(B_u) = B^\sigma$. Thus π_u is injective on $B_u^{\sigma_u}$. By Lemma 3.2 the map θ is injective on $\mathcal{B}_u$. Let $b \in \mathcal{B}_u$. Then $\pi_u(b) \in \mathcal{B}$ by Lemma 3.2, and hence $\pi_u(b - \pi_u(b)) = 0$. Therefore $b = \pi_u(b) \in \mathcal{B}$. This proves $\mathcal{B}_u = \mathcal{B}$ as desired. □

We refer the reader to [19] for basics on Hilbert C^*-modules. Since $E : B \to B^\sigma$ is a conditional expectation, B is a right semi-inner-product B^σ-module with the inner product $\langle \cdot, \cdot \rangle_{B^\sigma}$ given by $\langle x, y\rangle_{B^\sigma} = E(x^*y)$ [19, page 7]. Denote by H_B the completion, and by π_r the associated representation of B on H_B. Denote $\pi_r(B)$ by B_r.

Proposition 3.4. There exists a unique $*$-homomorphism $\sigma_r : B_r \to B_r \otimes A$ such that

$$\sigma_r \circ \pi_r = (\pi_r \otimes \mathrm{id}) \circ \sigma. \tag{14}$$

The homomorphism σ_r is injective and is an action of $\mathcal{G}$ on B_r. The map π_r is equivariant, and is injective on $\mathcal{B}$. The algebra of regular functions for σ_r is $\pi_r(\mathcal{B})$.

Proof. The uniqueness of such σ_r follows from the surjectivity of π_r. Consider the right Hilbert $(B^\sigma \otimes A)$-module $H_B \otimes A$. Denote by $B(H_B \otimes A)$ the C^*-algebra of adjointable operators of the Hilbert $(B^\sigma \otimes A)$-module $H_B \otimes A$. Then $B_r \otimes A \subseteq B(H_B \otimes A)$. The argument in the proof of [5, Lemma 3] shows that there is a unitary $U \in B(H_B \otimes A)$ satisfying $U(b \otimes a) = ((\pi_r \otimes \mathrm{id})(\sigma(b)))(1_B \otimes a)$ for all $a \in A, b \in B$. It follows that $U(\pi_r(b) \otimes 1_A) = ((\pi_r \otimes \mathrm{id})(\sigma(b)))U$ for all $b \in B$. Thus $U(B_r \otimes 1_A)U^{-1} \subseteq B_r \otimes A$. Define $\sigma_r : B_r \to B_r \otimes A$ by $\sigma_r(b') = U(b' \otimes 1_A)U^{-1}$. Then (14) follows. Clearly σ_r is injective. Since σ is an action of $\mathcal{G}$ on B, it follows easily that σ_r is an action of $\mathcal{G}$ on B_r. The equivariance of π_r follows from (14). By Lemma 3.2, $\pi_r(\mathcal{B}) = \mathcal{B}_r$. It is clear that π_r is injective on B^σ. Thus by Lemma 3.2 the map π_r is injective on $\mathcal{B}$. ☐

Definition 3.5. We call the action (B_u, σ_u) in Proposition 3.3 the *full action* associated to (B, σ), and call the action (B_r, σ_r) in Proposition 3.4 the *reduced action* associated to (B, σ). The action (B, σ) is said to be *full* (*reduced*, *co-amenable*, resp.) if π_u (π_r, both π_u and π_r, resp.) is an isomorphism.

Example 3.6.
(1) When B is finite dimensional, the action (B, σ) is co-amenable. This applies to the actions constructed in [40] and the adjoint action on $B(H)$ associated to any finite-dimensional representation of $\mathcal{G}$ on H [41, notation after Theorem 2.5].
(2) Consider the Cuntz algebra $\mathcal{O}_n$ [6] for an integer $n \geq 2$, that is, the universal C^*-algebra generated by isometries $S_1, \cdots, S_n$ satisfying $\sum_{j=1}^n S_jS_j^* = 1$. Since $\mathcal{O}_n$ is simple, any action of a compact quantum group on $\mathcal{O}_n$ is reduced. Given a compact quantum group $A = C(\mathcal{G})$ and an n-dimensional unitary representation $u = (u_{ij})_{ij}$ of $\mathcal{G}$, one has an action σ of $\mathcal{G}$ on $\mathcal{O}_n$

determined by $\sigma(S_i) = \sum_{j=1}^{n} S_j \otimes u_{ji}$ for all $1 \leq i \leq n$ [18, Theorem 1]. The regular subalgebra $\mathcal{B}$ for this action σ contains $S_1, \cdots, S_n$, thus σ is full and hence is co-amenable, because of the universal property of $\mathcal{O}_n$. This kind of actions has been considered for $\mathcal{G}$ being $\mathrm{SU}_q(2)$ [18, 24], $\mathrm{SU}_q(N)$ [28], and $A_u(Q)$ [41, Section 5].

(3) For the action $\Phi : A \to A \otimes A$ of $\mathcal{G}$ on A, the C^*-algebra for the associated full action is the C^*-algebra of the full quantum group [3, Section 3], while the C^*-algebra for the associated reduced action is the C^*-algebra of the reduced quantum group [3, Section 2]. Thus the action (A, Φ) is full (reduced resp.) exactly if $\mathcal{G}$ is a full (reduced resp.) compact quantum group.

Remark 3.7. Having isomorphic $(\mathcal{B}, \sigma|_{\mathcal{B}})$ is an equivalence relation between actions of $\mathcal{G}$ on unital C^*-algebras. Two actions are equivalent in this sense exactly if they have isomorphic full actions, exactly if they have isomorphic reduced actions. If (A_1, Φ_1) is another compact quantum group with $(\mathcal{A}_1, \Phi_1|_{\mathcal{A}_1})$ isomorphic to $(\mathcal{A}, \Phi|_{\mathcal{A}})$, then A_1 has also a natural action on B_u. Thus the class of the equivalence classes of actions of $\mathcal{G}$ depends only on $(\mathcal{A}, \Phi|_{\mathcal{A}})$.

Proposition 3.8. The following are equivalent:

(1) $\mathcal{G}$ is co-amenable,
(2) every action of $\mathcal{G}$ on a unital C^*-algebra is co-amenable,
(3) every ergodic action of $\mathcal{G}$ on a unital C^*-algebra is co-amenable.

Proof. (1)⇒(2). Let σ be an action of $\mathcal{G}$ on a unital C^*-algebra B. Then (B_r, σ_r) is also the reduced action associated to (B_u, σ_u). By Lemma 3.1 E is faithful on B_u. Thus the canonical homomorphism $B_u \to B_r$ is injective, and hence is an isomorphism. Therefore (B, σ) is co-amenable.

(2)⇒(3). This is trivial.

(3)⇒(1). This follows from Example 3.6(3). □

4. Ergodic actions

In this section we introduce a topology on the set of isomorphism classes of ergodic actions of $\mathcal{G}$ in Definition 4.3 and prove Theorem 4.4. At the end of this section we also discuss the behavior of this topology under taking Cartesian products of compact quantum groups.

Notation 4.1. Denote by $\mathrm{EA}(\mathcal{G})$ the set of isomorphism classes of ergodic actions of $\mathcal{G}$. Denote by $\mathrm{EA}^{\sim}(\mathcal{G})$ the quotient space of $\mathrm{EA}(\mathcal{G})$ modulo the equivalence relation in Remark 3.7.

What we shall do is to define a topology on $\mathrm{EA}^{\sim}(\mathcal{G})$, then pull it back to a topology on $\mathrm{EA}(\mathcal{G})$. For each $\gamma \in \hat{\mathcal{G}}$, let M_γ be the quantum dimension defined after Theorem 5.4 in [47]. One knows that M_γ is a positive number no less than d_γ and that $M_{\gamma_0} = 1$. Set N_γ to be the largest integer no bigger than M_γ^2/d_γ. Let (B, σ) be an ergodic action of $\mathcal{G}$. According to [5, Theorem 17], one has $\mathrm{mul}(B, \gamma) \leq N_\gamma$ for each $\gamma \in \hat{\mathcal{G}}$ (the assumption on the injectivity of σ in [5] is not used in the proof of Theorem 17 therein; this can be also seen by passing to the associated reduced action in Proposition 3.4 for which σ_r is always injective).

The pair $(\mathcal{B}, \sigma|_{\mathcal{B}})$ consists of the $*$-algebra $\mathcal{B}$ and the action $\sigma|_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \odot \mathcal{A}$. For each $\gamma \in \hat{\mathcal{G}}$, one has a linear basis $\mathscr{S}_\gamma$ of B_γ satisfying (9), where we take J_γ to be $\{1, \cdots, \mathrm{mul}(B, \gamma)\}$. If we choose such a basis $\mathscr{S}_\gamma$ for each $\gamma \in \hat{\mathcal{G}}$, then the action $\sigma|_{\mathcal{B}}$ is fixed by (9) and the pair $(\mathcal{B}, \sigma|_{\mathcal{B}})$ is determined by the $*$-algebra structure on $\mathcal{B}$ which in turn can be determined by the coefficients appearing in the multiplication and $*$-operation rules on these basis elements. In order to reduce the set of possible coefficients appearing this way, we put one more restriction on $\mathscr{S}_\gamma$. By the argument on [5, page 103], one can require $\mathscr{S}_\gamma$ to be an orthonormal basis of B_γ with respect to the inner product $\langle x, y\rangle = \omega(x^* y)$, that is,

$$\omega(e^*_{\gamma s j} e_{\gamma k i}) = \delta_{sk}\delta_{ji}. \tag{15}$$

We can always choose $e_{\gamma_0 11} = 1_B$. We shall call a basis $\mathscr{S}_\gamma$ satisfying all these conditions a *standard basis* of B_γ, and call the union $\mathscr{S}$ of a standard basis for each B_γ a standard basis of B.

Notation 4.2. Set

$$\hat{\mathcal{G}}^\flat = \hat{\mathcal{G}} \setminus \{\gamma_0\}, \quad \mathscr{M} = \{(\alpha, \beta, \gamma) \in \hat{\mathcal{G}}^\flat \times \hat{\mathcal{G}}^\flat \times \hat{\mathcal{G}} : \gamma \preceq \alpha \circledT \beta\}.$$

For each $\gamma \in \hat{\mathcal{G}}$, set

$$\begin{aligned} X_\gamma &= \{(\gamma, k, i) : 1 \leq k \leq N_\gamma, 1 \leq i \leq d_\gamma\}, \\ X'_\gamma &= \{(\gamma, k, i) : 1 \leq k \leq \mathrm{mul}(B, \gamma), 1 \leq i \leq d_\gamma\}. \end{aligned}$$

Denote by x_0 the unique element $(\gamma_0, 1, 1)$ in X_{γ_0}. Set

$$\begin{aligned} Y &= \cup_{(\alpha,\beta,\gamma)\in\mathscr{M}} X_\alpha \times X_\beta \times X_\gamma, \qquad & Z &= \cup_{\gamma\in\hat{\mathcal{G}}^\flat} X_\gamma \times X_{\gamma^c}, \\ Y' &= \cup_{(\alpha,\beta,\gamma)\in\mathscr{M}} X'_\alpha \times X'_\beta \times X'_\gamma, \qquad & Z' &= \cup_{\gamma\in\hat{\mathcal{G}}^\flat} X'_\gamma \times X'_{\gamma^c}. \end{aligned}$$

Fix a standard basis $\mathscr{S}$ of $\mathcal{B}$. Since we have chosen e_{x_0} to be 1_B, the algebra structure of $\mathcal{B}$ is determined by the linear expansion of $e_{x_1}e_{x_2}$ for all $x_1 \in X'_\alpha, x_2 \in X'_\beta, \alpha, \beta \in \hat{\mathcal{G}}^\flat$. By (8) we have $e_{x_1}e_{x_2} \in \sum_{\gamma \preceq \alpha \circledT \beta} B_\gamma$. Thus the coefficients of the expansion of $e_{x_1}e_{x_2}$ under $\mathscr{S}$ for all such

x_1, x_2 determine a scalar function on Y', that is, there exists a unique element $f \in \mathbb{C}^{Y'}$ such that for any $x_1 \in X'_\alpha, x_2 \in X'_\beta, \alpha, \beta \in \hat{\mathcal{G}}^\flat$,

$$e_{x_1} e_{x_2} = \sum_{(x_1,x_2,x_3)\in Y'} f(x_1,x_2,x_3) e_{x_3}. \tag{16}$$

Similarly, the $*$-structure of $\mathcal{B}$ is determined by the linear expansion of e_{x_1} for all $x_1 \in X'_\gamma, \gamma \in \hat{\mathcal{G}}^\flat$. By (8) we have $e^*_{x_1} \in B_{\gamma^c}$. Thus there exists a unique element $g \in \mathbb{C}^{Z'}$ such that for any $x_1 \in X'_\gamma, \gamma \in \hat{\mathcal{G}}^\flat$,

$$e^*_{x_1} = \sum_{(x_1,x_2)\in Z'} g(x_1,x_2) e_{x_2}. \tag{17}$$

Then (f, g) determines the isomorphism class of $(\mathcal{B}, \sigma|_{\mathcal{B}})$ and hence determines the equivalence class of (B, σ) in $\mathrm{EA}^\sim(\mathcal{G})$. Note that (f, g) does not determine the isomorphism class of (B, σ) in $\mathrm{EA}(\mathcal{G})$ unless (B, σ) is co-amenable. Since we are going to consider all ergodic actions of $\mathcal{G}$ in a uniform way, we extend f and g to functions on Y and Z respectively by

$$f|_{Y\setminus Y'} = 0, \quad g|_{Z\setminus Z'} = 0. \tag{18}$$

We shall say that (f, g) is the element in $\mathbb{C}^Y \times \mathbb{C}^Z$ associated to $\mathscr{S}$.

Denote by $\mathcal{P}$ the set of (f, g) in $\mathbb{C}^Y \times \mathbb{C}^Z$ associated to various bases of ergodic actions of $\mathcal{G}$. We say that (f_1, g_1) and (f_2, g_2) in $\mathcal{P}$ are *equivalent* if they are associated to standard bases of (B_1, σ_1) and (B_2, σ_2) respectively such that $(\mathcal{B}_1, \sigma_1|_{\mathcal{B}_1})$ and $(\mathcal{B}_2, \sigma_2|_{\mathcal{B}_2})$ are isomorphic. Then this is an equivalence relation on $\mathcal{P}$ and we can identify the quotient space of $\mathcal{P}$ modulo this equivalence relation with $\mathrm{EA}^\sim(\mathcal{G})$ naturally.

Definition 4.3. Endow $\mathbb{C}^Y \times \mathbb{C}^Z$ with the product topology. Define the topology on $\mathcal{P}$ as the relative topology, and define the topology on $\mathrm{EA}^\sim(\mathcal{G})$ as the quotient topology from $\mathcal{P} \to \mathrm{EA}^\sim(\mathcal{G})$. Also define the topology on $\mathrm{EA}(\mathcal{G})$ via setting the open subsets in $\mathrm{EA}(\mathcal{G})$ as inverse image of open subsets in $\mathrm{EA}^\sim(\mathcal{G})$ under the quotient map $\mathrm{EA}(\mathcal{G}) \to \mathrm{EA}^\sim(\mathcal{G})$.

Theorem 4.4. Both $\mathcal{P}$ and $\mathrm{EA}^\sim(\mathcal{G})$ are compact Hausdorff spaces. The space $\mathrm{EA}(\mathcal{G})$ is also compact, but it is Hausdorff if and only if $\mathcal{G}$ is co-amenable. Both quotient maps $\mathcal{P} \to \mathrm{EA}^\sim(\mathcal{G})$ and $\mathrm{EA}(\mathcal{G}) \to \mathrm{EA}^\sim(\mathcal{G})$ are open.

Remark 4.5. The equation (9) depends on the identification of $B(H_\gamma)$ with $M_{d_\gamma}(\mathbb{C})$, which in turn depends on the choice of an orthonormal basis of H_γ. Then $\mathcal{P}$ also depends on such choice. However, using Lemma 4.11 below one can show directly that the quotient topology on $\mathrm{EA}^{\sim}(\mathcal{G})$ does not depend on such choice. This will also follow from Corollary 5.16 below.

