

# APPENDIX TO V. MATHAI AND J. ROSENBERG'S PAPER "A NONCOMMUTATIVE SIGMA-MODEL"

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This short note is an appendix to [6].

Let  $\theta \in \mathbb{R}$ . Denote by  $A_\theta$  the rotation  $C^*$ -algebra generated by unitaries  $U$  and  $V$  subject to  $UV = e^{2\pi i\theta}VU$ , and by  $A_\theta^\infty$  its canonical smooth subalgebra. Denote by  $\text{tr}$  the canonical faithful tracial state on  $A_\theta$  determined by  $\text{tr}(U^m V^n) = \delta_{m,0}\delta_{n,0}$  for all  $m, n \in \mathbb{Z}$ . Denote by  $\delta_1$  and  $\delta_2$  the unbounded closed  $*$ -derivations of  $A_\theta$  defined on some dense subalgebras of  $A_\theta$  and determined by  $\delta_1(U) = 2\pi iU$ ,  $\delta_1(V) = 0$ , and  $\delta_2(U) = 0$ ,  $\delta_2(V) = 2\pi iV$ . The *energy* [9],  $E(u)$ , of a unitary  $u$  in  $A_\theta$  is defined as

$$(1) \quad E(u) = \frac{1}{2}\text{tr}(\delta_1(u)^*\delta_1(u) + \delta_2(u)^*\delta_2(u))$$

when  $u$  belongs to the domains of  $\delta_1$  and  $\delta_2$ , and  $\infty$  otherwise.

Rosenberg has the following conjecture [9, Conjecture 5.4].

**Conjecture 1.** For any  $m, n \in \mathbb{Z}$ , in the connected component of  $U^m V^n$  in the unitary group of  $A_\theta^\infty$ , the functional  $E$  takes its minimal value exactly at the scalar multiples of  $U^m V^n$ .

For a  $*$ -endomorphism  $\varphi$  of  $A_\theta^\infty$ , its *energy* [6],  $\mathcal{L}(\varphi)$ , is defined as  $2E(\varphi(U)) + 2E(\varphi(V))$ . Mathai and Rosenberg's Conjecture 3.1 in [6] about the minimal value of  $\mathcal{L}(\varphi)$  follows directly from Conjecture 1.

Denote by  $H$  the Hilbert space associated to the GNS representation of  $A_\theta$  for  $\text{tr}$ , and denote by  $\|\cdot\|_2$  its norm. We shall identify  $A_\theta$  as a subspace of  $H$  as usual. Then (1) can be rewritten as

$$E(u) = \frac{1}{2}(\|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2).$$

Now we prove Conjecture 1, and hence also prove Conjecture 3.1 of [6].

**Theorem 2.** *Let  $\theta \in \mathbb{R}$  and  $m, n \in \mathbb{Z}$ . Let  $u \in A_\theta$  be a unitary whose class in  $K_1(A_\theta)$  is the same as that of  $U^m V^n$ . Then  $E(u) \geq E(U^m V^n)$ , and "=" holds if and only if  $u$  is a scalar multiple of  $U^m V^n$ .*

*Proof.* We may assume that  $u$  belongs to the domains of  $\delta_1$  and  $\delta_2$ . Set  $a_j = u^*\delta_j(u)$  for  $j = 1, 2$ . For any closed  $*$ -derivation  $\delta$  defined on a dense subset of a unital  $C^*$ -algebra  $A$  and any tracial state  $\tau$  of  $A$  vanishing on the range of  $\delta$ , if unitaries  $v_1$

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and  $v_2$  in the domain of  $\delta$  have the same class in  $K_1(A)$ , then  $\tau(v_1^*\delta(v_1)) = \tau(v_2^*\delta(v_2))$  [7, page 281]. Thus

$$\mathrm{tr}(a_j) = \mathrm{tr}((U^m V^n)^* \delta_j(U^m V^n)) = \begin{cases} 2\pi i m & \text{if } j = 1; \\ 2\pi i n & \text{if } j = 2. \end{cases}$$

We have

$$\begin{aligned} \|\delta_j(u)\|_2^2 &= \|a_j\|_2^2 = \|\mathrm{tr}(a_j)\|_2^2 + \|a_j - \mathrm{tr}(a_j)\|_2^2 \\ &\geq \|\mathrm{tr}(a_j)\|_2^2 = |\mathrm{tr}(a_j)|^2 \\ &= \begin{cases} 4\pi^2 m^2 & \text{if } j = 1; \\ 4\pi^2 n^2 & \text{if } j = 2, \end{cases} \end{aligned}$$

and “=” holds if and only if  $a_j = \mathrm{tr}(a_j)$ . It follows that  $E(u) \geq 2\pi^2(m^2 + n^2)$ , and “=” holds if and only if  $\delta_1(u) = 2\pi i m u$  and  $\delta_2(u) = 2\pi i n u$ . Now the theorem follows from the fact that the elements  $a$  in  $A_\theta$  satisfying  $\delta_1(a) = 2\pi i m a$  and  $\delta_2(a) = 2\pi i n a$  are exactly the scalar multiples of  $U^m V^n$ .  $\square$

