

# ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C\*-ALGEBRAS

REVISED AND EXPANDED VERSION

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ABSTRACT. We examine the question of when the  $*$ -homomorphism  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  of full amalgamated free product C\*-algebras, arising from compatible inclusions of C\*-algebras  $A \subseteq \tilde{A}$ ,  $B \subseteq \tilde{B}$  and  $D \subseteq \tilde{D}$ , is an embedding. Results giving sufficient conditions for  $\lambda$  to be injective, as well of classes of examples where  $\lambda$  fails to be injective, are obtained. As an application, we give necessary and sufficient condition for the full amalgamated free product of finite dimensional C\*-algebras to be residually finite dimensional.

## 1. INTRODUCTION

Given C\*-algebras  $A$ ,  $B$  and  $D$  with injective  $*$ -homomorphisms  $\phi_A : D \rightarrow A$  and  $\phi_B : D \rightarrow B$ , the corresponding full amalgamated free product C\*-algebra (see [1] or [9, Chapter 5]) is the C\*-algebra  $\mathfrak{A}$ , equipped with injective  $*$ -homomorphisms  $\sigma_A : A \rightarrow \mathfrak{A}$  and  $\sigma_B : B \rightarrow \mathfrak{A}$  such that  $\sigma_A \circ \phi_A = \sigma_B \circ \phi_B$ , such that  $\mathfrak{A}$  is generated by  $\sigma_A(A) \cup \sigma_B(B)$  and satisfying the universal property that whenever  $\mathfrak{C}$  is a C\*-algebra and  $\pi_A : A \rightarrow \mathfrak{C}$  and  $\pi_B : B \rightarrow \mathfrak{C}$  are  $*$ -homomorphisms satisfying  $\pi_A \circ \phi_A = \pi_B \circ \phi_B$ , there is a  $*$ -homomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{C}$  such that  $\pi \circ \sigma_A = \pi_A$  and  $\pi \circ \sigma_B = \pi_B$ . This situation is illustrated by the following commuting diagram:

$$\begin{array}{ccccc}
 & & D & & \\
 & \nearrow \phi_A & & \searrow \phi_B & \\
 A & \xrightarrow{\sigma_A} & \mathfrak{A} & \xleftarrow{\sigma_B} & B \\
 & \searrow \pi_A & \downarrow \pi & \swarrow \pi_B & \\
 & & \mathfrak{C} & & 
 \end{array} \tag{1}$$

The full amalgamated free product C\*-algebra  $\mathfrak{A}$  is commonly denoted by  $A *_D B$ , although this notation hides the dependence of  $\mathfrak{A}$  on the embeddings  $\phi_A$  and  $\phi_B$ .

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**Question 1.1.** Let  $D, A, B, \tilde{D}, \tilde{A}$  and  $\tilde{B}$  be  $C^*$ -algebras and suppose there are injective  $*$ -homomorphisms making the following diagram commute:

$$\begin{array}{ccccc} \tilde{A} & \xleftarrow{\phi_{\tilde{A}}} & \tilde{D} & \xrightarrow{\phi_{\tilde{B}}} & \tilde{B} \\ \lambda_A \uparrow & & \lambda_D \uparrow & & \lambda_B \uparrow \\ A & \xleftarrow{\phi_A} & D & \xrightarrow{\phi_B} & B \end{array}$$

Let  $A *_D B$  and  $\tilde{A} *_{\tilde{D}} \tilde{B}$  be the corresponding full amalgamated free product  $C^*$ -algebras and let  $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$  be the  $*$ -homomorphism arising from  $\lambda_A$  and  $\lambda_B$  via the universal property. When is  $\lambda$  injective?