In order to prove Theorem 4.4, we need to characterize $\mathcal{P}$ and its equivalence relation more explicitly. We start with characterizing $\mathcal{P}$, that is, we consider which elements of $\mathbb{C}^Y \times \mathbb{C}^Z$ come from standard bases of ergodic actions of $\mathcal{G}$. For this purpose, we take $f(y)$ and $g(z)$ for $y \in Y, z \in Z$ as variables and try to find algebraic conditions they should satisfy in order to construct $(\mathcal{B}, \sigma|_{\mathcal{B}})$. Set

$$X = \cup_{\gamma \in \hat{\mathcal{G}}^\flat} X_\gamma, \quad \text{and} \quad X_0 = \cup_{\gamma \in \hat{\mathcal{G}}} X_\gamma.$$

Let $\mathscr{V}$ be a vector space with basis $\{v_x : x \in X_0\}$. We hope to construct $\mathcal{B}$ out of $\mathscr{V}$ such that v_x becomes e_x. Corresponding to (16)-(18) we want to make $\mathscr{V}$ into a $*$-algebra with identity v_{x_0} satisfying

$$v_{x_1} v_{x_2} = \sum_{(x_1, x_2, x_3) \in Y} f(x_1, x_2, x_3) v_{x_3} \tag{19}$$

for any $x_1, x_2 \in X$, and

$$v_{x_1}^* = \sum_{(x_1, x_2) \in Z} g(x_1, x_2) v_{x_2}, \tag{20}$$

for any $x_1 \in X$. Corresponding to (9), we also want a unital $*$-homomorphism $\sigma_{\mathscr{V}} : \mathscr{V} \to \mathscr{V} \odot \mathcal{A}$ satisfying

$$\sigma_{\mathscr{V}}(v_{\gamma k i}) = \sum_{1 \leq j \leq d_\gamma} v_{\gamma k j} \otimes u_{ji}^\gamma \tag{21}$$

for $(\gamma, k, i) \in X$. Thus consider the equations

$$(v_{x_1} v_{x_2}) v_{x_3} = v_{x_1}(v_{x_2} v_{x_3}), \qquad (v_{x_1}^*)^* = v_{x_1}, \quad (v_{x_1} v_{x_2})^* = v_{x_2}^* v_{x_1}^*,$$
$$\sigma_{\mathscr{V}}(v_{x_1} v_{x_2}) = \sigma_{\mathscr{V}}(v_{x_1}) \sigma_{\mathscr{V}}(v_{x_2}), \qquad (\sigma_{\mathscr{V}}(v_{x_1}))^* = \sigma_{\mathscr{V}}(v_{x_1}^*)$$

for all $x_1, x_2, x_2 \in X$. Expanding both sides of these equations formally using (19)-(21) and identifying the corresponding coefficients, we get a set $\mathscr{E}_1$ of polynomial equations in the variables $f(y), g(z)$ and their conjugates for $y \in Y, z \in Z$. For any $(f, g) \in \mathbb{C}^Y \times \mathbb{C}^Z$ satisfying $\mathscr{E}_1$, we have a conjugate-linear map $* : \mathscr{V} \to \mathscr{V}$ specified by (20). Set $I_{f,g}$ to be the kernel of $*$, and set $\mathscr{V}_{f,g} = \mathscr{V}/I_{f,g}$. Denote the quotient map $\mathscr{V} \to \mathscr{V}_{f,g}$ by $\phi_{f,g}$, and denote $\phi_{f,g}(v_x)$ by ν_x for $x \in X_0$. Then the formulas

$$\nu_{x_1} \nu_{x_2} = \sum_{(x_1, x_2, x_3) \in Y} f(x_1, x_2, x_3) \nu_{x_3}, \tag{22}$$

$$\nu_{x_1}^* = \sum_{(x_1, x_2) \in Z} g(x_1, x_2) \nu_{x_2}, \tag{23}$$

$$\sigma_{f,g}(\nu_{\gamma ki}) = \sum_{1\le j\le d_\gamma} \nu_{\gamma kj}\otimes u^\gamma_{ji} \tag{24}$$

corresponding to (19)-(21) determine a unital $*$-algebra structure of $\mathscr{V}_{f,g}$ with the identity ν_{x_0} and a unital $*$-homomorphism $\sigma_{f,g}:\mathscr{V}_{f,g}\to\mathscr{V}_{f,g}\odot\mathcal{A}$.

In order to make sure that (f,g) is associated to some standard basis of some ergodic action of $\mathcal{G}$, we need to also take care of (15). Note that $\omega|_{\mathcal{B}}$ is simply to take the coefficient at 1_B. For any $\gamma\in\hat{\mathcal{G}}^\flat$ and any $x_1,x_2\in X_\gamma$, expand $v^*_{x_2}v_{x_1}$ formally using (20) and (19) and denote by F_{x_1,x_2} the coefficient at $1_{\mathscr{V}}$. Then we want the existence of a non-negative integer $m_{\gamma,f,g}\le N_\gamma$ for each $\gamma\in\hat{\mathcal{G}}^\flat$, which one expects to be $\mathrm{mul}(B,\gamma)$, such that the value of $\mathrm{F}_{\gamma sj,\gamma ki}$ at (f,g) is $\delta_{sk}\delta_{ji}$ or 0 depending on $s,k\le m_{\gamma,f,g}$ or not. This condition can be expressed as the set $\mathscr{E}_2$ of the equations $\mathrm{F}_{x_1,x_2}=0$ for all $x_1,x_2\in X_\gamma$ with $x_1\neq x_2$, the equations $\mathrm{F}_{\gamma si,\gamma si}=\mathrm{F}_{\gamma sj,\gamma sj}$ for all $1\le i,j\le d_\gamma, 1\le s\le N_\gamma$, and the equations $\mathrm{F}_{\gamma s1,\gamma s1}\mathrm{F}_{\gamma k1,\gamma k1}=\mathrm{F}_{\gamma s1,\gamma s1}$ for all $1\le k\le s\le N_\gamma$ (and for all $\gamma\in\hat{\mathcal{G}}^\flat$). We also need to take care of (18). Thus denote by $\mathscr{E}_3$ the set of equations

$$f(x_1,x_2,x_3)=f(x_1,x_2,x_3)\mathrm{F}_{x_1,x_1}=f(x_1,x_2,x_3)\mathrm{F}_{x_2,x_2}=f(x_1,x_2,x_3)\mathrm{F}_{x_3,x_3}$$

for all $(x_1,x_2,x_3)\in Y$ (the last equation is vacuous when $x_3=x_0$), and the equations

$$g(x_1,x_2)=g(x_1,x_2)\mathrm{F}_{x_1,x_1}=g(x_1,x_2)\mathrm{F}_{x_2,x_2}$$

for all $(x_1,x_2)\in Z$.

Notation 4.6. Denote by $\mathscr{E}$ the union of $\mathscr{E}_1,\mathscr{E}_2$ and $\mathscr{E}_3$.

Clearly every element in $\mathcal{P}$ satisfies $\mathscr{E}$. This proves part of the following characterization of $\mathcal{P}$:

Proposition 4.7. *$\mathcal{P}$ is exactly the set of elements in $\mathbb{C}^Y\times\mathbb{C}^Z$ satisfying $\mathscr{E}$.*

Let $(f,g)\in\mathbb{C}^Y\times\mathbb{C}^Z$ satisfy $\mathscr{E}$. Set

$$X_{f,g}=\{(\gamma,s,i)\in X: 1\le s\le m_{\gamma,f,g}\},$$

which one expects to parameterize $\mathscr{S}\setminus\{e_{x_0}\}$. Since (f,g) satisfies $\mathscr{E}_3$, $\mathrm{span}\{v_x: x\in X\setminus X_{f,g}\}\subseteq I_{f,g}$. Thus ν_x's for $x\in X_{f,g}\cup\{x_0\}$ span $\mathscr{V}_{f,g}$. Clearly $\mathscr{V}_{f,g}$ is the direct sum of $\mathbb{C}\nu_{x_0}$ and $\mathrm{span}\{\nu_x: x\in X_\gamma\}$ for all $\gamma\in\hat{\mathcal{G}}^\flat$. Thus it makes sense to talk about the coefficient of ν at ν_{x_0} for any $\nu\in\mathscr{V}_{f,g}$. This defines a linear functional $\varphi_{f,g}$ on $\mathscr{V}_{f,g}$, which one expects to be ω. Clearly

$$\varphi_{f,g}(\cdot)\nu_{x_0}=(\mathrm{id}\otimes h)\sigma_{f,g}(\cdot) \tag{25}$$

on $\mathscr{V}_{f,g}$.

Lemma 4.8. Let $(f,g) \in \mathbb{C}^Y \times \mathbb{C}^Z$ satisfy $\mathscr{E}$. Then $\mathscr{V}_{f,g}$ has a universal C^*-algebra $B_{f,g}$. The canonical $*$-homomorphism $\mathscr{V}_{f,g} \to B_{f,g}$ is injective. Identifying $\mathscr{V}_{f,g}$ with its canonical image in $B_{f,g}$ one has

$$\|\nu_x\| \leq \sqrt{\|F_\gamma\| M_\gamma} \tag{26}$$

for any $x = (\gamma, k, i) \in X_{f,g}$, where F_γ denotes the element in $M_{d_\gamma}(\mathbb{C})$ defined after Theorem 5.4 in [47]. The set $\mathscr{S} := \{\nu_x : x \in X_{f,g} \cup \{x_0\}\}$ is a linear basis of $\mathscr{V}_{f,g}$.

Proof. We show first that for each $\nu \in \mathscr{V}_{f,g}$ there exists some $c_\nu \in \mathbb{R}$ such that $\|\pi(\nu)\| \leq c_\nu$ for any $*$-representation π of $\mathscr{V}_{f,g}$. Recalling that $\mathscr{S}$ spans $\mathscr{V}_{f,g}$, it suffices to prove the claim for $\nu = \nu_x$ for every $x \in X_{f,g}$. Say $x = (\gamma, k, i)$. Set

$$\begin{aligned} W_{\gamma k} &= F_\gamma^{-\frac{1}{2}} (\nu_{\gamma k 1}, \cdots, \nu_{\gamma k d_\gamma})^T \in M_{d_\gamma \times 1}(\mathscr{V}_{f,g}), \\ W'_{\gamma k} &= (W_{\gamma k}, 0, \cdots, 0) \in M_{d_\gamma \times d_\gamma}(\mathscr{V}_{f,g}). \end{aligned}$$

Note that $\{\nu \in \mathscr{V}_{f,g} : \sigma_{f,g}(\nu) = \nu \otimes 1_A\} = \mathbb{C}\nu_{x_0}$. The argument in [5, page 103] shows that

$$W^*_{\gamma k} W_{\gamma s} = \delta_{ks} M_\gamma \nu_{x_0} \tag{27}$$

for all $1 \leq k, s \leq m_{\gamma,f,g}$. Thus for any $*$-representation π of $\mathscr{V}_{f,g}$ we have $\|\pi(W'_{\gamma k})\| \leq \sqrt{M_\gamma}$ and hence

$$\|\pi(\nu_x)\| \leq \|F_\gamma^{\frac{1}{2}}\| \sqrt{M_\gamma} = \sqrt{\|F_\gamma\| M_\gamma}. \tag{28}$$

Next we show that $\mathscr{V}_{f,g}$ does have a $*$-representation. By [47, Theorem 5.7] one has $h(A^*_\alpha A_\beta) = 0$ for any $\alpha \neq \beta \in \hat{\mathcal{G}}$. Using (25) one sees that

$$\varphi_{f,g}(\nu^*_{x_2} \nu_{x_1}) = 0 \tag{29}$$

for all $x_1 \in X_\alpha, x_2 \in X_\beta, \alpha \neq \beta$. Using (29) and the assumption that (f,g) satisfies $\mathscr{E}_2$, one observes that

$$\varphi_{f,g}(\nu^* \nu) = \sum_{x \in X_{f,g} \cup \{x_0\}} |\lambda_x|^2 \geq 0 \tag{30}$$

for any $\nu = \sum_{x \in X_0} \lambda_x \nu_x \in \mathscr{V}_{f,g}$. Denote by H the Hilbert space completion of $\mathscr{V}_{f,g}$ with respect to the inner product $\langle \nu_1, \nu_2 \rangle = \varphi_{f,g}(\nu_1^* \nu_2)$, and by $H^{(d_\gamma)}$ the direct sum of d_γ copies of H. By (27) the multiplication by $W'_{\gamma s}$ extends to a bounded operator on $H^{(d_\gamma)}$. Then so does the multiplication by $((\nu_{\gamma k 1}, \cdots, \nu_{\gamma k d_\gamma})^T, 0, \cdots, 0) \in M_{d_\gamma \times d_\gamma}(\mathscr{V}_{f,g})$. Consequently, the multiplication by ν_x for $x = (\gamma, k, i) \in X_{f,g}$ extends to a bounded operator on H. Since $\mathscr{S}$ spans $\mathscr{V}_{f,g}$, the multiplication of $\mathscr{V}_{f,g}$ extends to a $*$-representation π of $\mathscr{V}_{f,g}$ on H.

Now we conclude that $\mathscr{V}_{f,g}$ has a universal C^*-algebra $B_{f,g}$. It follows from (30) that $\pi \circ \phi_{f,g}$ is injective on $\mathrm{span}\{v_x : x \in X_{f,g} \cup \{x_0\}\}$, where $\phi_{f,g} : \mathscr{V} \to \mathscr{V}_{f,g}$ is the quotient map. Thus $\mathscr{S}$ is a linear basis of $\mathscr{V}_{f,g}$, and the canonical $*$-homomorphism $\mathscr{V}_{f,g} \to B_{f,g}$ must be injective. The inequality (26) follows from (28). $\square$

For (f, g) as in Lemma 4.8, by the universality of $B_{f,g}$, the $*$-homomorphism $\mathscr{V}_{f,g} \overset{\sigma_{f,g}}{\to} \mathscr{V}_{f,g} \odot \mathcal{A} \hookrightarrow B_{f,g} \otimes A$ extends uniquely to a (unital) $*$-homomorphism $B_{f,g} \to B_{f,g} \otimes A$, which we still denote by $\sigma_{f,g}$.

Proposition 4.9. Let (f, g) be as in Lemma 4.8. Then $\sigma_{f,g}$ is an ergodic action of $\mathcal{G}$ on $B_{f,g}$. The algebra of regular functions for this action is $\mathscr{V}_{f,g}$. The set $\mathscr{S}$ is a standard basis of $\mathscr{V}_{f,g}$. The element in $\mathcal{P}$ associated to this basis is exactly (f, g).

Proof. By Lemma 4.8, $\mathscr{S}$ is a basis of $\mathscr{V}_{f,g}$. By (24) and the characterization of actions of $\mathcal{G}$ in terms of elements satisfying (9) in subsection 2.1, $\sigma_{f,g}$ is an ergodic action of $\mathcal{G}$ on $B_{f,g}$, and $(B_{f,g})_\gamma = \mathrm{span}\{\nu_x : x = (\gamma, k, i) \in X_{f,g}\}$, $\mathrm{mul}(B_{f,g}, \gamma) = m_{\gamma,f,g}$ for all $\gamma \in \hat{\mathcal{G}}^\flat$. Thus $\mathscr{V}_{f,g}$ is the algebra of regular functions. Denote by ω the unique $\mathcal{G}$-invariant state on $B_{f,g}$. By (25) ω extends $\varphi_{f,g}$. Since (f, g) satisfies $\mathscr{E}_2$, we have $\omega(\nu_x^* \nu_y) = \delta_{xy}$ for any $x = (\gamma, k, i), y = (\gamma, s, j) \in X_{f,g}$. Thus $\mathscr{S}$ is a standard basis of $\mathcal{B}_{f,g}$. Clearly the element in $\mathcal{P}$ associated to this basis is exactly (f, g). $\square$

Now Proposition 4.7 follows from Proposition 4.9.

We are ready to prove the compactness of $\mathcal{P}$.

Lemma 4.10. Let $(f, g) \in \mathcal{P}$. Then

$$|f(x_1, x_2, x_3)| \leq \sqrt{\|F_\alpha\| M_\alpha} \tag{31}$$

for any $(x_1, x_2, x_3) \in Y, x_1 \in X_\alpha$. And

$$|g(x_1, x_2)| \leq \sqrt{\|F_\alpha\| M_\alpha} \tag{32}$$

for any $(x_1, x_2) \in Z, x_1 \in X_\alpha$. The space $\mathcal{P}$ is compact.

Proof. Say, (f, g) is associated to a standard basis $\mathscr{S}$ for an ergodic action (B, σ) of $\mathcal{G}$. Let (H_B, π_r) be the GNS representation associated to the unique σ-invariant state ω of B. Then B_α and B_β are orthogonal to each other in H_B for distinct $\alpha, \beta \in \hat{\mathcal{G}}$ [5, Corollary 12]. In view of (15), $\mathscr{S}$ is an orthonormal basis of H_B. We may identify $\mathcal{B}$ with $\mathscr{V}_{f,g}$ naturally via $e_x \leftrightarrow \nu_x$. Then there is a $*$-homomorphism from $B_{f,g}$ in Lemma 4.8 to B extending this identification. Thus by (26) we have $\|e_x\| \leq \sqrt{\|F_\alpha\| M_\alpha}$ for any $x \in X'_\alpha, \alpha \in \hat{\mathcal{G}}$. For any $(x_1, x_2, x_3) \in Y', x_1 \in X'_\alpha$, by (16),

$$|f(x_1, x_2, x_3)| = |\langle e_{x_3}, e_{x_1} e_{x_2} \rangle| \leq \|e_{x_1}\| \leq \sqrt{\|F_\alpha\| M_\alpha}.$$

If $y \in Y \setminus Y'$, then $f(y) = 0$ by (18). This proves (31). The inequality (32) is proved similarly.

By Proposition 4.7 the space $\mathcal{P}$ is closed in $\mathbb{C}^Y \times \mathbb{C}^Z$. It follows from (31) and (32) that $\mathcal{P}$ is compact. □

Next we characterize the equivalence relation on $\mathcal{P}$. For this purpose, we need to consider the relation between two standard bases of $\mathcal{B}$. The argument in the proof of [30, Theorem 1.5] shows the first two assertions of the following lemma:

Lemma 4.11. Let $\gamma \in \hat{\mathcal{G}}$. If $b_i \in B, 1 \leq i \leq d_\gamma$ satisfy

$$\sigma(b_i) = \sum_{1 \leq j \leq d_\gamma} b_j \otimes u^\gamma_{ji} \tag{33}$$

for all $1 \leq i \leq d_\gamma$, then $b_i = E^\gamma_{i1}(b_1)$ (see (5)) for all $1 \leq i \leq d_\gamma$. Conversely, given $b \in E^\gamma_{11}(B)$, if we set $b_i = E^\gamma_{i1}(b)$, then $b_i \in B_\gamma, 1 \leq i \leq d_\gamma$ satisfy (33), and $b_1 = b$. For any $b_1, \cdots, b_{d_\gamma}$ ($b'_1, \cdots, b'_{d_\gamma}$ resp.) in B satisfying (33) ((33) with b_i replaced by b'_i resp.) we have

$$\omega(b^*_j b'_i) = \delta_{ji} \omega(b^*_1 b'_1) \tag{34}$$

for all $1 \leq i, j \leq d_\gamma$.

