An Example where $dist_{GH} > dist_q$

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Proposition 5.7 in [2] says that for compact quantum metric spaces (B_i, L_i), if their states spaces are affinely isometrically embedded into the states space $S(A)$ of some other compact quantum metric space, then

$$dist_q(B_1, B_2) \le dist_H^{S(A)}(S(B_1), S(B_2))$$

where $dist_H^{S(A)}$ is the Hausdorff distance for compact subsets in $S(A)$. This provides a useful way getting upper bounds of quantum Gromov-Hausdorff distance. In practice, it's quite easy to embed the states space of some quantum metric space into some other compact metric space. So we need to find out what kind of compact metric space could be the states space of a quamtum metric sapce.

Let (A, L_A) be a quantum metric space. Then clearly $S(A)$ with the metric ρ_{L_A} is a compact balanced convex metric space, i.e. the metric ρ_{L_A} is convex and balanced. Another important property of $S(A)$ is that the $\mathbb{R}-$valued affine continuous functions $Af(S(A))$ separate the points of $S(A)$. Conversly, let (X, ρ) be a compact balanced convex metric space on which $Af(X)$ separate the points of X. Then $Af(X)$ is an order-unit space. By some standard argument involving separating disjoint compact convex subsets, we see $X = S(Af(X))$ and the metric topology coincides with the w^*-topology. By Theorem 9.7 in [1], ρ comes from a norm M on $(Af(X))'^0 = \{f \in (Af(X))' : f(1_{Af(X)}) = 0\} = \mathbb{R}(X - X)$. Let $Af(X)_L = (Lip_\rho) \cap Af(S(A))$. By Lemma 9.9 in [1], $Af(X)_L$ is dense in $Af(X)$. So $S(Af(X)_L) = X$, and the w^*-topology on X as the states space of $Af(X)_L$ is the same as that from $Af(X)$ hence the same as the metric topology. By Theorem 9.8 in [1], ρ on X is induced by the Lip-norm L_ρ on $Af(X)_L$. Therefore $(Af(X)_L, L_\rho)$ is a quantum metric space and (X, ρ) is its states space. So the states spaces of quantum metric spaces are exactly compact balanced convex metric spaces whose affine coutinuous functions separate ponits.

Fix $d > 0$. Let $Y = \{y_1, y_2, y_3\}$ be a $3-$point set with metric $\rho_Y(y_1, y_2) = \rho_Y(y_3, y_2) = d, \rho_Y(y_1, y_3) = 2d$. And let $Z = \{z_1, z_2\}$ be a $2-$point set with metric $\rho_Z(z_1, z_2) = 3d$. Also let $C(Y)$ and $C(Z)$ be the $\mathbb{R}-$valued continuous functions on Y and Z with the Lip-norms L_{ρ_Y} and L_{ρ_Z}. Then the probability measure spaces $P(Y)$ and $P(Z)$ are the state spaces of $C(Y)$ and $C(Z)$ respectively. We're going to show that $dist_{GH}(Y, Z) = d$. And by using the characterization of states spaces of quantum metric spaces in above, we'll show $dist_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) = \frac{d}{2}$. Hence this is an example for $dist_{GH} > dist_q$.

Let's show $dist_{GH}(Y, Z) = d$ first. Suppose Y and Z are embedded into some metric space (W, ρ) and $dist_H^W(Y, Z) < d$. Then we have $\rho(y_1, z_1) < d$ or $\rho(y_1, z_2) < d$. Wlog, say

$\rho(y_1, z_1) < d$. Then we cann't have $\rho(y_3, z_1) < d$, otherwise $\rho(y_1, y_3) \leq \rho(y_1, z_1) + \rho(y_3, z_1) < d + d = 2d = \rho(y_1, y_3)$ which is impossible. So $\rho(y_3, z_2) < d$. We also have $\rho(y_2, z_1) < d$ or $\rho(y_2, z_2) < d$. Wlog, say $\rho(y_2, z_1) < d$, then $\rho(z_1, z_2) \leq \rho(z_1, y_2) + \rho(y_2, y_3) + \rho(y_3, z_2) < d + d + d = 3d = \rho(z_1, z_2)$, contradiction. Therefore $dist_{GH}(Y, Z) \geq d$. On the other hand, we may embed Y and Z into $\mathbb{R}$ as: $y_1 \to 0, y_2 \to d, y_3 \to 2d, z_1 \to 0, z_2 \to 3d$. Then $dist_{GH}(Y, Z) \leq dist_H^{\mathbb{R}}(Y, Z) = d$. So $dist_{GH}(Y, Z) = d$.