We prove in §2 that  $\lambda$  is injective when either (i)  $D = \tilde{D}$ , (or more precisely, when the  $*$ -homomorphism  $\lambda_D$  is surjective), or (ii) there are conditional expectations  $E_A : \tilde{A} \rightarrow A$  and  $E_B : \tilde{B} \rightarrow B$  that send  $\tilde{D}$  onto  $D$  and agree on  $\tilde{D}$ . Injectivity in the case  $D = \tilde{D}$  was previously proved by G.K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map  $\lambda$  is injective when  $D = \tilde{D}$  and when there are conditional expectations

$$\tilde{A} \xrightarrow{E_A^A} A \xrightarrow{E_D^A} D \xleftarrow{E_D^B} B \xleftarrow{E_B^{\tilde{B}}} \tilde{B};$$

an argument for the case  $D = \tilde{D} = \mathbf{C}$ , which uses Boca's results, is outlined in [3, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In §3, we consider some general conditions and give some concrete examples when  $\lambda$  fails to be injective. Finally, in §4, we apply this embedding result to extend a result from [5] about residual finite dimensionality of full amalgamated free products of finite dimensional  $C^*$ -algebras.

## 2. EMBEDDINGS OF FULL FREE PRODUCTS

The following result is of course well known. We include a proof for completeness.

**Lemma 2.1.** Let  $A$  be a  $C^*$ -subalgebra of a  $C^*$ -algebra  $\tilde{A}$  and let  $\pi : A \rightarrow B(\mathcal{H})$  be a  $*$ -representation. Then there is a Hilbert space  $\mathcal{K}$  and a  $*$ -representation  $\tilde{\pi} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  such that

$$\tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0, \quad (a \in A, h \in \mathcal{H}). \quad (2)$$

*Proof.* Since in general  $\pi$  is a direct sum of cyclic representations, we may without loss of generality assume  $\pi$  is a cyclic representation with cyclic vector  $\xi$ . Let  $\phi$  be the vector state  $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$  of  $A$ . Then  $\mathcal{H}$  is identified with  $L^2(A, \phi)$  and  $\pi$  is the associated GNS representation. Let  $\tilde{\phi}$  be an extension of  $\phi$  to a state of  $\tilde{A}$  and let  $\tilde{\mathcal{H}} = L^2(\tilde{A}, \tilde{\phi})$ . Then the inclusion  $A \hookrightarrow \tilde{A}$  gives rise to an isometry  $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$ , and we may thus write  $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$  for a Hilbert space  $\mathcal{K}$ . If  $\tilde{\pi} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  is the GNS representation associated to  $\tilde{\phi}$ , then (2) holds.  $\square$

The following result was first proved by G.K. Pedersen [10]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.

**Proposition 2.2.** *Let*

$$\tilde{A} \supseteq A \supseteq D \subseteq B \subseteq \tilde{B}$$

*be inclusions of  $C^*$ -algebras and let  $A *_D B$  and  $\tilde{A} *_D \tilde{B}$  be the corresponding full amalgamated free product  $C^*$ -algebras. Let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the  $*$ -homomorphism arising via the universal property from the inclusions  $A \hookrightarrow \tilde{A}$  and  $B \hookrightarrow \tilde{B}$ . Then  $\lambda$  is injective.*

*Proof.* Let  $\pi : A *_D B \rightarrow B(\mathcal{H})$  be a faithful  $*$ -homomorphism. We will find a Hilbert space  $\mathcal{K}$  and a  $*$ -homomorphism  $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \quad (x \in A *_D B, h \in \mathcal{H}). \quad (3)$$

This will imply  $\lambda$  is injective.

Let  $\pi_A : A \rightarrow B(\mathcal{H})$  and  $\pi_B : B \rightarrow B(\mathcal{H})$  be the  $*$ -representations obtained by composing  $\pi$  with the inclusions  $A \hookrightarrow A *_D B$  and  $B \hookrightarrow A *_D B$ . Let

$$\begin{aligned} \sigma_{A,0} : \tilde{A} &\rightarrow B(\mathcal{H} \oplus \mathcal{K}_{A,0}), \\ \sigma_{B,0} : \tilde{B} &\rightarrow B(\mathcal{H} \oplus \mathcal{K}_{B,0}) \end{aligned}$$

be  $*$ -representations obtained from Lemma 2.1 such that

$$\sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0, \quad (a \in A, h \in \mathcal{H}),$$

and similarly with  $A$  replaced by  $B$ . Note that  $0 \oplus \mathcal{K}_{A,0}$  is reducing for  $\sigma_{A,0}(D)$ . Using Lemma 2.1, find Hilbert spaces  $\mathcal{K}_{B,1}$  and  $\mathcal{K}_{A,1}$  and  $*$ -representations