Proof. We just need to prove (34). By [47, Theorem 5.7] we have

$$h({u^\gamma_{lj}}^* u^\gamma_{ni}) = \frac{1}{M_\gamma} f_{-1}(u^\gamma_{nl}) \delta_{ji}, \tag{35}$$

where f_{-1} is the linear functional on $\mathcal{A}$ defined in [47, Theorem 5.6]. Thus

$$\begin{aligned} \omega(b^*_j b'_i) 1_B &= (\mathrm{id} \otimes h)(\sigma(b^*_j b'_i)) \overset{(33)}{=} (\mathrm{id} \otimes h)(\sum_{1 \leq l, n \leq d_\gamma} b^*_l b'_n \otimes {u^\gamma_{lj}}^* u^\gamma_{ni}) \\ &\overset{(35)}{=} \sum_{1 \leq l, n \leq d_\gamma} b^*_l b'_n \frac{1}{M_\gamma} f_{-1}(u^\gamma_{nl}) \delta_{ji}. \end{aligned}$$

Therefore

$$\omega(b^*_j b'_i) 1_B = \delta_{ji} \omega(b^*_1 b'_1) 1_B,$$

which proves (34). □

By Lemma 4.11, for $\gamma \in \hat{\mathcal{G}}^\flat$, there is a 1-1 correspondence between standard bases of B_γ and orthonormal bases of $E^\gamma_{11}(B)$ with respect to the inner product $\langle b, b'\rangle = \omega(b^*b')$. It also follows from Lemma 4.11 that $\dim(E^\gamma_{11}(B)) = \mathrm{mul}(B,\gamma)$. Denote by $\mathcal{U}_n$ the unitary group of $M_n(\mathbb{C})$. Then $\prod_{\gamma\in\hat{\mathcal{G}}^\flat}\mathcal{U}_{\mathrm{mul}(B,\gamma)}$ has a right free transitive action on the set of standard bases of $\mathcal{B}$ via acting on the set of orthonormal bases of $E^\gamma_{11}(B)$ for each $\gamma \in \hat{\mathcal{G}}^\flat$. For $n \leq m$ identify $\mathcal{U}_n$ with the subgroup of $\mathcal{U}_m$ consisting of elements with 1_{m-n} at the lower-right corner. Denote $\prod_{\gamma\in\hat{\mathcal{G}}^\flat}\mathcal{U}_{N_\gamma}$ by $\mathcal{U}$, equipped with the product topology. Then $\mathcal{U}$ has a natural partial right (not necessarily free) action τ on $\mathcal{P}$, that is, $\xi \in \mathcal{U}$ acts at $t \in \mathcal{P}$ exactly if $\xi \in \prod_{\gamma\in\hat{\mathcal{G}}^\flat}\mathcal{U}_{m_{\gamma,t}}$, where $m_{\gamma,t}$ was defined in the paragraph before Notation 4.6, and the image $t{\cdot}\xi$ is the element in $\mathcal{P}$ associated to the standard basis $\mathscr{S}{\cdot}\xi$ of $\mathcal{B}_t$, where $\mathscr{S}{\cdot}\xi$ is the image of the action of ξ at the standard basis $\mathscr{S}$ of $\mathcal{B}_t$ in Proposition 4.9. Clearly the orbits of this partial action are exactly the fibres of the quotient map $\mathcal{P} \to \mathrm{EA}^\sim(\mathcal{G})$, equivalently, exactly the equivalence classes in $\mathcal{P}$ introduced before Definition 4.3. Thus we may identify $\mathrm{EA}^\sim(\mathcal{G})$ with the quotient space $\mathcal{P}/\mathcal{U}$.

Lemma 4.12. The quotient map $\mathcal{P} \to \mathcal{P}/\mathcal{U}$ is open. The quotient topology on $\mathcal{P}/\mathcal{U}$ is compact Hausdorff.

Proof. Denote by π the quotient map $\mathcal{P} \to \mathcal{P}/\mathcal{U}$. To show the openness of π, it suffices to show that $\pi^{-1}(\pi(V))$ is open for every open subset V of $\mathcal{P}$. Let $t \in V$ and $\xi \in \mathcal{U}$ such that $t{\cdot}\xi$ is defined. Say $\xi = (\xi_\gamma)_{\gamma\in\hat{\mathcal{G}}^\flat}$. Let J be a finite subset of $\hat{\mathcal{G}}$. Replacing ξ_γ by 1_{N_γ} for $\gamma \in \hat{\mathcal{G}}^\flat \setminus J$ we get an element $\xi' \in \mathcal{U}$. Notice that when $t' \in \mathcal{P}$ is close enough to $t{\cdot}\xi$, $t'{\cdot}(\xi')^{-1}$ is defined. Moreover, the restrictions of $t'{\cdot}(\xi')^{-1}$ on $(X_J \times X_J \times X_J)\cap Y$ and $(X_J \times X_J)\cap Z$ converge to the restrictions of t as t' converges to $t{\cdot}\xi$, where $X_J = \cup_{\gamma\in J}X_\gamma$. Clearly we can find a large enough finite subset J of $\hat{\mathcal{G}}$ such that when t' is close enough to $t\cdot\xi$, the element $t'\cdot(\xi')^{-1}$ is in V. Then $t' = (t'{\cdot}(\xi')^{-1}){\cdot}\xi'$ is in $\pi^{-1}(\pi(V))$. Therefore $\pi^{-1}(\pi(V))$ is open, and hence π is open.

Denote by $\mathbb{D}$ the domain of τ, i.e., the subset of $\mathcal{P}\times\mathcal{U}$ consisting of elements (t,ξ) for which $t{\cdot}\xi$ is defined. From the equations in $\mathscr{E}_2$ it is clear that $\mathbb{D}$ is closed in $\mathcal{P}\times\mathcal{U}$. By Lemma 4.10 the space $\mathcal{P}$ is compact. Since $\mathcal{U}$ is also compact, so is $\mathbb{D}$. It is also clear that τ is continuous in the sense that the map $\mathbb{D} \to \mathcal{P}$ sending (t,ξ) to $t{\cdot}\xi$ is continuous. Thus the set $\{(t,t') \in \mathcal{P}\times\mathcal{P} : \pi(t) = \pi(t')\} = \{(t, t{\cdot}\xi) \in \mathcal{P}\times\mathcal{P} : (t,\xi) \in \mathbb{D}\}$ is closed in $\mathcal{P}\times\mathcal{P}$. Since π is open, a standard argument shows that the quotient topology on $\mathcal{P}/\mathcal{U}$ is compact Hausdorff. □

Since $\mathrm{EA}^{\sim}(\mathcal{G})$ is Hausdorff by Lemma 4.12, $\mathrm{EA}(\mathcal{G})$ is Hausdorff exactly if the quotient map $\mathrm{EA}(\mathcal{G}) \to \mathrm{EA}^{\sim}(\mathcal{G})$ is a bijection, exactly if $\mathcal{G}$ is co-amenable by Proposition 3.8. Then Theorem 4.4 follows from Lemmas 4.10 and 4.12.

Notice that the function $t \mapsto m_{\gamma,t}$ is continuous on $\mathcal{P}$ for each $\gamma \in \hat{\mathcal{G}}^{\flat}$. Thus we have

Proposition 4.13. The multiplicity function $\mathrm{mul}(\cdot, \gamma)$ is continuous on both $\mathrm{EA}(\mathcal{G})$ and $\mathrm{EA}^{\sim}(\mathcal{G})$ for each $\gamma \in \hat{\mathcal{G}}$.

To end this section, we discuss the behavior of $\mathrm{EA}(\mathcal{G})$ when we take Cartesian products of compact quantum groups. Let $\{A_\lambda = C(\mathcal{G}_\lambda)\}_{\lambda\in\Lambda}$ be a family of compact quantum groups indexed by a set Λ. Then $\otimes_\lambda A_\lambda$ has a unique compact quantum group structure such that the embeddings $A_\mu \hookrightarrow \otimes_\lambda A_\lambda$ for $\mu \in \Lambda$ are all morphisms between compact quantum groups [39, Theorem 1.4, Proposition 2.6], which we shall denote by $C(\prod_\lambda \mathcal{G}_\lambda)$. The Haar measure of $\otimes_\lambda A_\lambda$ is the tensor product $\otimes_\lambda h_\lambda$ of the Haar measures h_λ of A_λ [39, Proposition 2.7].

If $\sigma_\lambda : B_\lambda \to B_\lambda \otimes A_\lambda$ is an action of $\mathcal{G}_\lambda$ on a unital C^*-algebra B_λ for each λ, then the unique $*$-homomorphism $\otimes_\lambda\sigma_\lambda : \otimes_\lambda B_\lambda \to (\otimes_\lambda B_\lambda) \otimes (\otimes_\lambda A_\lambda)$ extending all σ_λ's is easily seen to be an action of $\prod_\lambda \mathcal{G}_\lambda$. Using the canonical conditional expectation $\otimes_\lambda B_\lambda \to (\otimes_\lambda B_\lambda)^{\otimes_\lambda\sigma_\lambda}$, one checks easily that $(\otimes_\lambda B_\lambda)^{\otimes_\lambda\sigma_\lambda} = \otimes_\lambda B_\lambda^{\sigma_\lambda}$. In particular, $\otimes_\lambda\sigma_\lambda$ is ergodic if and only if every σ_λ is.

Proposition 4.14. Let $\{A_\lambda = C(\mathcal{G}_\lambda)\}_{\lambda\in\Lambda}$ be a family of compact quantum groups indexed by a set Λ. The map $\prod_\lambda \mathrm{EA}(\mathcal{G}_\lambda) \to \mathrm{EA}(\prod_\lambda \mathcal{G}_\lambda)$ sending the isomorphism classes of $(B_\lambda, \sigma_\lambda)$'s to the isomorphism class of $(\otimes_\lambda B_\lambda, \otimes_\lambda\sigma_\lambda)$ descends to a map $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda)$, that is, there exists a (unique) map $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda)$ such that the diagram

$$\begin{array}{ccc} \prod_\lambda \mathrm{EA}(\mathcal{G}_\lambda) & \longrightarrow & \mathrm{EA}(\prod_\lambda \mathcal{G}_\lambda) \\ \downarrow & & \downarrow \\ \prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) & \longrightarrow & \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda) \end{array} \tag{36}$$

commutes. Moreover, both of these maps are injective and continuous, where both $\prod_\lambda \mathrm{EA}(\mathcal{G}_\lambda)$ and $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda)$ are endowed with the product topology.

Proof. Denote by $\prod_\lambda^{\sim} \widehat{\mathcal{G}_\lambda}$ the subset of $\prod_\lambda \widehat{\mathcal{G}_\lambda}$ consisting of elements whose all but finitely many components are classes of trivial representations. For any $\gamma \in \prod_\lambda^{\sim} \widehat{\mathcal{G}_\lambda}$, say $\gamma_{\lambda_1}, \cdots, \gamma_{\lambda_n}$ are the nontrivial components of γ, the element $u^{\gamma_{\lambda_1}}_{1(n+1)} u^{\gamma_{\lambda_2}}_{2(n+2)} \cdots u^{\gamma_{\lambda_n}}_{n(2n)}$ (in the leg numbering notation) is an irreducible unitary representation of $\prod_\lambda \mathcal{G}_\lambda$. Moreover, this map $\prod^{\sim}_\lambda \widehat{\mathcal{G}_\lambda} \to \widehat{\prod_\lambda \mathcal{G}_\lambda}$ is bijective [39, Theorem 2.11], and hence we may identify these two sets. Fixing an orthonormal basis of H_{γ_λ} we take the tensor products of the bases of $H_{\gamma_{\lambda_1}}, \cdots, H_{\gamma_{\lambda_n}}$ as an orthonormal basis of H_γ.

Let $\mathscr{S}_\lambda$ be a standard basis of B_λ. Say, it consists of a standard basis $\mathscr{S}_{\alpha_\lambda}$ of $(B_\lambda)_{\alpha_\lambda}$ for each $\alpha_\lambda \in \widehat{\mathcal{G}_\lambda}$. Denote by ω_λ the σ_λ-invariant state on B_λ. Then $\otimes_\lambda \omega_\lambda$ is the $\prod_\lambda \sigma_\lambda$-invariant state of $\otimes_\lambda B_\lambda$. Using the characterization of ergodic actions in terms of elements satisfying (9) in subsection 2.1, one sees that the algebra of regular functions for $\otimes_\lambda \sigma_\lambda$ is $\odot_\lambda \mathcal{B}_\lambda$ and that the tensor products of $\mathscr{S}_{\gamma_{\lambda_1}}, \cdots, \mathscr{S}_{\gamma_{\lambda_n}}$ is a standard basis of $(\otimes_\lambda B_\lambda)_\gamma$. This shows the existence of the map $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda)$ making (36) commute. Taking the union of the above standard basis of $(\otimes_\lambda B_\lambda)_\gamma$, we also get a standard basis of $\otimes_\lambda B_\lambda$, which we shall denote by $\prod^{\sim}_\lambda \mathscr{S}_\lambda$. For any fixed λ_0, if we take all $\gamma \in \prod^{\sim}_\lambda \widehat{\mathcal{G}_\lambda}$ whose components are trivial at all $\lambda \neq \lambda_0$ and take the sum of the corresponding spectral subspaces of $\otimes_\lambda B_\lambda$, we get $\mathcal{B}_{\lambda_0}$. Taking norm closure, we get B_{λ_0}. This proves the injectivity of the maps $\prod_\lambda \mathrm{EA}(\mathcal{G}_\lambda) \to \mathrm{EA}(\prod_\lambda \mathcal{G}_\lambda)$ and $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda)$.

Clearly the map $\prod_\lambda \mathcal{P}(\mathcal{G}_\lambda) \to \mathcal{P}(\prod_\lambda \mathcal{G}_\lambda)$ sending $(t_\lambda)_{\lambda\in\Lambda}$ to the element of $\mathcal{P}(\prod_\lambda \mathcal{G}_\lambda)$ associated to the standard basis $\prod^{\sim}_\lambda \mathscr{S}_{t_\lambda}$ is continuous, where $\mathscr{S}_{t_\lambda}$ is the standard basis of B_{t_λ} in Proposition 4.9. Note that the diagram

$$\begin{array}{ccc} \prod_\lambda \mathcal{P}(\mathcal{G}_\lambda) & \longrightarrow & \mathcal{P}(\prod_\lambda \mathcal{G}_\lambda) \\ \downarrow & & \downarrow \\ \prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) & \longrightarrow & \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda) \end{array} \tag{37}$$

commutes, where the left vertical map is the product map. By Theorem 4.4 the map $\mathcal{P}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\mathcal{G}_\lambda)$ is open for each λ. Thus the product map $\prod_\lambda \mathcal{P}(\mathcal{G}_\lambda) \to \prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda)$ is open. It follows from the commutativity of the diagram (37) that the map $\prod_\lambda \mathrm{EA}^{\sim}(\mathcal{G}_\lambda) \to \mathrm{EA}^{\sim}(\prod_\lambda \mathcal{G}_\lambda)$ is continuous. Then the continuity of the map $\prod_\lambda \mathrm{EA}(\mathcal{G}_\lambda) \to \mathrm{EA}(\prod_\lambda \mathcal{G}_\lambda)$ follows from the commutativity of the diagram (36). □

5. Semi-continuous fields of ergodic actions

In this section we prove Theorems 5.11 and 5.12, from which we deduce Theorems 1.1 and 1.3.

We start with discussion of semi-continuous fields of C^*-algebras.

Notation 5.1. For a field $\{C_t\}_{t\in T}$ of C^*-algebras over a locally compact Hausdorff space T, we denote by $\prod_t C_t$ the C^*-algebra of bounded cross-section (for the supremum norm), and by $\prod_t^{\sim} C_t$ the C^*-algebra of bounded cross-sections vanishing at infinity on T.

Note that both $\prod_t C_t$ and $\prod_t^{\sim} C_t$ are Banach modules over the C^*-algebra $C_\infty(T)$ of continuous $\mathbb{C}$-valued functions on T vanishing at infinity. We use Rieffel's definition of semi-continuous fields of C^*-algebras [32, Definition 1.1]. We find that it is convenient to extend the definition slightly.

Definition 5.2. Let $\{C_t\}_{t\in T}$ be a field of C^*-algebras over a locally compact Hausdorff space T, and let C be a C^*-subalgebra of $\prod_t^{\sim} C_t$. We say that $(\{C_t\}_{t\in T}, C)$ is a *topological* field of C^*-algebras if

(1) the evaluation map π_t from C to C_t is surjective for each $t\in T$,
(2) C is a $C_\infty(T)$-submodule of $\prod_t^{\sim} C_t$.