We're going to show $dist_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) = \frac{d}{2}$ now. Let

$$B_Y = \{f \in C(Y) : L_{\rho_Y}(f) \leq 1, \| f \| \leq diam(P(Y)) = 2d\}$$

Then B_Y with the metric ρ_{B_Y} induced from the norm in $C(Y)$ is a compact metric space. Also let

$$X_{a,b} = \{g \in C(B_Y) : \| g \| \leq a, L_{\rho_{B_Y}}(g) \leq b\}$$

for all $a, b > 0$. Then $X_{a,b}$ with the metric $\rho_{X_{a,b}}$ induced from the norm in $C(B_Y)$ is a compact balanced convex metric space. Obviously the evaluations at ponits of B_Y are in $Af(X_{a,b})$ and separate the points in $X_{a,b}$. So by our characterization of states spaces of quantum metric spaces in above, $(X_{a,b}, \rho_{X_{a,b}})$ is the states space of some quantum metric space. Think of points in $P(Y) = S(C(Y))$ as functions on $C(Y)$ and then restrict to B_Y, we get a map from $P(Y)$ to $X_{2d,1}$. Clearly this map is affine and isometric. So we have embedded $P(Y)$ into $X_{2d,1}$ and may identify it with its image. For any $a \geq 2d, b \geq 1$, since $X_{2d,1} \subseteq X_{a,b}$, we may also think of $P(Y)$ as in $X_{a,b}$. Especially, $P(Y)$ is in $X_{4d,2}$.

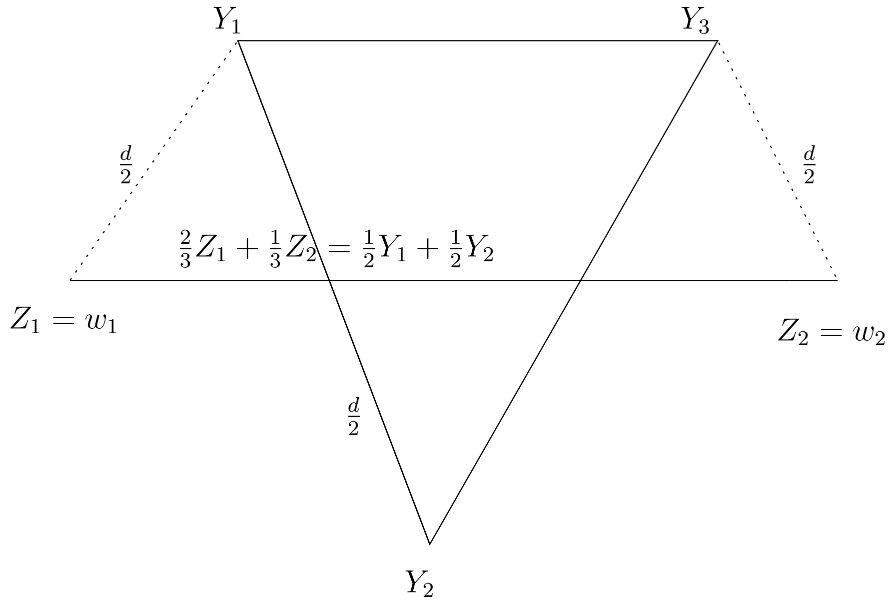
Let $w_1 = Y_1 + \frac{Y_2 - Y_3}{2}, w_2 = Y_3 + \frac{Y_2 - Y_1}{2}$. Then clearly $w_1, w_2 \in X_{4d,2}$ and

$$\rho_{X_{4d,2}}(w_1, w_2) = \| w_1 - w_2 \|_{C(B_Y)} = \| \frac{3}{2}(Y_1 - Y_3) \|_{C(B_Y)} = \frac{3}{2}\rho_Y(Y_1, Y_3) = 3d$$