$$\begin{aligned} \sigma_{B,1} : \tilde{B} &\rightarrow B(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}) \\ \sigma_{A,1} : \tilde{A} &\rightarrow B(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}) \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,1}(d)(k \oplus 0) &= \sigma_{A,0}(d)(0 \oplus k), & (d \in D, k \in \mathcal{K}_{A,0}), \\ \sigma_{A,1}(d)(k \oplus 0) &= \sigma_{B,0}(d)(0 \oplus k), & (d \in D, k \in \mathcal{K}_{B,0}). \end{aligned}$$

Proceeding recursively, for every integer  $n \geq 2$  we find  $*$ -representations

$$\begin{aligned} \sigma_{B,n} : \tilde{B} &\rightarrow B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}), \\ \sigma_{A,n} : \tilde{A} &\rightarrow B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}) \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,n}(d)(k \oplus 0) &= \sigma_{A,n-1}(d)(0 \oplus k), & (d \in D, k \in \mathcal{K}_{A,n-1}), \\ \sigma_{A,n}(d)(k \oplus 0) &= \sigma_{B,n-1}(d)(0 \oplus k), & (d \in D, k \in \mathcal{K}_{B,n-1}). \end{aligned}$$

We now define the Hilbert spaces

$$\begin{aligned}\tilde{\mathcal{H}}_A &= \overbrace{\mathcal{H} \oplus \mathcal{K}_{A,0}}^{\sigma_{A,0}} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\ \tilde{\mathcal{H}}_B &= \overbrace{\mathcal{H} \oplus \mathcal{K}_{B,0}}^{\sigma_{B,0}} \oplus \overbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}^{\sigma_{B,1}} \oplus \overbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}^{\sigma_{B,2}} \oplus \cdots,\end{aligned}\tag{4}$$

where the brackets indicate where the constructed representations act, and we let  $\sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\tilde{\mathcal{H}}_A)$  and  $\sigma_{\tilde{B}} : \tilde{B} \rightarrow B(\tilde{\mathcal{H}}_B)$  be the  $*$ -representations

$$\begin{aligned}\sigma_{\tilde{A}} &= \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots, \\ \sigma_{\tilde{B}} &= \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,\end{aligned}$$

where the summands act as indicated by brackets in (4). Consider the unitary  $U : \tilde{\mathcal{H}}_A \rightarrow \tilde{\mathcal{H}}_B$  mapping the summands in  $\tilde{\mathcal{H}}_A$  identically to the corresponding summands in  $\tilde{\mathcal{H}}_B$  as indicated by the arrows below:

$$\begin{array}{ccccccccccc} \tilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,2} & \oplus & \cdots \\ \downarrow U & & \downarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \\ \tilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,2} & \oplus & \cdots \end{array}$$

Let  $\mathcal{K} = \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$  and identify  $\mathcal{H} \oplus \mathcal{K}$  with  $\tilde{\mathcal{H}}_A$ . Then we have the  $*$ -representations  $\tilde{\pi}_{\tilde{A}} = \sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  and  $\tilde{\pi}_{\tilde{B}} : \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ , the latter defined by  $\tilde{\pi}_{\tilde{B}}(\cdot) = U^* \sigma_{\tilde{B}}(\cdot) U$ . By construction, the restrictions of  $\tilde{\pi}_{\tilde{A}}$  and  $\tilde{\pi}_{\tilde{B}}$  to  $D$  agree, and we have

$$\begin{aligned}\tilde{\pi}_{\tilde{A}}(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0, & (a \in A, h \in \mathcal{H}), \\ \tilde{\pi}_{\tilde{B}}(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0, & (b \in B, h \in \mathcal{H}).\end{aligned}$$

Letting  $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  be the  $*$ -homomorphism obtained from  $\tilde{\pi}_{\tilde{A}}$  and  $\tilde{\pi}_{\tilde{B}}$  via the universal property, we have that (3) holds.  $\square$

For a  $C^*$ -algebra  $A$ , unital or not, let  $A^u$  denote the unitization of  $A$ . Thus, as a vector space,  $A^u = A \oplus \mathbf{C}$  with multiplication defined by  $(a, \mu) \cdot (a', \mu') = (aa' + \mu a + \mu' a, \mu\mu')$ . We identify  $A$  with the ideal  $A \oplus 0$  of  $A^u$ , which has codimension 1.