We say that $(\{C_t\}_{t\in T}, C)$ is *upper semi-continuous* (*lower semi-continuous*, *continuous*, resp.) if furthermore for each $c\in C$ the function $t\mapsto \|\pi_t(c)\|$ is upper semi-continuous (lower semi-continuous, continuous, resp.). In such case we say that $(\{C_t\}_{t\in T}, C)$ is *semi-continuous.*

Remark 5.3. If we have two upper semi-continuous fields of C^*-algebras $(\{C_t\}_{t\in T}, C_1)$ and $(\{C_t\}_{t\in T}, C_2)$ over T with the same fibres and $C_1\subseteq C_2$, then $C_1 = C_2$ [11, Proposition 2.3]. This is not true for lower semi-continuous fields of C^*-algebras. For example, let T be a compact Hausdorff space and let H be a Hilbert space. Take $C_t = B(H)$ for each t. Set C_1 to be the set of all cross-sections c such that $t\mapsto \pi_t(c)$ is norm continuous, while set C_2 to be the set of all norm-bounded cross-sections c such that both $t\mapsto \pi_t(c)$ and $t\mapsto (\pi_t(c))^*$ are continuous with respect to the strong operator topology in $B(H)$. Then $C_1 \subsetneqq C_2$ when T is the one-point compactification of $\mathbb{N}$ and H is infinite-dimensional.

Definition 5.4. By a *homomorphism* φ between two topological fields of C^*-algebras $(\{C_t\}_{t\in T}, C)$ and $(\{B_t\}_{t\in T}, B)$ over a locally compact Hausdorff space T we mean a $*$-homomorphism $\varphi_t : C_t \to B_t$ for each $t\in T$ such that the pointwise $*$-homomorphism $\prod_t \varphi_t : \prod_t C_t \to \prod_t B_t$ sends C into B.

Lemma 5.5. Let $\{C_t\}_{t\in T}$ be a field of C^*-algebras over a locally compact Hausdorff space T, and let $\mathscr{C}$ be a linear subspace of $\prod_t C_t$. Then a section $c'\in \prod_t^{\sim} C_t$ is in $C := \overline{C_\infty(T)\mathscr{C}}$ if and only if for any $t_0\in T$ and $\varepsilon > 0$, there exist a neighborhood U of t_0 and $c\in\mathscr{C}$ such that $\|\pi_t(c - c')\| < \varepsilon$ throughout U. If furthermore $\pi_t(\mathscr{C})$ is dense in C_t for each t and $\mathscr{C}\mathscr{C}, \mathscr{C}^* \subseteq C$, then $(\{C_t\}_{t\in T}, C)$ is a topological field of C^*-algebras over T, which we shall call the *topological field generated by* $\mathscr{C}$. If furthermore the function $t \mapsto \|\pi_t(c)\|$ is upper semi-continuous (lower semi-continuous, continuous, resp.) for each $c \in C$, then $(\{C_t\}_{t\in T}, C)$ is upper semi-continuous (lower semi-continuous, continuous, resp.).

Proof. The "only if" part is obvious. The "if" part follows from a partition-of-unity argument. The second and the third assertions follow easily. ☐

Let $(\{C_t\}_{t\in T}, C)$ be a topological field of C^*-algebras over a locally compact Hausdorff space T. If Θ is another locally compact Hausdorff space and $p : \Theta \to T$ is a continuous map, then we have the pull-back field $\{C_{p(\theta)}\}_{\theta\in\Theta}$ of C^*-algebras over Θ. There is a natural $*$-homomorphism $p^* : \prod_t C_t \to \prod_\theta C_{p(\theta)}$ sending c to $\{\pi_{p(\theta)}(c)\}_{\theta\in\Theta}$. We will call the topological field generated by $p^*(C)$ in Lemma 5.5 the *pull-back* of $(\{C_t\}_{t\in T}, C)$ under p. In particular, if Θ is a closed or open subset of T and p is the embedding, we get *the restriction of* $(\{C_t\}_{t\in T}, C)$ *on* Θ. Clearly the pull-back and restriction of homomorphisms between topological fields are also homomorphisms.

Lemma 5.6. Let $(\{C_t\}_{t\in T}, C)$ be a semi-continuous field of unital C^*-algebras over a locally compact Hausdorff space T such that the section $\{f(t)1_{C_t}\}_{t\in T}$ is in C for each $f \in C_\infty(T)$. Then for any bounded function g on T vanishing at infinity, the section $\{g(t)1_{C_t}\}_{t\in T}$ is in C if and only if $g \in C_\infty(T)$.

Proof. Via restricting to compact subsets of T, we may assume that T is compact. The "if" part is given by assumption. To prove the "only if" part, it suffices to show that when the section $\{g(t)1_{C_t}\}_{t\in T}$ is in C and $g(t_0) = 0$ for some $t_0 \in T$, we have $g(t) \to 0$ as $t \to t_0$. Replacing g by g^*g, we may assume that g is nonnegative. When the field is upper semi-continuous, the function $t \mapsto \|g(t)1_{C_t}\| = g(t)$ is upper semi-continuous at t_0 and hence $g(t) \to 0$ as $t \to t_0$. When the field is lower semi-continuous, the function $t \mapsto \|(\|g\| - g(t))1_{C_t}\| = \|g\| - g(t)$ is lower semi-continuous at t_0 and hence $g(t) \to 0$ as $t \to t_0$. ☐

Lemma 5.7. Let $(\{C_t\}_{t\in T}, C)$ be a topological field of C^*-algebras over a locally compact Hausdorff space T. Let D be a C^*-algebra. Then there is a natural injective $*$-homomorphism $\varphi : C \otimes D \to \prod_t^{\sim}(C_t \otimes D)$ determined by $\pi'_s(\varphi(c \otimes d)) = \pi_s(c) \otimes d$ for all $c \in C$, $d \in D$, and $s \in T$, where π_s and π'_s denote the coordinate maps $\prod_t C_t \to C_s$ and $\prod_t(C_t \otimes D) \to C_s \otimes D$ respectively. Identifying $C \otimes D$ with $\varphi(C \otimes D)$, the pair $(\{C_t \otimes D\}_{t\in T}, C \otimes D)$ is also a topological field of C^*-algebras over T.

Proof. For each $s \in T$ we have the $*$-homomorphism $\pi_s \otimes \mathrm{id} : (\prod_t C_t) \otimes D \to C_s \otimes D$. Then we have the product $*$-homomorphism $(\prod_t C_t) \otimes D \to \prod_t (C_t \otimes D)$. Denote by φ the restriction of this homomorphism to $C \otimes D$. We have $\pi'_s(\varphi(c \otimes d)) = \pi_s(c) \otimes d$ for all $c \in C$, $d \in D$, and $s \in T$. Clearly this identity also determines φ.

To show that φ is injective, we may assume that C_s is contained in the algebra of bounded linear operators on H_s for some Hilbert space H_t for each $s \in T$, and D is contained in the algebra of bounded linear operators on K for some Hilbert space K. Denote the Hilbert space direct sum $\oplus_t H_t$ by H_T. Then $\prod_t C_t$ can be identified with the algebra of bounded linear operators c on H_T satisfying that c preserves H_s for each $s \in T$ and the restriction of c on H_s is in C_s for each $s \in T$. Now $C \otimes D$ is naturally a C^*-algebra of bounded linear operators on the Hilbert space tensor product $H_T \otimes K = \oplus_{t \in T}(H_t \otimes K)$. It is easily checked that for every $g \in C \otimes D$, g preserves $H_s \otimes K$ for each $s \in T$, the restriction of c on H_s is equal to $\pi'_s(\varphi(g))$ for each $s \in T$, and the function $t \mapsto \|\pi'_t(\varphi(g))\|$ on T vanishes at infinity (check this for $g' \in C \odot D$ first, then approximate $g \in C \otimes D$ by $g' \in C \odot D$). It follows that φ is injective and maps $C \otimes D$ into $\prod_t^{\sim}(C_t \otimes D)$.

Clearly the restriction of π'_s on $\varphi(C \otimes D)$ is onto $C_s \otimes D$ for each $s \in T$. Since C is a $C_\infty(T)$-module, $\varphi(C \odot D)$ is easily seen to be a C_∞-submodule of $\prod_t (C_t \otimes D)$. It follows that $\varphi(C \otimes D)$ is a C_∞-submodule of $\prod_t (C_t \otimes D)$. Thus the pair $(\{C_t \otimes D\}_{t \in T}, \varphi(C \otimes D))$ is a topological field of C^*-algebras over T. □

From now on, for a topological field $(\{C_t\}_{t \in T}, C)$ of C^*-algebras over a locally compact Hausdorff space T and a C^*-algebra D, we shall take $(\{C_t \otimes D\}_{t \in T}, C \otimes D)$ to be the topological field of C^*-algebras over T in Lemma 5.7.

In general, for a continuous field $(\{C_t\}_{t \in T}, C)$ of C^*-algebras over a compact metrizable space T and a C^*-algebra D, the topological field $(\{C_t \otimes D\}_{t \in T}, C \otimes D)$ of C^*-algebras may fail to be continuous [17, Theorem A]. The following lemma tells us that if a field $(\{C_t\}_{t \in T}, C)$ over a locally compact Hausdorff space T can be *subtrivialized* in the sense that there is a C^*-algebra B containing each C_t as a C^*-subalgebra so that the elements of C are exactly the continuous maps $T \to B$ vanishing at ∞ whose images at each t are in C_t, then the field $(\{C_t \otimes D\}_{t \in T}, C \otimes D)$ can also be subtrivialized and hence is continuous.

Lemma 5.8. Let $(\{C_t\}_{t \in T}, C)$ be a topological field of C^*-algebras over a locally compact Hausdorff space T. Suppose that there is a C^*-algebra B containing each C_t as a C^*-subalgebra so that the elements of C are exactly the continuous maps $T \to B$ vanishing at ∞ whose images at each t are in C_t. Let D be a C^*-algebra, and identify $C \otimes D$ with a C^*-subalgebra of $\prod_t^{\sim}(C_t \otimes D)$ as in Lemma 5.7. Then elements of $C \otimes D$ are exactly the continuous maps $T \to B \otimes D$ vanishing at ∞ whose images at each t are in $C_t \otimes D$.

Proof. Denote by W the continuous maps $T \to B \otimes D$ vanishing at ∞ whose images at each t are in $C_t \otimes D$. This is a C^*-subalgebra of $\prod_t^{\sim}(C_t \otimes D)$.

Denote by π'_t the coordinate map $C \otimes D \to C_t \otimes D$ for each $t \in T$. Then $\pi'_t(f \otimes d) = f(t) \otimes d$ for all $t \in T$, $f \in C$, and $d \in D$. It is easy to check that $C \odot D \subseteq W$. Thus $C \otimes D \subseteq W$.

Let $w \in W$ and let $\varepsilon > 0$. For any $s \in T$, we can find some $\sum_j b_j \otimes d_j \in C_s \odot D$ satisfying $\|w(s) - \sum_j b_j \otimes d_j\| < \varepsilon$. Take $f_j \in C$ with $f_j(s) = b_j$. Then $\|w(t) - (\sum_j f_j \otimes d_j)(t)\| < \varepsilon$ for $t = s$ and hence for all t in some neighborhood of s by continuity. Note that both C and W are Banach modules over $C_\infty(T)$. Now a standard partition of unity argument shows that we can find some $g \in C \odot D$ with $\|w - g\| < \varepsilon$. Thus $C \otimes D$ is dense in W and hence $C \otimes D = W$. □

Next we discuss semi-continuous fields of ergodic actions of $\mathcal{G}$. The following definition is a natural generalization of Rieffel's definition of upper semi-continuous fields of actions of locally compact groups [32, Definition 3.1].

Definition 5.9. By a *topological field of actions* of $\mathcal{G}$ on unital C^*-algebras we mean a topological field $(\{B_t\}_{t\in T}, B)$ of unital C^*-algebras over a locally compact Hausdorff space T, and an action σ_t of $\mathcal{G}$ on B_t for each $t \in T$ such that the section $\{f(t)1_{B_t}\}_{t\in T}$ is in B for each $f \in C_\infty(T)$ and $\{\sigma_t\}_{t\in T}$ is a homomorphism from $(\{B_t\}_{t\in T}, B)$ to $(\{B_t \otimes A\}_{t\in T}, B \otimes A)$. If the field $(\{B_t\}_{t\in T}, B)$ is actually upper semi-continuous (lower semi-continuous, continuous, resp.), then we will say that the field of actions is *upper semi-continuous* (*lower semi-continuous*, *continuous*, resp.). If each σ_t is ergodic, we say that this is a *field of ergodic actions.*

Clearly the pull-back of a topological (upper semi-continuous, lower semi-continuous, continuous, resp.) field of actions of $\mathcal{G}$ on unital C^*-algebras is a topological (upper semi-continuous, lower semi-continuous, continuous, resp.) field of actions of $\mathcal{G}$.

Lemma 5.10. Let $(\{(B_t, \sigma_t)\}_{t\in T}, B)$ be a semi-continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T. Then for any $b \in B$ the function $t \mapsto \omega_t(\pi_t(b))$ is continuous on T, where ω_t is the unique σ_t-invariant state on B_t. Denote by $(B_{t,r}, \sigma_{t,r})$ the reduced action associated to (B_t, σ_t) and by $\pi_{t,r}$ the canonical $*$-homomorphism $B_t \to B_{t,r}$. Denote by π_r the $*$-homomorphism $\prod_t B_t \to \prod_t B_{t,r}$ given pointwisely by $\pi_{t,r}$. Then $(\{(B_{t,r}, \sigma_{t,r})\}_{t\in T}, \pi_r(B))$ is a lower semi-continuous field of ergodic actions of $\mathcal{G}$ over T.

Proof. We prove the continuity of the function $t \mapsto \omega_t(\pi_t(b))$ first. Via taking restrictions to compact subsets of T we may assume that T is compact. The cross-section $t \mapsto \omega_t(\pi_t(b))1_{B_t}$ is simply $((\mathrm{id} \otimes h) \circ (\prod_t \sigma_t))(b)$, which is in B. Thus the function $t \mapsto \omega_t(\pi_t(b))$ is continuous by Lemma 5.6.

Next we show that $(\{(B_{t,r}, \sigma_{t,r})\}_{t\in T}, \pi_r(B))$ is a lower semi-continuous field of actions. Clearly $\pi_r(B)$ is a C^*-subalgebra and $C_\infty(T)$-submodule of $\prod_t^{\sim} B_{t,r}$, and the evaluation map $\pi_t : \pi_r(B) \to B_{t,r}$ is surjective for each t. Since $\prod_t \sigma_{t,r} \circ \prod_t \pi_{t,r} = \prod_t(\pi_{t,r} \otimes \mathrm{id}) \circ \prod_t \sigma_t$, one sees that $\prod_t \sigma_{t,r}$ sends $\pi_r(B)$ into $\pi_r(B) \otimes A$. We are left to show that the function $t \mapsto \|\pi_t(\pi_r(b))\|$ is lower semi-continuous for each $b \in B$. Note that for any $b \in B$ and $t \in T$, the norm of $\pi_t(\pi_r(b))$ is the smallest number K such that $\omega_t(\pi_t(b_1^* b^* b b_1))^{\frac{1}{2}} \leq K\omega_t(\pi_t(b_1^* b_1))^{\frac{1}{2}}$ for all $b_1 \in B$. It follows easily that the function $t \mapsto \|\pi_t(\pi_r(b))\|$ is lower semi-continuous over T for each $b \in B$. This completes the proof of Lemma 5.10. $\square$

It is well-known that there is a continuous field of ergodic actions of the n-dimensional torus $\mathbb{T}^d$ over the compact space of isomorphism classes of faithful ergodic actions of $\mathbb{T}^d$ such that the isomorphism class of the fibre at each point is exactly the point (see [1, Theorem 1.1] for a proof for the case $n = 2$; the proof for the higher-dimensional case is similar). We have not been able to extend this to arbitrary compact quantum groups. What we find is that there are two natural semi-continuous fields of ergodic actions of $\mathcal{G}$ over $\mathcal{P}$ such that the equivalence class of the fibre at each $t \in \mathcal{P}$ is the image of t under the quotient map $\mathcal{P} \to \mathrm{EA}^{\sim}(\mathcal{G})$ defined before Definition 4.3. By Propositions 4.7 and 4.9, for each $t \in \mathcal{P}$, the pair $(\mathscr{V}_t, \sigma_t)$ defined after the formula (24) is isomorphic to the regular part of some ergodic action of $\mathcal{G}$. By Propositions 3.3 and 3.4 there exist (unique up to isomorphisms) a full action $(B_{t,u}, \sigma_{t,u})$ and a reduced action $(B_{t,r}, \sigma_{t,r})$ of $\mathcal{G}$ whose regular parts are exactly $(\mathscr{V}_t, \sigma_t)$. In fact, one can take (B_t, σ_t) in Proposition 4.9 as $(B_{t,u}, \sigma_{t,u})$. Recall the the quotient map $\phi_t : \mathscr{V} \to \mathscr{V}_t$ defined before (22) for each $t \in \mathcal{P}$.

Theorem 5.11. The set of cross-sections $\{\phi_t(v)\}_{t\in\mathcal{P}}$ over $\mathcal{P}$ for $v \in \mathscr{V}$ is in $\prod_t B_{t,u}$ ($\prod_t B_{t,r}$ resp.). It generates an upper (lower resp.) semi-continuous field $(\{B_{t,u}\}_{t\in\mathcal{P}}, B_u)$ $((\{B_{t,r}\}_{t\in\mathcal{P}}, B_r)$ resp.) of C^*-algebras over $\mathcal{P}$. Moreover, the field $(\{(B_{t,u}, \sigma_{t,u})\}_{t\in\mathcal{P}}, B_u)$ $((\{(B_{t,r}, \sigma_{t,r})\}_{t\in\mathcal{P}}, B_r)$ resp.) is an upper (lower resp.) semi-continuous field of full (reduced resp.) ergodic actions of $\mathcal{G}$. If $\mathcal{G}$ is co-amenable, then these two fields coincide and are continuous.