When  $\theta \in \mathbb{R}$  is irrational, the  $C^*$ -algebra  $A_\theta$  is simple [10, Theorem 3.7], has real rank zero [1, Theorem 1.5], and is an  $AT$ -algebra [5, Theorem 4]. It is a result of Elliott that for any pair of  $AT$ -algebras with real rank zero, every homomorphism between their graded  $K$ -groups preserving the graded dimension range is induced by a  $*$ -homomorphism between them [4, Theorem 7.3]. The graded dimension range of a unital simple  $AT$ -algebra  $A$  is the subset  $\{(g_0, g_1) \in K_0(A) \oplus K_1(A) : 0 \not\leq g_0 \leq [1_A]_0\} \cup (0, 0)$  of the graded  $K$ -group  $K_0(A) \oplus K_1(A)$  [8, page 51]. It follows that, when  $\theta$  is irrational, for any group endomorphism  $\psi$  of  $K_1(A_\theta)$ , there is a unital  $*$ -endomorphism  $\varphi$  of  $A_\theta$  inducing  $\psi$  on  $K_1(A_\theta)$ . It is an open question when one can choose  $\varphi$  to be smooth in the sense of preserving  $A_\theta^\infty$ , though it was shown in [2, 3] that if  $\theta$  is irrational and  $\varphi$  restricts to a  $*$ -automorphism of  $A_\theta^\infty$ , then  $\psi$  must be an automorphism of the rank-two free abelian group  $K_1(A_\theta)$  with determinant 1. When  $\psi$  is the zero endomorphism, from Theorem 2 one might guess that  $\mathcal{L}(\varphi)$  could be arbitrarily small. It is somehow surprising, as we show now, that in fact there is a common positive lower bound for  $\mathcal{L}(\varphi)$  for all  $0 < \theta < 1$ . This answers a question Rosenberg raised at the Noncommutative Geometry workshop at Oberwolfach in September 2009.

**Theorem 3.** *Suppose that  $0 < \theta < 1$ . For any unital  $*$ -endomorphism  $\varphi$  of  $A_\theta$ , one has  $\mathcal{L}(\varphi) \geq 4(3 - \sqrt{5})\pi^2$ .*

Theorem 3 is a direct consequence of the following lemma.

**Lemma 4.** *Let  $\theta \in \mathbb{R}$  and let  $u, v$  be unitaries in  $A_\theta$  with  $uv = \lambda vu$  for some  $\lambda \in \mathbb{C} \setminus \{1\}$ . Then  $E(u) + E(v) \geq 2(3 - \sqrt{5})\pi^2$ .*

*Proof.* We have

$$\mathrm{tr}(uv) = \mathrm{tr}(\lambda vu) = \lambda \mathrm{tr}(uv),$$

and hence  $\operatorname{tr}(uv) = 0$ . Thus

$$\begin{aligned} -\operatorname{tr}(u)\operatorname{tr}(v) &= \operatorname{tr}(uv - \operatorname{tr}(u)\operatorname{tr}(v)) = \operatorname{tr}((u - \operatorname{tr}(u))v) + \operatorname{tr}(\operatorname{tr}(u)(v - \operatorname{tr}(v))) \\ &= \operatorname{tr}((u - \operatorname{tr}(u))v). \end{aligned}$$

We may assume that both  $u$  and  $v$  belong to the domains of  $\delta_1$  and  $\delta_2$ . For any  $m, n \in \mathbb{Z}$ , denote by  $a_{m,n}$  the Fourier coefficient  $\langle u, U^m V^n \rangle$  of  $u$ . Then  $a_{0,0} = \operatorname{tr}(u)$ , and

$$\begin{aligned} (2\pi)^2 \|u - \operatorname{tr}(u)\|_2^2 &= \sum_{m,n \in \mathbb{Z}, m^2+n^2>0} |2\pi a_{m,n}|^2 \\ &\leq \sum_{m,n \in \mathbb{Z}, m^2+n^2>0} |2\pi a_{m,n}|^2 (m^2 + n^2) \\ &= \|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2 = 2E(u). \end{aligned}$$

Thus

$$|\operatorname{tr}(u)|^2 = \|\operatorname{tr}(u)\|_2^2 = \|u\|_2^2 - \|u - \operatorname{tr}(u)\|_2^2 \geq 1 - \frac{1}{2\pi^2} E(u),$$

and

$$|\operatorname{tr}((u - \operatorname{tr}(u))v)| \leq \|(u - \operatorname{tr}(u))v\|_2 = \|u - \operatorname{tr}(u)\|_2 \leq \left(\frac{1}{2\pi^2} E(u)\right)^{1/2}.$$

Similarly,  $|\operatorname{tr}(v)|^2 \geq 1 - \frac{1}{2\pi^2} E(v)$ .

Write  $\frac{1}{2\pi^2} E(u)$  and  $\frac{1}{2\pi^2} E(v)$  as  $t$  and  $s$  respectively. We just need to show that  $t + s \geq 3 - \sqrt{5}$ . If  $t \geq 1$  or  $s \geq 1$ , then this is trivial. Thus we may assume that  $1 - t, 1 - s > 0$ . Then

$$(1 - t)(1 - s) \leq |\operatorname{tr}(u)\operatorname{tr}(v)|^2 \leq t.$$

Equivalently,  $t(1 - s) \geq 1 - (t + s)$ . Without loss of generality, we may assume  $s \geq t$ . Write  $t + s$  as  $w$ . Then

$$t(1 - w/2) \geq t(1 - s) \geq 1 - (t + s) = 1 - w,$$

and hence

$$w = t + s \geq \frac{1 - w}{1 - w/2} + \frac{w}{2}.$$

It follows that  $w^2 - 6w + 4 \leq 0$ . Thus  $w \geq 3 - \sqrt{5}$ .  $\square$

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