**Lemma 2.3.** *Let  $A \supseteq D \subseteq B$  be inclusions of  $C^*$ -algebras. Consider the unitizations and corresponding inclusions*

$$\begin{array}{ccccc} A^u & \longleftarrow & D^u & \longrightarrow & B^u \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & D & \longrightarrow & B. \end{array}$$

Let  $\lambda : A *_D B \rightarrow A^u *_D B^u$  be the resulting  $*$ -homomorphism between full amalgamated free products. Then there is an isomorphism  $\pi : A^u *_D B^u \rightarrow (A *_D B)^u$  such

that  $\pi \circ \lambda : A *_D B \rightarrow (A *_D B)^u$  is the canonical embedding arising in the definition of the unitization.

*Proof.* Since any  $*$ -representations of  $A$  and  $B$  that agree on  $D$  extend to  $*$ -representations of  $A^u$  and  $B^u$  that agree on  $D^u$ , the  $*$ -homomorphism  $\lambda$  is injective. Let  $e \in A^u *_D B^u$  be the unit of  $A^u$ , which is of course identified with the units of  $B^u$  and  $D^u$ . Clearly,  $A^u *_D B^u$  is generated by the image of  $\lambda$  together with  $e$ . One easily sees

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu\lambda(x') + \mu'\lambda(x) + \mu\mu'e.$$

Moreover, if  $\rho : A^u *_D B^u \rightarrow \mathbf{C}$  is the  $*$ -homomorphism arising from the unital  $*$ -homomorphisms  $A^u \rightarrow \mathbf{C}$  and  $B^u \rightarrow \mathbf{C}$ , then  $\rho(e) = 1$  and  $\lambda(A *_D B) \subseteq \ker \rho$ . Hence  $\lambda(A *_D B)$  has codimension 1 in  $A^u *_D B^u$ . Now  $\pi$  can be defined by  $\pi(\lambda(x) + \mu e) = (x, \mu)$ .  $\square$

**Proposition 2.4.** *Suppose*

$$\begin{array}{ccccc} \tilde{A} & \longleftarrow & \tilde{D} & \longrightarrow & \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & D & \longrightarrow & B \end{array} \quad (5)$$

is a commuting diagram of inclusions of  $C^*$ -algebras. Let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the resulting  $*$ -homomorphism of full free product  $C^*$ -algebras. Suppose there are conditional expectations  $E_A : \tilde{A} \rightarrow A$ ,  $E_D : \tilde{D} \rightarrow D$  and  $E_B : \tilde{B} \rightarrow B$  onto  $A$ ,  $D$  and  $B$ , respectively, such that the diagram

$$\begin{array}{ccccc} \tilde{A} & \longleftarrow & \tilde{D} & \longrightarrow & \tilde{B} \\ \downarrow E_A & & \downarrow E_D & & \downarrow E_B \\ A & \longleftarrow & D & \longrightarrow & B \end{array} \quad (6)$$

commutes. Then  $\lambda$  is injective.

*Proof.* By appealing to Lemma 2.3, we may without loss of generality assume all the algebras and  $*$ -homomorphisms in (5) are unital. Let  $\pi : A *_D B \rightarrow B(\mathcal{H})$  be a faithful, unital  $*$ -representation. As in the proof of Proposition 2.2, in order to show  $\lambda$  is injective, we will find a Hilbert space  $\mathcal{K}$  and a  $*$ -homomorphism  $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0, \quad (x \in A *_D B, h \in \mathcal{H}). \quad (7)$$