Proof. Consider generators w_x for $x \in X_0$, $\theta(y)$ for $y \in Y$ and $\zeta(z)$ for $z \in Z$ subject to the following relations:

(1) w_{x_0} is the identity,
(2) the equations (22) and (23) with $\nu_x, f(y), g(z)$ replaced by w_x, $\theta(y)$, $\zeta(z)$ respectively,
(3) the equations in $\mathscr{E}$ with $f(y), \overline{f(y)}, g(z), \overline{g(z)}$ replaced by $\theta(y)$, $\theta(y)^*$, $\zeta(z)$, $\zeta(z)^*$ respectively,
(4) $\theta(y)$ and $\zeta(z)$ are in the center.

These relations have $*$-representations since $B_{f,g,u}$ for any $(f, g) \in \mathcal{P}$ has generators satisfying these conditions. Consider an irreducible representation π of these relations. Because of (4), $\pi(\theta(y))$ and $\pi(\zeta(z))$ have to be scalars. Say $\pi(\theta(y)) = f(y)$ and $\theta(\zeta(z)) = g(z)$. Then $(f, g) \in \mathbb{C}^Y \times \mathbb{C}^Z$ satisfies the equations in $\mathscr{E}$ because of (3). Thus the inequalities (31) and (32) hold with $|f(y)|$ and $|g(z)|$ replaced by $\|\pi(\theta(y))\|$ and $\|\pi(\zeta(z))\|$ respectively. Also, there is a $*$-homomorphism from $\mathscr{V}_{f,g}$ to the C^*-algebra generated by $\pi(w_x), \pi(\theta(y)), \pi(\zeta(z))$ sending ν_x to $\pi(w_x)$. Thus (26) holds with ν_x replaced by $\pi(w_x)$. Consequently, above generators and relations do have a universal C^*-algebra B_u.

In particular, there is a surjective $*$-homomorphism $\pi_{f,g} : B_u \to B_{f,g,u}$ for each $(f, g) \in \mathcal{P}$ sending w_x, $\theta(y)$, $\zeta(z)$ to $\phi_{f,g}(v_x)$, $f(y)\phi_{f,g}(v_{x_0})$, $g(z)\phi_{f,g}(v_{x_0})$ respectively. These $*$-homomorphisms π_t's for $t \in \mathcal{P}$ combine to a $*$-homomorphism $\pi : B_u \to \prod_t B_{t,u}$. In above we have seen that every irreducible $*$-representation of B_u factors through π_t for some $t \in \mathcal{P}$. Thus π is faithful and we may identify B_u with $\pi(B_u)$. Since $B_{f,g,u}$ is the universal C^*-algebra of $\mathscr{V}_{f,g}$, one sees easily that $\ker(\pi_{f,g})$ is generated by $\theta(y) - f(y)w_{x_0}$ and $\zeta(z) - g(z)w_{x_0}$. Since $\theta(y) - f'(y)w_{x_0} \to \theta(y) - f(y)w_{x_0}$ and $\zeta(z) - g'(z)w_{x_0} \to \zeta(z) - g(z)w_{x_0}$ as $(f', g') \to (f, g)$, the function $t \mapsto \|\pi_t(b)\|$ is upper semi-continuous on $\mathcal{P}$ for each $b \in B_u$. Thanks to the Stone-Weierstrass theorem, the unital C^*-subalgebra of B_u generated by $\theta(y)$ and $\zeta(z)$ is exactly $C(\mathcal{P})$. Thus B_u is a $C(\mathcal{P})$-submodule of $\prod_t B_{t,u}$. Therefore $(\{B_{t,u}\}_{t\in\mathcal{P}}, B_u)$ is an upper semi-continuous field of C^*-algebras over $\mathcal{P}$. Clearly it is generated by the sections $\{\phi_t(v)\}_{t\in\mathcal{P}}$ for $v \in \mathscr{V}$.

The formula (23) tells us that $\prod_t \sigma_{t,u}$ sends the section $\{\phi_t(v_x)\}_{t\in\mathcal{P}}$ into $B_u \otimes A$ for each $x \in X_0$. Since B_u is generated by such sections and $C(\mathcal{P})$, $\prod_t(\sigma_{t,u})$ sends B_u into $B_u \otimes A$. Thus $(\{(B_{t,u}, \sigma_{t,u})\}_{t\in\mathcal{P}}, B_u)$ is an upper semi-continuous field of ergodic actions of $\mathcal{G}$.

The assertions about the reduced actions follow from Lemma 5.10. The assertion about the case $\mathcal{G}$ is co-amenable follows from Proposition 3.8. □

Theorem 5.12. Let $(\{(B_t, \sigma_t)\}_{t\in T}, B)$ be a semi-continuous field of ergodic actions of $\mathcal{G}$ over a locally compact Hausdorff space T. Let $t_0 \in T$. Then the following are equivalent:

(1) the map $T \to \mathrm{EA}(\mathcal{G})$ sending each t to the isomorphism class of (B_t, σ_t) is continuous at t_0,
(2) the map $T \to \mathrm{EA}^{\sim}(\mathcal{G})$ sending each t to the equivalence class of (B_t, σ_t) is continuous at t_0,
(3) $\limsup_{t\to t_0} \mathrm{mul}(B_t, \gamma) \le \mathrm{mul}(B_{t_0}, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$,
(4) $\lim_{t\to t_0} \mathrm{mul}(B_t, \gamma) = \mathrm{mul}(B_{t_0}, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$.

Lemma 5.13. Let the notation be as in Theorem 5.12. Let $\gamma \in \hat{\mathcal{G}}$, and let $c_{\gamma si}, 1 \le i \le \mathrm{mul}(B_{t_0}, \gamma), 1 \le i \le d_\gamma$ be a standard basis of $(B_{t_0})_\gamma$. Then there is a linear map $\varphi_t : (B_{t_0})_\gamma \to (B_t)_\gamma$ for all $t \in T$ such that the section $t \mapsto \varphi_t(c)$ is in B for every $c \in (B_{t_0})_\gamma$, that $\varphi_{t_0} = \mathrm{id}$, and that $\varphi_t(c_{\gamma si}), 1 \le s \le \mathrm{mul}(B_{t_0}, \gamma), 1 \le i \le d_\gamma$ satisfy (9) and (15) (with $e_{\gamma si}$ and ω replaced by $\varphi_t(c_{\gamma si})$ and the unique σ_t-invariant state ω_t respectively) throughout a neighborhood of t_0.

Proof. We may assume that T is compact. Denote by σ the restriction of $\prod_t \sigma_t$ on B. Recall the map E^γ_{ij} defined via (5). Then E^γ_{ij} is also defined on B for the unital $*$-homomorphism $\sigma : B \to B \otimes A$. Set $m = \mathrm{mul}(B_{t_0}, \gamma)$ and $S = \{c_{\gamma s1} : 1 \le s \le m\}$. For each $c \in S$ take $b \in B$ with $\pi_{t_0}(b) = c$. Then $\pi_t(E^\gamma_{11}(b)) = E^\gamma_{11}(\pi_t(b))$ is in $E^\gamma_{11}(B_t)$ for each $t \in T$. By Lemma 4.11 S is a linear basis of $E^\gamma_{11}(B_{t_0})$. Set ψ_t to be the linear map $E^\gamma_{11}(B_{t_0}) \to E^\gamma_{11}(B_t)$ sending each $c \in S$ to $\pi_t(E^\gamma_{11}(b))$. By Lemma 4.11 we have $\psi_{t_0} = \mathrm{id}$. By Lemma 5.10 the function $t \mapsto \omega_t(\pi_t(b'))$ is continuous on T for any $b' \in B$, where ω_t is the unique σ_t-invariant state on B_t. Consequently, for any $c_1, c_2 \in S$, we have

$$\omega_t(\psi_t(c_1)^*\psi_t(c_2)) \to \omega_{t_0}(\psi_{t_0}(c_1)^*\psi_{t_0}(c_2)) = \omega_{t_0}(c_1c_2) = \delta_{c_1c_2}$$

as $t \to t_0$. Shrinking T if necessary, we may assume that the matrix $Q_t = (\omega_t(\psi_t(c_{\gamma k1})^*\psi_t(c_{\gamma s1})))_{ks} \in M_m(\mathbb{C})$ is invertible for all $t \in T$. Set $(c_{t,1}, \cdots, c_{t,m}) = (\psi_t(c_{\gamma 11}), \cdots, \psi_t(c_{\gamma m1}))Q_t^{-\frac{1}{2}}$. Then $c_{t,k} \in E^\gamma_{11}(B_t)$ and $\omega_t(c^*_{t,k}c_{t,s}) = \delta_{ks}$ for all $t \in T$. Note that the section $t \mapsto c_{t,s}$ is in B for each $1 \le s \le m$. Thus the section $t \mapsto E^\gamma_{i1}(c_{t,s})$ is in B for all $1 \le s \le m, 1 \le i \le d_\gamma$. Set φ_t to be the linear map $(B_{t_0})_\gamma \to B_t$ sending

$c_{\gamma si}$ to $E^{\gamma}_{i1}(c_{t,s})$. Then the section $t \mapsto \varphi_t(c)$ is in B for every $c \in (B_{t_0})_\gamma$. By Lemma 4.11 these maps have the other desired properties. □

Remark 5.14. Using Remark 5.3 one can show easily that for an upper semi-continuous field $(\{(B_t, \sigma_t)\}_{t\in T}, B)$ of actions of $\mathcal{G}$ over a compact Hausdorff space T, the $*$-homomorphism $(\prod_t \sigma_t)|_B : B \to B \otimes A$ is an action of $\mathcal{G}$ on B. Using the well-known fact that upper semi-continuous fields of unital C^*-algebras over a compact Hausdorff space T satisfying the hypothesis in Lemma 5.6 correspond exactly to unital C^*-algebras containing $C(T)$ in the centers, one can show further that upper semi-continuous fields of ergodic actions of $\mathcal{G}$ over T correspond exactly to actions of $\mathcal{G}$ on unital C^*-algebras whose fixed point algebras are $C(T)$ and are in the centers.

As a corollary of Lemma 5.13 we get

Lemma 5.15. Let the notation be as in Theorem 5.12. The function $t \mapsto \mathrm{mul}(B_t, \gamma)$ is lower semi-continuous on T for each $\gamma \in \hat{\mathcal{G}}$.

We are ready to prove Theorem 5.12.

Proof of Theorem 5.12. (1) $\Longleftrightarrow$ (2) follows from the definition of the topology on $\mathrm{EA}(\mathcal{G})$. (2)$\Rightarrow$(3) follows from Proposition 4.13. (3)$\Rightarrow$(4) follows from Lemma 5.15. We are left to show (4)$\Rightarrow$(2). Assume (4). Fix a standard basis $\mathscr{S}$ of $\mathcal{B}_{t_0}$, consisting of a standard basis $\mathscr{S}_\gamma$ of $(B_{t_0})_\gamma$ for each $\gamma \in \hat{\mathcal{G}}$. Let J be a finite subset of $\hat{\mathcal{G}}$. Then $\mathrm{mul}(B_t, \gamma) = \mathrm{mul}(B_{t_0}, \gamma)$ for each $\gamma \in J$ throughout some neighborhood U of t_0. By Lemma 5.13, shrinking U if necessary, we can find a linear map $\varphi_t : (B_{t_0})_J \to (B_t)_J$ for all $t \in T$, where $(B_t)_J = \sum_{\gamma\in J}(B_t)_\gamma$, such that the section $t \mapsto \varphi_t(c)$ is in B for every $c \in (B_{t_0})_J$, that $\varphi_{t_0} = \mathrm{id}$, and that $\varphi_t(\mathscr{S}_\gamma)$ is a standard basis of $(B_t)_\gamma$ for all $\gamma \in J$ and $t \in U$. For each $t \in U$, extend these bases of $(B_t)_\gamma$ for $\gamma \in J$ to a standard basis $\mathscr{S}_t$ of $\mathcal{B}_t$. Set (f_t, g_t) to be the element in $\mathcal{P}$ associated to $\mathscr{S}_t$ via (16)-(18). Suppose that $\alpha, \beta \in J \setminus \{\gamma_0\}$. By Lemma 5.10 the function $t \mapsto \omega_t(\pi_t(b))$ is continuous for each $b \in B$, where ω_t is the unique σ_t-invariant state on B_t. Then one sees easily that the function $t \mapsto f_t(x_1, x_2, x_3)$ is continuous over U for any $x_1 \in X_\alpha, x_2 \in X_\beta, x_3 \in X_\gamma, \gamma \in J$. Similarly, if $\alpha, \bar{\alpha} \in J\setminus\{\gamma_0\}$, then the function $t \mapsto g_t(x_1, x_2)$ is continuous over U for any $x_1 \in X_\alpha, x_2 \in X_{\bar{\alpha}}$. Since J is an arbitrary finite subset of $\hat{\mathcal{G}}$, this means that for any neighborhood W of (f_{t_0}, g_{t_0}) in $\mathcal{P}$, we can find a neighborhood V of t_0 in T and choose a standard basis of $\mathcal{B}_t$ for each $t \in V$ such that the associated element in $\mathcal{P}$ is in W. Therefore (2) holds. □

Now Theorems 1.1 and 1.3 follow from Theorems 4.4, 5.12 and 5.11. In fact we have a stronger assertion:

Corollary 5.16. The topology on $\mathrm{EA}^{\sim}(\mathcal{G})$ defined in Definition 4.3 is the unique Hausdorff topology on $\mathrm{EA}^{\sim}(\mathcal{G})$ such that the implication (4)⇒(2) in Theorem 5.12 holds for all upper semi-continuous (lower semi-continuous resp.) fields of ergodic actions of $\mathcal{G}$ over compact Hausdorff spaces. If $\mathcal{G}$ is co-amenable, then the topology on $\mathrm{EA}(\mathcal{G})$ defined in Definition 4.3 is the unique Hausdorff topology on $\mathrm{EA}(\mathcal{G})$ such that the implication (4)⇒(2) in Theorem 5.12 holds for all continuous fields of ergodic actions of $\mathcal{G}$ over compact Hausdorff spaces.

When A is separable and co-amenable, one can describe the topology on $\mathrm{EA}(\mathcal{G})$ more explicitly in terms of continuous fields of actions:

Theorem 5.17. Suppose that A is separable and co-amenable. Then both $\mathcal{P}$ and $\mathrm{EA}(\mathcal{G})$ are metrizable. The isomorphism classes of a sequence $\{(B_n, \sigma_n)\}_{n\in\mathbb{N}}$ of ergodic actions of $\mathcal{G}$ converge to that of $(B_\infty, \sigma_\infty)$ in $\mathrm{EA}(\mathcal{G})$ if and only if there exists a continuous field of ergodic actions of $\mathcal{G}$ over the one-point compactification $\mathbb{N}\cup\{\infty\}$ of $\mathbb{N}$ with fibre (B_n, σ_n) at n for $1 \leq n \leq \infty$ and $\lim_{n\to\infty} \mathrm{mul}(B_n, \gamma) = \mathrm{mul}(B_\infty, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$.

Proof. Denote by (π_A, H_A) the GNS representation of A associated to h. Since A is separable, so is H_A. Note that the subspaces A_γ are nonzero and orthogonal to each other in H_A for $\gamma \in \hat{\mathcal{G}}$. Thus $\hat{\mathcal{G}}$ is countable. Then Y and Z are both countable. Therefore $\mathcal{P}$ and $\mathrm{EA}(\mathcal{G})$ are metrizable. The "if" part follows from Theorem 5.12. Suppose that the isomorphism class of (B_n, σ_n) converges to that of $(B_\infty, \sigma_\infty)$ in $\mathrm{EA}(\mathcal{G})$ as $n \to \infty$. By Proposition 4.13 we have $\lim_{n\to\infty} \mathrm{mul}(B_n, \gamma) = \mathrm{mul}(B_\infty, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$. Also the map $\xi : \mathbb{N}\cup\{\infty\} \to \mathrm{EA}(\mathcal{G})$ sending $1 \leq n \leq \infty$ to the isomorphism class of (B_n, σ_n) is continuous. By Theorem 4.4 the quotient map $\mathcal{P} \to \mathrm{EA}(\mathcal{G})$ is open. Thus ξ lifts up to a continuous map $\eta : \mathbb{N}\cup\{\infty\} \to \mathcal{P}$. The pull-back of the continuous field of ergodic actions of $\mathcal{G}$ over $\mathcal{P}$ in Theorem 5.11 via η is a continuous field of ergodic actions of $\mathcal{G}$ over $\mathbb{N}\cup\{\infty\}$ with the desired fibres. This proves the "only if" part. □

6. Podleś spheres

In this section we prove Theorem 1.2.

Fix $q \in [-1, 1]$. The quantum SU(2) group $A = C(\mathrm{SU}_q(2))$ [38, 46] is defined as the universal C^*-algebra generated by α and β subject to the condition that

$$u = \begin{pmatrix} \alpha & -q\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

is a unitary in $M_2(A)$. The comultiplication $\Phi : A \to A$ is defined in such a way that u is a representation of A.