So by mapping $Z_1 \to w_1, Z_2 \to w_2$, we may embed Z and hence $P(Z)$ into $X_{4d,2}$. Intuitively, the positions of Y and Z are shown in the following graph:

Now

$$\rho_{X_{4d,2}}(Z_1, Y_1) = \rho_{X_{4d,2}}(w_1, Y_1) = \| w_1 - Y_1 \|_{C(B_Y)} = \| \frac{Y_2 - Y_3}{2} \|_{C(B_Y)} = \frac{\rho_Y(Y_2, Y_3)}{2} = \frac{d}{2}$$



Similarly, we have $\rho_{X_{4d,2}}(Z_2, Y_3) = \frac{d}{2}$. Let $u = \frac{2}{3}Z_1 + \frac{1}{3}Z_2 \in P(Z)$. Then

$$\begin{aligned}
& \rho_{X_{4d,2}}(u, Y_2) \\
= \; & \rho_{X_{4d,2}}(\frac{2}{3}Z_1 + \frac{1}{3}Z_2, Y_2) \\
= \; & \rho_{X_{4d,2}}(\frac{2}{3}w_1 + \frac{1}{3}w_2, Y_2) \\
= \; & \| \frac{2}{3}w_1 + \frac{1}{3}w_2 - Y_2 \|_{C(B_Y)} \\
= \; & \| \frac{2}{3}(Y_1 + \frac{Y_2 - Y_3}{2}) + \frac{1}{3}(Y_3 + \frac{Y_2 - Y_1}{2}) - Y_2 \|_{C(B_Y)} \\
= \; & \| \frac{Y_1 - Y_2}{2} \|_{C(B_Y)} \\
= \; & \frac{\rho_Y(Y_1, Y_2)}{2} \\
= \; & \frac{d}{2}
\end{aligned}$$

Since Y and Z are the extreme points of $P(Y)$ and $P(Z)$ respectively, and the metric in $X_{4d,2}$ is convex, we see $dist_H^{X_{4d,2}}(P(Y), P(Z)) \leq \frac{d}{2}$. So $dist_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) \leq \frac{d}{2}$. On the other hand, by Lemma 13.6 in [2], $dist_q(C(Y), C(Z)) \geq \frac{|diam(C(Y), L_{\rho_Y}) - diam(C(Z), L_{\rho_Z})|}{2} = \frac{d}{2}$. Therefore $dist_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) = \frac{d}{2}$.

Let CM be the metric space of isometry classes of compact metric spaces. Also let QCM be the metric space of isometry classes of compact quantum metric spaces. Then associating a compact metric space with its Lipschitz $\mathbb{R}-$valued functions is an injective map from CM to QCM. Denote this map by P. Then our example shows P isn't isometric. However, things are not too bad. In fact, $P(CM)$ is a closed subspace of QCM. And $P^*(dist_q)$ gives the same topology on CM as $dist_{GH}$ does.

It would be interesting to know whether P^{-1} is Lipschitz or not. And if it is, what's the Lipschitz number? In our example, $\frac{dist_{GH}(Y,Z)}{dist_q(P(Y),P(Z))} = 2$. Certainly 2 isn't the Lipschitz number. Let Y' be the interval $[0, 2d]$ in $\mathbb{R}$ with the usual metric. Then similarly we can show $dist_{GH}(Y', Z) = \frac{3}{2}d$ and $dist_q(P(Y'), P(Z)) = \frac{d}{2}$. Hence $\frac{dist_{GH}(Y',Z)}{dist_q(P(Y),P(Z))} = 3$. However, no example for bigger number is known yet.

References

[1] Rieffel, M. A.: *Metrics on state spaces.* Doc. Math. **4** (1999), 559–660. arXiv:math.OA/9906151.

[2] Rieffel, M. A.: *Gromov-Hausdorff distance for quantum metric spaces* (2000). arXiv:math.QA/0011063.