Let  $\pi_A : A \rightarrow B(\mathcal{H})$  and  $\pi_B : B \rightarrow B(\mathcal{H})$  be the  $*$ -representations obtained by composing  $\pi$  with the inclusions  $A \hookrightarrow A *_D B$  and  $B \hookrightarrow A *_D B$ , and let  $\pi_D : D \rightarrow B(\mathcal{H})$  be their common restriction to  $D$ . Consider the canonical left action of  $\tilde{D}$  on the right Hilbert  $D$ -module  $L^2(\tilde{D}, E_D)$ , which is obtained from  $\tilde{D}$  by separation and completion with respect to the  $D$ -valued inner product  $\langle \tilde{d}_1, \tilde{d}_2 \rangle = E_D(\tilde{d}_1^* \tilde{d}_2)$ . Consider the Hilbert space  $L^2(\tilde{D}, E_D) \otimes_D \mathcal{H}$ , where the left action of  $D$  on  $\mathcal{H}$  is via  $\pi_D$ . Since  $\pi_D$  is unital,  $\mathcal{H}$  embeds as a subspace, and we can write

$$L^2(\tilde{D}, E_D) \otimes_D \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D. \quad (8)$$

Consider the left action of  $\tilde{D}$  on the Hilbert space  $\mathcal{H} \oplus \mathcal{K}_D$ . The subspace  $\mathcal{H}$  is reducing for the restriction of  $\sigma_D$  to  $D$ , and we have  $\sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0$  for every  $d \in D$  and  $h \in \mathcal{H}$ .

In a similar way, consider the Hilbert spaces

$$L^2(\tilde{A}, E_A) \otimes_A \mathcal{H}, \quad L^2(\tilde{B}, E_B) \otimes_B \mathcal{H} \quad (9)$$

and the associated left actions  $\sigma_{A,0}$  of  $\tilde{A}$ , respectively  $\sigma_{B,0}$  of  $\tilde{B}$ . As the diagram (6) commutes, the Hilbert space (8) embeds canonically as a subspace of both spaces (9). We may thus write

$$\begin{aligned} L^2(\tilde{A}, E_A) \otimes_A \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0} \\ L^2(\tilde{B}, E_B) \otimes_B \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0}, \end{aligned}$$

the subspace  $\mathcal{H} \oplus \mathcal{K}_D \oplus 0$  is reducing for the restrictions of  $\sigma_{A,0}$  and  $\sigma_{B,0}$  to  $\tilde{D}$ , and we have  $\sigma_{A,0}(\tilde{d})(\eta \oplus 0) = (\sigma_D(\tilde{d})\eta) \oplus 0 = \sigma_{B,0}(\tilde{d})(\eta \oplus 0)$  for every  $\tilde{d} \in \tilde{D}$  and  $\eta \in \mathcal{H} \oplus \mathcal{K}_D$ . Moreover,  $\mathcal{H} \oplus 0 \oplus 0$  is reducing for the restrictions of  $\sigma_{A,0}$  to  $A$  and  $\sigma_{B,0}$  to  $B$ , and we have

$$\begin{aligned} \sigma_{A,0}(a)(h \oplus 0 \oplus 0) &= (\pi_A(a)h) \oplus 0 \oplus 0 & (a \in A, h \in \mathcal{H}) \\ \sigma_{B,0}(b)(h \oplus 0 \oplus 0) &= (\pi_B(b)h) \oplus 0 \oplus 0 & (b \in B, h \in \mathcal{H}). \end{aligned}$$

Let  $\sigma_{A,0,\tilde{D}}$  denote the action of  $\tilde{D}$  on  $\mathcal{K}_{A,0}$  obtained by restricting  $\sigma_{A,0}$  to  $\tilde{D}$  and compressing, and similarly for  $\sigma_{B,0,\tilde{D}}$ .

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces  $\mathcal{K}_{A,n-1}$  and  $\mathcal{K}_{B,n-1}$  have been constructed with actions  $\sigma_{A,n-1,\tilde{D}}$  and  $\sigma_{B,n-1,\tilde{D}}$ , respectively, of  $\tilde{D}$ , use Lemma 2.1 to construct Hilbert spaces  $\mathcal{K}_{B,n}$  and  $\mathcal{K}_{A,n}$  and  $*$ -homomorphisms

$$\begin{aligned} \sigma_{B,n} : \tilde{B} &\rightarrow B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}) \\ \sigma_{A,n} : \tilde{A} &\rightarrow B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}), \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,n}(\tilde{d})(k \oplus 0) &= (\sigma_{A,n-1,\tilde{D}}(\tilde{d})k) \oplus 0 & (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{A,n-1}) \\ \sigma_{A,n}(\tilde{d})(k \oplus 0) &= (\sigma_{B,n-1,\tilde{D}}(\tilde{d})k) \oplus 0 & (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{B,n-1}). \end{aligned}$$