Below we assume $0 < |q| < 1$. The quantum group $\mathrm{SU}_q(2)$ is co-amenable [25][2, Corollary 6.2][3, Theorem 2.12]. Let

$$T_q = \{c(1), c(2), \cdots\} \cup [0, 1],$$

where

$$c(n) = -q^{2n}/(1 + q^{2n})^2.$$

For $t \in T_q$ with $t \le 0$, Podleś quantum sphere $C(S^2_{qt})$ [29] is defined as the universal C^*-algebra generated by a_t, b_t subject to the relations

$$\begin{aligned} a_t^* &= a_t, & b_t^* b_t &= a_t - a_t^2 + t, \\ b_t a_t &= q^2 a_t b_t, & b_t b_t^* &= q^2 a_t - q^4 a_t^2 + t. \end{aligned} \tag{38}$$

For $t \in T_q$ with $t \ge 0$, $C(S^2_{qt})$ is defined as the universal C^*-algebra generated by a_t, b_t subject to the relations

$$\begin{aligned} a_t^* &= a_t, \quad b_t^* b_t = (1 - t^2)a_t - a_t^2 + t^2, \\ b_t a_t &= q^2 a_t b_t, \quad b_t b_t^* = (1 - t^2)q^2 a_t - q^4 a_t^2 + t^2. \end{aligned} \tag{39}$$

The action $\sigma_t : C(S^2_{qt}) \to C(S^2_{qt}) \otimes A$ is determined by

$$\begin{aligned} \sigma_t(a_t) &= a_t \otimes 1_A + c_t \otimes \beta^*\beta + b_t^* \otimes \alpha^*\beta + b_t \otimes \beta^*\alpha, \\ \sigma_t(b_t) &= -q b_t^* \otimes \beta^2 + c_t \otimes \alpha\beta + b_t \otimes \alpha^2, \end{aligned} \tag{40}$$

where c_t is $1_{C(S^2_{qt})} - (1+q^2)a_t$ or $(1-t^2)1_{C(S^2_{qt})} - (1+q^2)a_t$ depending on $t \le 0$ or $t \ge 0$. As in [13], here we reparametrize the family for $0 \le c \le \infty$ in [29] for the parameters $0 \le t \le 1$ by $t = 2\sqrt{c}/(1 + \sqrt{1 + 4c})$ (and $c = (t^{-1} - t)^{-2}$), and rescale the generators A, B in [29] by $a_t = (1-t^2)A$, $b_t = (1 - t^2)B$ for $0 \le t < 1$.

Proposition 6.1. There is a unique continuous field of C^*-algebras over T_q with fibre $C(S^2_{qt})$ at each $t \in T_q$ such that the sections $t \mapsto a_t$ and $t \mapsto b_t$ are in the algebra B of continuous sections. Moreover, the field $\{\sigma_t\}_{t \in T_q}$ of ergodic actions of $\mathrm{SU}_q(2)$ is continuous.

Proof. The uniqueness is clear. We start to show that there exits an upper semi-continuous field $(\{C(S^2_{qt})\}_{t \in T_q}, B)$ of C^*-algebras over T_q such that the sections $t \mapsto a_t$ and $t \mapsto b_t$ are in B. For this purpose, by Lemma 5.5 it suffices to show that the function $\eta_p : t \mapsto \|p(a_t, b_t, a_t^*, b_t^*)\|$ is upper semi-continuous over T_q for any noncommutative polynomial p in four variables. Denote by T_q' the set of the non-positive numbers in T_q. We prove the upper semi-continuity of η_p over T'_q first.

We claim that there exists a universal C^*-algebra generated by a, b, x subject to the relations

(1) the equations in (38) with a_t, b_t, t replaced by a, b, x respectively,
(2) the inequality $\|x\| \leq |c(1)|$,
(3) $x = x^*$ is in the center.

Clearly $C(S^2_{qt})$ for $t \in T'_q$ has generators satisfying these conditions. Let a, b, x be bounded linear operators on a Hilbert space satisfying these relations. We have

$$\begin{aligned}(41) \qquad & (1+q^2)(b^*b + q^{-2}bb^*) \\ \overset{(38)}{=} \ & (1+q^2)(a - a^2 + x) + (1+q^2)(a - q^2a^2 + q^{-2}x) \\ = \ & -(1-(1+q^2)a)^2 + (1 + (1+q^2)^2q^{-2}x).\end{aligned}$$

Thus

$$\begin{aligned}\|(1-(1+q^2)a)^2\|, \|(1+q^2)b^*b\| &\leq \|1 + (1+q^2)^2q^{-2}x\| \\ &\leq 1 + (1+q^2)^2q^{-2}|c(1)| = 2,\end{aligned}$$

and hence

$$\begin{aligned}\|a\| &\leq (1+q^2)^{-1}(1+2^{\frac{1}{2}}), \\ \|b\| &\leq (1+q^2)^{-\frac{1}{2}}2^{\frac{1}{2}}.\end{aligned}$$

Therefore there does exist a universal C^*-algebra C generated by a, b, x subject to these relations. An argument similar to that in the proof of Theorem 5.11 shows that η_p is upper semi-continuous over T'_q.

The upper semi-continuity of η_p over $[0,1]$ is proved similarly, replacing (41) by

$$\begin{aligned}(42) \qquad & (1+q^2)(b^*b + q^{-2}bb^*) \\ = \ & -((1-x^2) - (1+q^2)a)^2 + ((1-x^2)^2 + (1+q^2)^2q^{-2}x^2).\end{aligned}$$

This proves the existence of the desired upper semi-continuous field of C^*-algebras over T_q. Note that B is generated as a C^*-algebra by $C(T_q)$ and the sections $t \mapsto a_t$ and $t \mapsto b_t$. From (40) one sees immediately that $(\{(C(S^2_{qt}), \sigma_t)\}_{t \in T_q}, B)$ is an upper semi-continuous field of ergodic actions of $\mathrm{SU}_q(2)$. Since $\mathrm{SU}_q(2)$ is co-amenable, by Proposition 3.8 and Lemma 5.10 this is actually a continuous field of actions. □

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. It is customary to index $\widehat{\mathrm{SU}_q(2)}$ by $0, \frac{1}{2}, 1, 1 + \frac{1}{2}, \cdots$ [46, remark after the proof of Theorem 5.8]. Say $\widehat{\mathrm{SU}_q(2)} =$

$\{\mathbf{d}_0, \mathbf{d}_{\frac{1}{2}}, \mathbf{d}_1, \mathbf{d}_{1+\frac{1}{2}}, \cdots\}$. Then $\mathrm{mul}(C(S^2_{qt}), \mathbf{d}_k) = 1$, $\mathrm{mul}(C(S^2_{qt}), \mathbf{d}_{k+\frac{1}{2}}) = 0$ for $k = 0, 1, 2, \cdots$ when $t \geq 0$. And $\mathrm{mul}(C(S^2_{qt}), \mathbf{d}_l) = 1$ or 0 depending on $l \in \{0, 1, \cdots, n-1\}$ or not when $t = c(n)$ [30, the note after Proposition 2.5]. Thus the multiplicity function $t \mapsto \mathrm{mul}(C(S^2_{qt}), \gamma)$ is continuous over T_q for any $\gamma \in \widehat{\mathrm{SU}_q(2)}$. Then Theorem 1.2 follows from Proposition 6.1 and Theorem 5.12. $\square$

7. Ergodic actions of full multiplicity of compact groups

In this section we show that the topology of Landstad and Wassermann on the set $\mathrm{EA}(G)_{\mathrm{fm}}$ of isomorphism classes of ergodic actions of full multiplicity of a compact group G coincides with the relative topology of $\mathrm{EA}_{\mathrm{fm}}$ in $\mathrm{EA}(G)$.

Throughout this section we let $\mathcal{G} = G$ be a compact Hausdorff group. An ergodic action (B, σ') of G is said to be of *full multiplicity* if $\mathrm{mul}(B, \gamma) = d_\gamma$ for all $\gamma \in \hat{G}$. Denote by $\mathrm{EA}(G)_{\mathrm{fm}}$ the set of isomorphism classes of ergodic actions of full multiplicity of G. By Proposition 4.13 $\mathrm{EA}(G)_{\mathrm{fm}}$ is a closed subset of $\mathrm{EA}(G)$.

Landstad [20] and Wassermann [43] showed independently that $\mathrm{EA}(G)_{\mathrm{fm}}$ can be identified with the set of equivalence classes of dual cocycles. Let us recall the notation in [20]. Denote by $\mathcal{L}(G)$ the von Neumann algebra generated by the left regular representation of G on $L^2(G)$. One has a natural decomposition $L^2(G) \cong \oplus_{\gamma\in\hat{G}} H_\gamma$ as unitary representations of G. Then $\mathcal{L}(G) = \prod_{\gamma\in\hat{G}} B(H_\gamma)$ under this decomposition. Denote by 1_{γ_0} the identity of $B(H_{\gamma_0})$ for the trivial representation γ_0. One has the normal $*$-homomorphism $\delta : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ (tensor product of von Neumann algebras) and the normal $*$-anti-isomorphism $\nu : \mathcal{L}(G) \to \mathcal{L}(G)$ determined by

$$\delta(x) = x \otimes x \quad \text{and} \quad \nu(x) = x^{-1} \quad \text{for } x \in G.$$

Denote by σ the flip automorphism of $\mathcal{L}(G) \otimes \mathcal{L}(G)$ determined by $\sigma(a \otimes b) = b \otimes a$ for all $a, b \in \mathcal{L}(G)$. One also has Takesaki's unitary W in $B(L^2(G)) \otimes \mathcal{L}(G)$ defined by $(Wf)(x, y) = f(x, xy)$ for $f \in C(G \times G)$ and $x, y \in G$. A *normalized dual cocycle* [20, page 376] is a unitary $w \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfying

$$\begin{aligned}
&(w \otimes I)((\delta \otimes \mathrm{id})(w)) = (I \otimes w)((\mathrm{id} \otimes \delta)(w)),\\
&(\nu \otimes \nu)(w) = \sigma(w^*), \qquad\qquad w(I \otimes 1_{\gamma_0}) = I \otimes 1_{\gamma_0},\\
&w(1_{\gamma_0} \otimes I) = 1_{\gamma_0} \otimes I, \qquad\qquad w\delta(1_{\gamma_0}) = \delta(1_{\gamma_0}),\\
&(\mathrm{id} \otimes \nu)(w\sigma(w^*)) = \sigma(w)w^*, \quad (\mathrm{id} \otimes \nu)(wW^*) = Ww^*.
\end{aligned}$$

Denote by C^2 the set of all normalized dual cocycles. Also denote by H the group of unitaries ξ in $\mathcal{L}(G)$ satisfying $\xi = \nu(\xi^*)$ and $\xi 1_{\gamma_0} = 1_{\gamma_0}$ (on page 376 of [20] only the condition $\xi = \nu(\xi^*)$ is mentioned, but in order for $\alpha_\xi(w)$ below to satisfy $\alpha_\xi(w)(I \otimes 1_{\gamma_0}) = I \otimes 1_{\gamma_0}$, one has to require $\xi 1_{\gamma_0} = 1_{\gamma_0}$; this can be seen using the formula $\delta(x)(I \otimes 1_{\gamma_0}) = x \otimes 1_{\gamma_0}$ for all $x \in \mathcal{L}(G)$). Then H has a left action α on C^2 via $\alpha_\xi(w) = (\xi \otimes \xi) w \delta(\xi^*)$. The result of Landstad and Wassermann says that $\mathrm{EA}(G)_{\mathrm{fm}}$ can be identified with C^2/H [20, Remark 3.13] in a natural way.

Note that the unitary groups of $\mathcal{L}(G) \otimes \mathcal{L}(G)$ and $\mathcal{L}(G)$ are both compact Hausdorff groups with the weak topology. Clearly C^2 and H are closed subsets of the unitary groups of $\mathcal{L}(G) \otimes \mathcal{L}(G)$ and $\mathcal{L}(G)$ respectively. Thus C^2 is a compact Hausdorff space and H is a compact Hausdorff group, with the relative topologies. It is also clear that the action α is continuous. Therefore C^2/H equipped with the quotient topology is a compact Hausdorff space.

In order to show that the quotient topology on C^2/H coincides with the relative topology of $\mathrm{EA}(G)_{\mathrm{fm}}$ in $\mathrm{EA}(G)$, we need to recall the map $C^2 \to \mathrm{EA}(G)_{\mathrm{fm}}$ constructed in the proof of [20, Theorem 3.9]. Let $w \in C^2$. Set $U = wW^* \in B(L^2(G)) \otimes \mathcal{L}(G)$. Recall that for each $\gamma \in \hat{G}$ we fixed an orthonormal basis of H_γ and identified $B(H_\gamma)$ with $M_{d_\gamma}(\mathbb{C})$. Let $e^\gamma_{ij}, 1 \le i, j \le d_\gamma$ be the matrix units of $M_{d_\gamma}(\mathbb{C})$ as usual. Then we may write U as $\sum_{\gamma \in \hat{G}} \sum_{1 \le i,j \le d_\gamma} b_{\gamma ij} \otimes e^\gamma_{ij}$ for $b_{\gamma ij} \in B(L^2(G))$. The conjugation of the right regular representation of G on $L^2(G)$ restricts on an ergodic action α of G on the C^*-algebra B generated by $b_{\gamma ij}$ for all $\gamma \in \hat{G}, 1 \le i, j \le d_\gamma$. The isomorphism class of α is the image of w under the map $C^2 \to \mathrm{EA}(G)_{\mathrm{fm}}$. Furthermore, each $U_\gamma = \sum_{1 \le i,j \le d_\gamma} b_{\gamma ij} \otimes e^\gamma_{ij}$ is a *unitary* u^γ*-eigenoperator* meaning that U_γ is a unitary in $B \otimes B(H_\gamma)$ satisfying

$$(\alpha_x \otimes \mathrm{id})(U_\gamma) = U_\gamma (1_B \otimes u^\gamma(x)) \tag{43}$$

for all $x \in G$. If we let $\sigma : B \to B \otimes C(G) = C(G, B)$ be the $*$-homomorphism associated to α, i.e., $(\sigma(b))(x) = \alpha_x(b)$, then (43) simply means $(\sigma \otimes \mathrm{id})(U_\gamma) = (U_\gamma)_{13}(\tau(u^\gamma))_{23}$, where $(U_\gamma)_{13}$ and $(\tau(u^\gamma))_{23}$ are in the leg numbering notation and $\tau : B(H_\gamma) \otimes C(G) \to C(G) \otimes B(H_\gamma)$ is the flip. It follows that (43) is equivalent to (9) with $e_{\gamma ki}$ replaced by $b_{\gamma ki}$. Then $\sum_{1 \le j \le d_\gamma} b_{\gamma ij} b^*_{\gamma kj}$ is easily seen to be σ-invariant and hence is in $\mathbb{C}1_B$. One checks easily that $U_\gamma U^*_\gamma = 1_B \otimes 1_{B(H_\gamma)}$ means that

$$\omega\Big(\sum_{1 \le j \le d_\gamma} b_{\gamma ij} b^*_{\gamma kj}\Big) = \delta_{ki} \tag{44}$$

for all $1 \leq k, i \leq d_\gamma$, where ω is the unique α-invariant state on B. Since G is a compact group, ω is a trace [14, Theorem 4.1]. From Lemma 4.11 one sees that (44) is equivalent to (15) with $e_{\gamma ki}$ replaced by $d_\gamma^{1/2} b_{\gamma ki}$. Using $W(I \otimes 1_{\gamma_0}) = w(I \otimes 1_{\gamma_0}) = I \otimes 1_{\gamma_0}$ one gets $U(I \otimes 1_{\gamma_0}) = I \otimes 1_{\gamma_0}$. Thus $b_{\gamma_0 11} = 1_B$. Therefore $d_\gamma^{1/2} b_{\gamma ij}$ for $\gamma \in \hat{G}, 1 \leq i, j \leq d_\gamma$ is a standard basis of B. Denote by $\psi(w)$ the associated element in $\mathcal{P}$. Then the diagram

$$\begin{array}{ccccc} C^2 & & \xrightarrow{\quad\psi\quad} & & \mathcal{P} \\ \downarrow & & & & \downarrow \\ C^2/H & \longrightarrow & \mathrm{EA}(G)_{\mathrm{fm}} & \hookrightarrow & \mathrm{EA}(G) = \mathrm{EA}^{\sim}(G) \end{array}$$

commutes, where we identify $\mathrm{EA}(G)$ with $\mathrm{EA}^{\sim}(G)$ since G is co-amenable. It was showed in the proof of [20, Theorem 3.9] that one has

$$U_{12}U_{13} = (I \otimes w)((\mathrm{id} \otimes \delta)(U)) \quad \text{and} \quad (\mathrm{id} \otimes \nu)(U) = U^*,$$

where U_{12} and U_{13} are in the leg numbering notation. It follows that the map ψ is continuous. Consequently, the relative topology on $\mathrm{EA}(G)_{\mathrm{fm}}$ in $\mathrm{EA}(G)$ coincides with the quotient topology coming from $C^2 \to \mathrm{EA}(G)_{\mathrm{fm}}$.

8. Induced Lip-norm

In this section we prove Theorem 1.4.

We recall first Rieffel's construction of Lip-norms from ergodic actions of compact groups. Let G be a compact group. A *length function* on G is a continuous function $l : G \to \mathbb{R}_+$ such that

$$\begin{aligned} l(xy) &\leq l(x) + l(y) \text{ for all } x, y \in G \\ l(x^{-1}) &= l(x) \text{ for all } x \in G \\ l(x) &= 0 \text{ if and only if } x = e_G. \end{aligned}$$

Given an ergodic action α of G on a unital C^*-algebra B, Rieffel showed that the seminorm L_B on B defined by

$$L_B(b) = \sup\{\frac{\|\alpha_x(b) - b\|}{l(x)} : x \in G, x \neq e_G\} \qquad (45)$$

is a Lip-norm [33, Theorem 2.3].

Note that there is a 1-1 correspondence between length functions on G and left-invariant metrics on G inducing the topology of G, via $\rho(x, y) = l(x^{-1}y)$ and $l(x) = \rho(x, e_G)$. Since a quantum metric on (the non-commutative space corresponding to) a unital C^*-algebra is a Lip-norm on this C^*-algebra, a length function for a compact quantum group $A = C(\mathcal{G})$ should be a Lip-norm L_A on A satisfying certain compatibility condition with the group structure. The proof of [33, Proposition 2.2] shows that L_B in above is finite on any α-invariant finite-dimensional subspace of B, and hence is finite on $\mathcal{B}$. If one applies this observation to the action of G on $C(G)$ corresponding to the right translation of G on itself, then we see that the Lipschitz seminorm $L_{C(G)}$ on $C(G)$ associated to the above metric ρ via

$$L_{C(G)}(a) = \sup_{x \neq y} \frac{|a(x) - a(y)|}{\rho(x, y)} = \sup_{x \neq e_G} \sup_{y} \frac{|a(yx) - a(y)|}{l(x)}$$

is finite on the algebra of regular functions in $C(G)$. This leads to the following definition:

Definition 8.1. We say that a Lip-norm L_A on a compact quantum group $A = C(\mathcal{G})$ is *regular* if L_A is finite on the algebra $\mathcal{A}$ of regular functions.

It turns out that a regular Lip-norm is sufficient for us to induce Lip-norms on C^*-algebras carrying ergodic actions of co-amenable compact quantum groups. We leave the discussion of the left and right invariance of L_A to the end of this section.

Remark 8.2. If a unital C^*-algebra B has a Lip-norm, then $S(B)$ with the weak-$*$ topology is metrizable and hence B is separable. Conversely, if B is a separable unital C^*-algebra, then for any countable subset W of B, there exist Lip-norms on B being finite on W [36, Proposition 1.1]. When $A = C(\mathcal{G})$ is separable, $\mathcal{A}$ is a countable-dimensional vector space, and hence A has regular Lip-norms.

Example 8.3. Let Γ be a discrete group. Then the reduced group C^*-algebra $C^*_r(\Gamma)$ is a compact quantum group with $\Phi(g) = g \otimes g$ for $g \in \Gamma$. Its algebra of regular functions is $\mathbb{C}\Gamma$. Let l be a length function on Γ. Denote by D the (possibly unbounded) linear operator of pointwise multiplication by l on $\ell^2(\Gamma)$. One may consider the seminorm L defined on $\mathbb{C}\Gamma$ as $L(a) = \|[D, a]\|$ and extend it to $C^*_r(\Gamma)$ via setting $L = \infty$ on $C^*_r(\Gamma) \setminus \mathbb{C}\Gamma$. The seminorm L so defined is always finite on $\mathbb{C}\Gamma$, and hence is regular if it is a Lip-norm. This is the case for $\Gamma = \mathbb{Z}^d$ when l is a word-length, or the restriction to $\mathbb{Z}^d$ of a norm on $\mathbb{R}^d$ [36, Theorem 0.1], and for Γ being a hyperbolic group when l is a word-length [27, Corollary 4.4].

Now we try to extend (45) to ergodic actions of compact quantum groups. Let $\sigma : B \to B \otimes C(G) = C(G, B)$ be the $*$-homomorphism associated to α, i.e., $(\sigma(b))(x) = \alpha_x(b)$ for $b \in B$ and $x \in G$. For any

$b \in B_{\rm sa}$, we have

$$\begin{aligned} L_B(b) &= \sup_{x \neq e_G} \sup_{y} \frac{\|\alpha_{yx}(b) - \alpha_y(b)\|}{l(x)} \\ &= \sup_{x \neq e_G} \sup_{y} \sup_{\varphi \in S(B)} \frac{|\varphi(\alpha_{yx}(b)) - \varphi(\alpha_y(b))|}{l(x)} \\ &= \sup_{\varphi \in S(B)} L_{C(G)}(b * \varphi), \end{aligned}$$

where $S(B)$ denotes the state space of B. Note that for quantum metrics, only the restriction of L_B on $B_{\rm sa}$ is essential. Thus the above formula leads to our definition of the (possibly $+\infty$-valued) seminorm L_B on B in (1) for any ergodic action $\sigma : B \to B \otimes A$ of a compact quantum group $A = C(\mathcal{G})$ equipped with a regular Lip-norm L_A.

Throughout the rest of this section we assume that L_A is a regular Lip-norm on A.

Lemma 8.4. We have

$$\|a - h(a)1_A\| \leq 2r_A L_A(a) \tag{46}$$

for all $a \in A_{\rm sa}$.

Proof. By Proposition 2.5 we can find $a' \in \mathbb{C}1_A$ such that $\|a - a'\| \leq r_A L_A(a)$. Then $\|h(a)1_A - a'\| = |h(a - a')| \leq \|a - a'\| \leq r_A L_A(a)$. Thus $\|a - h(a)1_A\| \leq \|a - a'\| + \|a' - h(a)1_A\| \leq 2r_A L_A(a)$. ☐

Lemma 8.5. Let L_B be the seminorm on a unital C^*-algebra B defined via (1) for an action $\sigma : B \to B \otimes A$ of $\mathcal{G}$ on B. Assume that A has bounded counit e. Then for any $b \in B_{\rm sa}$ we have $\|b - E(b)\| \leq 2r_A L_B(b)$, where $E : B \to B^\sigma$ is the canonical conditional expectation.

Proof. Let $\varphi \in S(B)$. Note that $h(b * \varphi) = \varphi(E(b))$. We have

$$\begin{aligned} \|b * \varphi - \varphi(E(b))1_A\| &= \|b * \varphi - h(b * \varphi)1_A\| \\ &\overset{(46)}{\leq} 2r_A L_A(b * \varphi) \overset{(1)}{\leq} 2r_A L_B(b). \end{aligned}$$

Thus

$$\sup_{\varphi \in S(B)} \|(b - E(b)) * \varphi\| = \sup_{\varphi \in S(B)} \|b * \varphi - \varphi(E(b))1_A\| \leq 2r_A L_B(b).$$

Therefore by Remark 2.2 we have

$$\begin{aligned} \|b - E(b)\| &= \|e * (b - E(b))\| = \sup_{\varphi \in S(B)} |\varphi(e * (b - E(b)))| \\ &= \sup_{\varphi \in S(B)} |e((b - E(b)) * \varphi)| \leq \sup_{\varphi \in S(B)} \|(b - E(b)) * \varphi\| \\ &\leq 2r_A L_B(b) \end{aligned}$$

as desired. ☐

For any $J \subseteq \hat{\mathcal{G}}$ denote $\sum_{\gamma \in J} A_\gamma$ and $\sum_{\gamma \in J} B_\gamma$ by A_J and B_J respectively.

Lemma 8.6. Assume that A has faithful Haar measure. For any $\varepsilon > 0$ and $\phi \in S(A)$ there exist $\psi \in S(A)$ and a finite subset $J \subseteq \hat{\mathcal{G}}$ such that ψ vanishes on A_γ for all $\gamma \in \hat{\mathcal{G}} \setminus J$ and

$$|(\phi - \psi)(a)| \leq \varepsilon L_A(a) \tag{47}$$

for all $a \in A_{\mathrm{sa}}$.

Proof. Denote by W the set of states of A consisting of convex combinations of states of the form $h(a^*(\cdot)a)$ for $a \in \mathcal{A}$ with $h(a^*a) = 1$. Let $\psi \in W$. Clearly there exists a finite subset $F \subseteq \hat{\mathcal{G}}$ such that if $h(A_F^* a' A_F) = 0$ for some $a' \in A$ then $\psi(a') = 0$. By the faithfulness of h on $\mathcal{A}$ and the Peter-Weyl theory [47, Theorems 4.2 and 5.7], for any $a' \in \mathcal{A}$ and any finite subset $J' \subseteq \hat{\mathcal{G}}$, $h(A_{J'}^* a') \neq 0$ if and only if $h(a' A_{J'}^*) \neq 0$. Denote by F' the set of equivalence classes of irreducible unitary subrepresentations of the tensor products $u^\alpha \circledT u^\beta$ of all $\alpha \in F$ and $\beta \in F^c = \{\gamma^c : \gamma \in F\}$. Denote $(F')^c$ by J. Suppose that ψ does not vanish on A_γ for some $\gamma \in \hat{\mathcal{G}}$. Then $h(A_F^* A_\gamma A_F) \neq 0$. Thus

$$h(A_\gamma A_{F'}) \overset{(8)}{\supseteq} h(A_\gamma A_F A_{F^c}) = h(A_\gamma A_F A_F^*) \supsetneqq \{0\}.$$

Since $h(A_\alpha A_\beta) = 0$ for all $\alpha \neq \beta^c$ in $\hat{\mathcal{G}}$ [47, Theorem 5.7], we get $\gamma \in J$.

Now we just need to find $\psi \in W$ such that (47) holds for all $a \in A$. Since h is faithful, the GNS representation (π_A, H_A) of A associated to h is faithful. Thus convex combinations of vector states from (π_A, H_A) are weak-$*$ dense in $S(A)$ [45, Lemma T.5.9]. Note that $\mathcal{A}$ is dense in A. Therefore W is weak-$*$ dense in $S(A)$. Take $R \geq r_A$. Since $\mathcal{D}_R(A)$ is totally bounded by Proposition 2.5, we can find $\psi \in W$ such that

$$|(\phi - \psi)(a)| \leq \varepsilon \tag{48}$$

for all $a \in \mathcal{D}_R(A)$. By Proposition 2.5 we have $\mathcal{E}(A) = \mathcal{D}_R(A) + \mathbb{R} \cdot 1_A$. Therefore (48) holds for all $a \in \mathcal{E}(A)$, from which (47) follows. ☐

The next lemma is an analogue of [34, Lemmas 8.3 and 8.4] and [21, Lemma 10.8].

Lemma 8.7. Let B and L_B be as in Lemma 8.5. Assume that A is co-amenable. Let $\varepsilon > 0$ and take ψ and J in Lemma 8.6 for ϕ being the counit e. Denote by P_ψ the linear map $B \to B$ sending $b \in B$ to $\psi * b$. Then $P_\psi(B) \subseteq B_J$ and

$$\|P_\psi(b)\| \leq \|b\|, \quad \text{and} \quad \|b - P_\psi(b)\| \leq \varepsilon L_B(b) \tag{49}$$

for all $b \in B_{\rm sa}$.

Proof. Since ψ vanishes on A_γ for all $\gamma \in \hat{\mathcal{G}} \setminus J$ and $\sigma(B_\beta) \subseteq B_\beta \odot A_\beta$ for all $\beta \in \hat{\mathcal{G}}$, we have $P_\psi(B_\beta) \subseteq B_J$ for all $\beta \in \hat{\mathcal{G}}$. Note that B_β is finite dimensional and $\mathcal{B} = \sum_{\beta \in \hat{\mathcal{G}}} B_\beta$ is dense in B. Thus $P_\psi(B) \subseteq B_J$.

For any $b \in B$ clearly $\|P_\psi(b)\| \le \|b\|$. If $b \in B_{\rm sa}$, by Remark 2.2 we have

$$\begin{aligned} \|b - P_\psi(b)\| &= \|e * (b - P_\psi(b))\| = \sup_{\varphi \in S(B)} |\varphi(e * (b - P_\psi(b)))| \\ &= \sup_{\varphi \in S(B)} |e(b * \varphi) - \psi(b * \varphi)| \overset{(47)}{\le} \sup_{\varphi \in S(B)} \varepsilon L_A(b * \varphi) \\ &\overset{(1)}{=} \varepsilon L_B(b). \end{aligned}$$

This finishes the proof of Lemma 8.7. □

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We verify the conditions in Proposition 2.5. For any $b \in B$ and φ in $S(B)$ we have $b^* * \varphi = (b * \varphi)^*$. Since L_A satisfies the reality condition (10), so does L_B. For any $b \in \mathcal{B}$, $\{b * \varphi : \varphi \in S(B)\}$ is bounded and contained in a finite dimensional subspace of $\mathcal{A}$ since $\sigma(\mathcal{B}) \subseteq \mathcal{B} \odot \mathcal{A}$. Then L_B is finite on $\mathcal{B}$ because of the regularity of L_A. Clearly L_B vanishes on $\mathbb{C}1_B$. By Lemma 8.5 we have $\|\cdot\|^\sim \le 2r_A \tilde{L}_B$ on $(\tilde{B})_{\rm sa}$. For any $\varepsilon > 0$ let P_ψ and J be as in Lemma 8.7. Then $P_\psi(\mathcal{D}_1(B))$ is a bounded subset of the finite dimensional space B_J. Thus $P_\psi(\mathcal{D}_1(B))$ is totally bounded. Since $\varepsilon > 0$ is arbitrary, $\mathcal{D}_1(B)$ is also totally bounded. Therefore Theorem 1.4 follows from Proposition 2.5. □

Now we consider the invariance of a (possibly $+\infty$-valued) seminorm on B with respect to an action σ of $\mathcal{G}$. We consider first the case $\mathcal{G} = G$ is a compact group. For any action of $A = C(G)$ on B, there is a strongly continuous action α of G on B such that for any $b \in B$, the element $\sigma(b) \in B \otimes A = C(G, B)$ is given by $(\sigma(b))(x) = \alpha_x(b)$ for all $x \in G$. If a seminorm L_B on B is lower semi-continuous, which is the case if L_B is defined via (45), and is α-invariant, then for any $\psi \in S(A)$ corresponding to a Borel probability measure μ on G, we have

$$L_B(\psi * b) = L_B(\int_G \alpha_x(b)\, d\mu(x)) \le L_B(b)$$

for all $b \in B$. Conversely, if $L_B(\psi * b) \le L_B(b)$ for all $b \in B$ and $\psi \in S(B)$, taking ψ to be the evaluation at $x \in G$, one sees immediately that L_B is α-invariant. Note that the essential information about the quantum metric is the restriction of L_B on B_{sa}. This leads to the following

Definition 8.8. Let $A = C(\mathcal{G})$ be a compact quantum group. We say that a (possibly $+\infty$-valued) seminorm L_A on A is *right-invariant* (*left-invariant* resp.) if

$$L_A(\psi * a) \leq L_A(a) \quad (L_A(a * \psi) \leq L_A(a) \text{ resp. })$$

for all $a \in A_{\text{sa}}$ and $\psi \in S(A)$. For an action $\sigma : B \to B \otimes A$ of $\mathcal{G}$ on a unital C^*-algebra B, we say that a (possibly $+\infty$-valued) seminorm L_B on B is *invariant* if

$$L_B(\psi * b) \leq L_B(b)$$

for all $b \in B_{\text{sa}}$ and $\psi \in S(A)$.

Proposition 8.9. Let L_A be a regular Lip-norm on A. Define (possibly $+\infty$-valued) seminorms L'_A and L''_A on A by

$$L'_A(a) = \sup_{\varphi \in S(A)} L_A(\varphi * a),$$

and

$$L''_A(a) = \sup_{\varphi \in S(A)} L_A(a * \varphi)$$

for $a \in A$. Assume that A has bounded counit. Then L'_A (L''_A resp.) is a right-invariant (left-invariant resp.) regular Lip-norm on A, and $L'_A \geq L_A$ ($L''_A \geq L_A$ resp.). If L_A is left-invariant (right-invariant resp.), then so is L'_A (L''_A resp.).

Proof. An argument similar to that in the proof of Theorem 1.4 shows that L'_A satisfies the reality condition (10), vanishes on $\mathbb{C}1_A$, and is finite on $\mathcal{A}$. Taking φ to be the counit we see that $L'_A \geq L_A$. It follows immediately from Proposition 2.5 that L'_A is a regular Lip-norm on A. For any $a \in A_{\text{sa}}$ and $\psi \in S(A)$ we have

$$L'_A(\psi * a) = \sup_{\varphi \in S(A)} L_A(\varphi * (\psi * a)) = \sup_{\varphi \in S(A)} L_A((\varphi * \psi) * a) \leq L'_A(a),$$

where $\varphi * \psi$ is the state on A defined via $(\varphi * \psi)(a') = (\varphi \otimes \psi)(\Phi(a'))$ for $a' \in A$. Therefore L'_A is right-invariant. Assume that L_A is left-invariant. Then for any $a \in A_{\text{sa}}$ and $\psi \in S(A)$ we have

$$\begin{aligned} L'_A(a * \psi) &= \sup_{\varphi \in S(A)} L_A(\varphi * (a * \psi)) = \sup_{\varphi \in S(A)} L_A((\varphi * a) * \psi) \\ &\leq \sup_{\varphi \in S(A)} L_A(\varphi * a) = L'_A(a). \end{aligned}$$

Thus L'_A is also left-invariant. The assertions about L''_A are proved similarly. ∎

Using Remark 8.2 and applying the construction in Proposition 8.9 twice, we get

Corollary 8.10. Every separable compact quantum group with bounded counit has a bi-invariant regular Lip-norm.

An argument similar to that in the proof of Proposition 8.9 shows

Proposition 8.11. Let σ be an action of $\mathcal{G}$ on a unital C^*-algebra B. If L_A is a right-invariant regular Lip-norm on A, then L_B defined via (1) is invariant.

9. Quantum distance

In this section we introduce the quantum distance $\mathrm{dist}_{\mathrm{e}}$ between ergodic actions of $\mathcal{G}$, and prove Theorem 1.5.

Throughout this section, A will be a co-amenable compact quantum group with a fixed regular Lip-norm L_A. For any ergodic action (B, σ) of $\mathcal{G}$, we endow B with the Lip-norm L_B in Theorem 1.4.

In [15, 16, 21, 22, 34] several quantum Gromov-Hausdorff distances are introduced, applying to quantum metric spaces in various contexts as order-unit spaces, operator systems, and C^*-algebras. They are all applicable to C^*-algebraic compact quantum metric spaces, which we are dealing with now. Among these distances, the unital version $\mathrm{dist}_{\mathrm{nu}}$ of the one introduced in [22, Remark 5.5] is the strongest one, which we recall below from [16, Section 5]. To simplify the notation, for fixed unital C^*-algebras B_1 and B_2, when we take infimum over unital C^*-algebras C containing both B_1 and B_2, we mean to take infimum over all unital injective $*$-homomorphisms of B_1 and B_2 into some unital C^*-algebra C. We denote by $\mathrm{dist}_{\mathrm{H}}^{C}$ the Hausdorff distance between subsets of C. Recall that $\mathcal{E}(B) := \{b \in B_{\mathrm{sa}} : L_B(b) \leq 1\}$. For any C^*-algebraic compact quantum metric spaces (B_1, L_{B_1}) and (B_2, L_{B_2}), the distance $\mathrm{dist}_{\mathrm{nu}}(B_1, B_2)$ is defined as

$$\mathrm{dist}_{\mathrm{nu}}(B_1, B_2) \quad = \quad \inf \mathrm{dist}_{\mathrm{H}}^{C}(\mathcal{E}(B_1), \mathcal{E}(B_2)),$$

where the infimum is taken over all unital C^*-algebras C containing B_1 and B_2. Note that $\mathrm{dist}_{\mathrm{nu}}(B_1, B_2)$ is always finite since $\mathcal{D}_R(\mathcal{B})$ is totally bounded and $\mathcal{E}(B) = \mathcal{D}_R(\mathcal{B}) + \mathbb{R} \cdot 1_B$ for any $R \geq r_B$ by Proposition 2.5. These distances become zero whenever there is a $*$-isomorphism $\varphi : B_1 \to B_2$ preserving the Lip-norms on the self-adjoint parts. In particular, as the following example shows, these distances may not distinguish the actions when the Lip-norms L_{B_i} come from ergodic actions of $\mathcal{G}$ on B_i.

Example 9.1. Let l' be a length function on the circle S^1. Set l to be the length function on the two-torus $\mathbb{T}^2$ defined as $l(x,y) = l'(x) + l'(y)$ for $x, y \in S^1$. Then $l(x,y) = l(x^{-1}, y)$ for all $(x,y) \in \mathbb{T}^2$. Let $\theta \in \mathbb{R}$, and let B_θ be the non-commutative two-torus generated by unitaries u_θ and v_θ satisfying $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. Then $\mathbb{T}^2$ has a strongly continuous action α_θ on B_θ specified by $\alpha_{\theta,(x,y)}(u_\theta) = xu_\theta$ and $\alpha_{\theta,(x,y)}(v_\theta) = yv_\theta$. Consider the $*$-isomorphism $\psi : B_\theta \to B_{-\theta}$ determined by $\psi(u_\theta) = (u_{-\theta})^{-1}$ and $\psi(v_\theta) = v_{-\theta}$. Then ψ preserves the Lip-norms defined via (45) for the actions α_θ and $\alpha_{-\theta}$ of $\mathbb{T}^2$, and hence B_θ and $B_{-\theta}$ have distances zero under all the quantum distances defined in [15, 16, 21, 22, 34]. However, when $0 < \theta < 1/2$, the actions $(B_\theta, \alpha_\theta)$ and $(B_{-\theta}, \alpha_{-\theta})$ are not isomorphic, as can be seen from the fact that $\mathbb{C}u_\theta = \{b \in B_\theta : \alpha_{\theta,(x,y)}(b) = xb \text{ for all } (x,y) \in \mathbb{T}^2\}$ and $\mathbb{C}v_\theta = \{b \in B_\theta : \alpha_{\theta,(x,y)}(b) = yb \text{ for all } (x,y) \in \mathbb{T}^2\}$.

Notation 9.2. For any C^*-algebra C we denote $C \oplus (C \otimes A)$ by $C^\sharp$. For any action $\sigma : B \to B \otimes A$ of $\mathcal{G}$ on a unital C^*-algebra B and any subset $\mathcal{X}$ of B we denote by $\mathcal{X}_\sigma$ the graph

$$\{(b, \sigma(b)) \in B^\sharp : b \in \mathcal{X}\}$$

of $\sigma|_{\mathcal{X}}$.

We are going to introduce a quantum distance between ergodic actions of $\mathcal{G}$ to distinguish the actions. Modifying the above definition of $\mathrm{dist}_{\mathrm{nu}}$, we just need to add one term to take care of the actions:

Definition 9.3. Let (B_1, σ_1) and (B_2, σ_2) be ergodic actions of A. We set

$$\mathrm{dist}_{\mathrm{e}}(B_1, B_2) = \inf \mathrm{dist}_{\mathrm{H}}^{C^\sharp}((\mathcal{E}(B_1))_{\sigma_1}, (\mathcal{E}(B_2))_{\sigma_2}),$$

where the infimum is taken over all unital C^*-algebras C containing both B_1 and B_2.

Clearly $\mathrm{dist}_{\mathrm{e}} \geq \mathrm{dist}_{\mathrm{nu}}$. An argument similar to that in the proof of [22, Theorem 3.15] yields

Proposition 9.4. The distance $\mathrm{dist}_{\mathrm{e}}$ is a metric on $\mathrm{EA}(\mathcal{G})$.

We relate first continuous fields of ergodic actions of $\mathcal{G}$ to the distance $\mathrm{dist}_{\mathrm{e}}$.

Proposition 9.5. Suppose that L_A is left-invariant. Let $(\{(B_t, \sigma_t)\}_{t\in T}, B)$ be a continuous field of ergodic actions of $\mathcal{G}$ over a compact metric space T. Let $t_0 \in T$. If $\lim_{t\to t_0} \mathrm{mul}(B_t, \gamma) = \mathrm{mul}(B_{t_0}, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$, then $\mathrm{dist}_{\mathrm{e}}(B_t, B_{t_0}) \to 0$ as $t \to t_0$.

To simplify the notation, we shall write L_t for L_{B_t} below.

Lemma 9.6. Let the notation be as in Proposition 9.5. Let J be a finite subset of $\hat{\mathcal{G}}$, and let $b \in B$ such that $\pi_t(b) \in (B_t)_J$ for each $t \in T$. Then the function $t \mapsto L_t(\pi_t(b))$ is continuous on T.

Proof. Let $s \in T$. To prove the continuity of $t \mapsto L_t(\pi_t(b))$ at $t = s$, it suffices to show that for any sequence $t_n \to s$ one has $L_{t_n}(\pi_{t_n}(b)) \to L_s(\pi_s(b))$. By Remark 8.2 each B_t is separable. Taking restriction to the closure of this sequence, we may assume that B is separable. Since A is co-amenable, any unital C^*-algebra admitting an ergodic action of A is nuclear [10]. Every separable continuous field of unital nuclear C^*-algebras over a compact metric space can be subtrivialized [4, Theorem 3.2]. Thus we can find a unital C^*-algebra C and unital embeddings $B_t \to C$ for all $t \in T$ such that (via identifying each B_t with its image in C) elements in B are exactly those continuous maps $T \to C$ whose images at each t are in B_t.

Let $\varphi_s \in S(B_s)$. Extend it to a state of C and let φ_t be the restriction on B_t for each $t \in T$. Then $\varphi_t \in S(B_t)$ for each $t \in T$ and $\varphi_t(\pi_t(c)) \to \varphi_s(\pi_s(c))$ as $t \to s$ for any $c \in B$. Say,

$$\sigma_t(\pi_t(b)) = \sum_{\gamma \in J} \sum_{1 \leq i,j \leq d_\gamma} c_{\gamma ij}(t) \otimes u^{\gamma}_{ij}$$

for all $t \in T$. Then clearly the sections $t \mapsto c_{\gamma ij}(t)$ are in B. Thus $\pi_t(b) * \varphi_t$ converges to $\pi_s(b) * \varphi_s$ in A_J as $t \to s$. Since A_J is finite dimensional, L_A is continuous on A_J. Therefore $L_A(\pi_t(b) * \varphi_t)$ converges to $L_A(\pi_s(b) * \varphi_s)$ as $t \to s$. Then it follows easily that the function $t \mapsto L_t(\pi_t(b))$ is lower semi-continuous at s.

Let $\varepsilon > 0$. Take a sequence $t_1, t_2, \cdots$ in T converging to s such that

$$\varepsilon + L_{t_n}(\pi_{t_n}(b)) \geq \limsup_{t \to s} L_t(\pi_t(b))$$

for each $n \geq 1$. Take $\varphi_{t_n} \in S(B_{t_n})$ for each $n \geq 1$ such that

$$\varepsilon + L_A(\pi_{t_n}(b) * \varphi_{t_n}) \geq L_{t_n}(\pi_{t_n}(b)).$$

Since B is separable, passing to a subsequence if necessary, we may assume that $\varphi_{t_n} \circ \pi_{t_n}$ converges to some state ψ of B (in the weak-$*$ topology) as $n \to \infty$. Then $\psi = \varphi_s \circ \pi_s$ for some $\varphi_s \in S(B_s)$ by the upper semi-continuity of the field $(\{B_t\}_{t \in T}, B)$. We have $\varphi_{t_n}(\pi_{t_n}(c)) \to \varphi_s(\pi_s(c))$ as $n \to \infty$ for any $c \in B$. As in the second paragraph of the proof, $L_A(\pi_{t_n}(b) * \varphi_{t_n})$ converges to $L_A(\pi_s(b) * \varphi_s)$ as $n \to \infty$. Therefore,

$$2\varepsilon + L_s(\pi_s(b)) \geq 2\varepsilon + L_A(\pi_s(b) * \varphi_s) \geq \limsup_{t \to t_0} L_t(\pi_t(b)).$$

Thus the function $t \mapsto L_t(\pi_t(b))$ is upper semi-continuous at s and hence continuous at s. ∎

Lemma 9.7. Let V be a finite-dimensional vector space, and let W be a linear subspace of V. Let T be a topological space. Let $\|\cdot\|_t$ be a norm on V and L_t be a seminorm on V vanishing exactly on W for each $t \in T$ such that the functions $t \mapsto \|v\|_t$ and $t \mapsto L_t(v)$ are upper semicontinuous and continuous respectively on T for every $v \in V$. Let $t_0 \in T$, and let $\varepsilon > 0$. Then

$$\mathrm{dist}_{\mathrm{H}}^{\|\cdot\|_t}(\mathcal{E}_{t_0}(V), \mathcal{E}_t(V)) \leq \varepsilon \tag{50}$$

throughout some neighborhood U of t_0, where $\mathcal{E}_t(V) = \{v \in V : L_t(v) \leq 1\}$.

Proof. Via considering V/W we may assume that $W = \{0\}$. For any $\delta > 0$, a standard compactness argument shows that

$$\begin{aligned} \|\cdot\|_t &\leq (1+\delta)\|\cdot\|_{t_0}, \\ \frac{1}{1+\delta}L_{t_0} \leq \quad L_t &\leq (1+\delta)L_{t_0} \end{aligned}$$

throughout some neighborhood U_δ of t_0. Then we can find some $R > 0$ such that $\|\cdot\|_t \leq RL_t(\cdot)$ throughout U_1. Fix $\delta = R/\varepsilon$. Let $t \in U_1 \cap U_\delta$ and $v \in \mathcal{E}_{t_0}(V)$. Then $v/(1+\delta) \in \mathcal{E}_t(V)$, and

$$\|v - v/(1+\delta)\|_t = \frac{\delta}{1+\delta}\|v\|_t \leq \delta\|v\|_{t_0} \leq \delta R = \varepsilon.$$

Similarly, for any $t \in U_1 \cap U_\delta$ and $v \in \mathcal{E}_t(V)$, we have $v/(1+\delta) \in \mathcal{E}_{t_0}(V)$ and $\|v - v/(1+\delta)\|_t \leq \varepsilon$. This proves (50). □

We are ready to prove Proposition 9.5.

Proof of Proposition 9.5. As in the first paragraph of the proof of Lemma 9.6 we may assume that there is a unital C^*-algebra C containing each B_t as a unital C^*-subalgebra and that elements in B are exactly those continuous maps $T \to C$ whose images at each t are in B_t. Let $\varepsilon > 0$. Pick $\psi \in S(A)$ and $J \subseteq \hat{\mathcal{G}}$ in Lemma 8.6 for ϕ being the counit. We may assume that $\gamma_0 \in J$ and $\gamma^c \in J$ for each $\gamma \in J$. Then $1_{B_t} \in (B_t)_J$ and $((B_t)_J)^* = (B_t)_J$. By Proposition 8.11 L_t is invariant for all $t \in T$. By Lemma 8.7 we have

$$\mathrm{dist}_{\mathrm{H}}^{C}(\mathcal{E}(B_t), \mathcal{E}((B_t)_J)) \leq \varepsilon \tag{51}$$

for all $t \in T$, where $\mathcal{E}((B_t)_J) := \mathcal{E}(B_t) \cap (B_t)_J$. Suppose that $\lim_{t \to t_0} \mathrm{mul}(B_t, \gamma) = \mathrm{mul}(B_{t_0}, \gamma)$ for all $\gamma \in \hat{\mathcal{G}}$. By Lemma 5.13 there are a neighborhood U of t_0 and a linear isomorphism $\varphi_t : (B_{t_0})_J \to (B_t)_J$ for each $t \in U$ such that $\varphi_{t_0} = \mathrm{id}$, $\varphi_t((B_{t_0})_\gamma) = (B_t)_\gamma$ for each $\gamma \in J$ and $t \in U$, and the map $t \mapsto \varphi_t(v) \in C$ is continuous over U for all $v \in (B_{t_0})_J$. Replacing φ_t by $(\varphi_t + \varphi_t^*)(\varphi_t(1_{B_{t_0}}) + \varphi_t(1_{B_{t_0}})^*)^{-1}$ and shrinking U if necessary, we may assume that φ_t is unital and Hermitian throughout U. By Lemma 9.6 we know that $\{\|\cdot\|_C \circ \varphi_t\}_{t\in U}$ and $\{L_t \circ \varphi_t\}_{t\in U}$ are continuous families of norms and seminorms on $(B_{t_0})_J$. By Lemma 9.7, shrinking U if necessary, we have

$$\mathrm{dist}^C_{\mathrm{H}}(\varphi_t(\mathcal{X}),\, \mathcal{E}((B_t)_J)) < \varepsilon \tag{52}$$

throughout U, where $\mathcal{X} = \mathcal{E}((B_{t_0})_J)$. Putting (51) and (52) together, we get

$$\mathrm{dist}^C_{\mathrm{H}}(\mathcal{E}(B_t),\, \varphi_t(\mathcal{X})) < 2\varepsilon \tag{53}$$

throughout U. Note that

$$\mathrm{dist}^C_{\mathrm{H}}(\mathcal{Y},\, \mathcal{Z}) = \mathrm{dist}^{C^\sharp}_{\mathrm{H}}(\mathcal{Y}_{\sigma_t},\, \mathcal{Z}_{\sigma_t})$$

for any subsets $\mathcal{Y}, \mathcal{Z}$ of B_t. Thus

$$\mathrm{dist}^{C^\sharp}_{\mathrm{H}}((\mathcal{E}(B_t))_{\sigma_t},\, (\varphi_t(\mathcal{X}))_{\sigma_t}) < 2\varepsilon \tag{54}$$

throughout U. By Lemma 5.8 we may identify elements of $B \otimes A$ with the continuous maps $T \to C \otimes A$ whose images at each t are in $B_t \otimes A$. Since $\mathcal{D}_R((B_{t_0})_J)$ is totally bounded and $\mathcal{X} = \mathcal{D}_R((B_{t_0})_J) + \mathbb{R}\cdot 1_{B_{t_0}}$ for any $R \geq 2r_A$ by Lemma 8.5, shrinking U if necessary, we may assume that $\|\sigma_t(\varphi_t(x)) - \sigma_{t_0}(x)\|_{C\otimes A}$, $\|\varphi_t(x) - x\|_C < \varepsilon$ for all $x \in \mathcal{X}$ and $t \in U$. Then

$$\mathrm{dist}^{C^\sharp}_{\mathrm{H}}((\varphi_t(\mathcal{X}))_{\sigma_t},\, \mathcal{X}_{\sigma_{t_0}}) < \varepsilon \tag{55}$$

throughout U. Putting (54) and (55) together, we get

$$\mathrm{dist}_{\mathrm{e}}(B_t, B_{t_0}) \;\leq\; \mathrm{dist}^{C^\sharp}_{\mathrm{H}}((\mathcal{E}(B_t))_{\sigma_t},\, (\mathcal{E}(B_{t_0}))_{\sigma_{t_0}}) < 6\varepsilon$$

throughout U. This finishes the proof of Proposition 9.5. □

Remark 9.8. Since $\mathrm{dist}_{\mathrm{e}} \geq \mathrm{dist}_{\mathrm{nu}}$ and $\mathrm{dist}_{\mathrm{nu}}$ is the strongest one among the quantum distances defined in [15, 16, 21, 22, 34], Proposition 9.5 also holds with $\mathrm{dist}_{\mathrm{e}}$ replaced by any of them.

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. By Proposition 9.4 $\mathrm{dist}_{\mathrm{e}}$ is a metric on $\mathrm{EA}(\mathcal{G})$. By Theorem 5.17 and Proposition 9.5 the topology on $\mathrm{EA}(\mathcal{G})$ defined in Definition 4.3 is stronger than that induced by $\mathrm{dist}_{\mathrm{e}}$. By Theorem 4.4 the former is compact. Thus these two topologies coincide. □

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