Then let  $\sigma_{B,n,\tilde{D}}$  be the action of  $\tilde{D}$  on  $\mathcal{K}_{B,n}$  obtained from the restriction of  $\sigma_{B,n}$  to  $\tilde{D}$  by compressing, and similarly define the action  $\sigma_{A,n,\tilde{D}}$  of  $\tilde{D}$  on  $\mathcal{K}_{A,n}$ .

We may now define the Hilbert spaces

$$\begin{aligned}
 \tilde{\mathcal{H}}_A &= \overbrace{\mathcal{H} \oplus \mathcal{K}_D}^{\sigma_D} \oplus \mathcal{K}_{A,0} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\
 \tilde{\mathcal{H}}_B &= \underbrace{\mathcal{H} \oplus \mathcal{K}_D}_{\sigma_{B,0}} \oplus \mathcal{K}_{B,0} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots,
 \end{aligned} \tag{10}$$

where the brackets indicate where the constructed representations act. We let  $\sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\tilde{\mathcal{H}}_A)$  and  $\sigma_{\tilde{B}} : \tilde{B} \rightarrow B(\tilde{\mathcal{H}}_B)$  be the  $*$ -representations

$$\begin{aligned}
 \sigma_{\tilde{A}} &= \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots, \\
 \sigma_{\tilde{B}} &= \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,
 \end{aligned}$$

where the summands act as indicated by brackets in (10). Consider the unitary  $U : \tilde{\mathcal{H}}_A \rightarrow \tilde{\mathcal{H}}_B$  mapping the summands in  $\tilde{\mathcal{H}}_A$  identically to the corresponding summands in  $\tilde{\mathcal{H}}_B$  as indicated by the arrows below:

$$\begin{array}{ccccccccccc}
 \tilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,2} & \oplus & \cdots \\
 U \downarrow & & \downarrow & & \downarrow & & & & & & & & & & & & & \\
 \tilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,2} & \oplus & \cdots
 \end{array}$$

Let  $\mathcal{K} = \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$  and identify  $\mathcal{H} \oplus \mathcal{K}$  with  $\tilde{\mathcal{H}}_A$ . Then we have the  $*$ -representations  $\tilde{\pi}_{\tilde{A}} = \sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  and  $\tilde{\pi}_{\tilde{B}} : \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ , the latter defined by  $\tilde{\pi}_{\tilde{B}}(\cdot) = U^* \sigma_{\tilde{B}}(\cdot) U$ . By construction, the restrictions of  $\tilde{\pi}_{\tilde{A}}$  and  $\tilde{\pi}_{\tilde{B}}$  to  $\tilde{D}$  agree, and we have

$$\begin{aligned}
 \tilde{\pi}_{\tilde{A}}(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0 & (a \in A, h \in \mathcal{H}) \\
 \tilde{\pi}_{\tilde{B}}(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0 & (b \in B, h \in \mathcal{H}).
 \end{aligned}$$

Letting  $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$  be the  $*$ -homomorphism obtained from  $\tilde{\pi}_{\tilde{A}}$  and  $\tilde{\pi}_{\tilde{B}}$  via the universal property, we have that (7) holds.  $\square$

### 3. EXAMPLES OF NON-EMBEDDING

In this section, we give some examples when the map  $\lambda$  of Question 1.1 fails to be injective. (In contrast, it is known [2] that in the more stringent situation of *reduced* amalgamated free products, the map analogous to  $\lambda$  is always injective.)

We begin with a trivial class of examples.

**Examples 3.1.** Let  $A$  and  $B$  be  $C^*$ -subalgebras of a  $C^*$ -algebra  $E$  with  $A \not\subseteq B$  and  $B \not\subseteq A$ . Let  $D = A \cap B$ ,  $\tilde{A} = E$  and  $\tilde{D} = \tilde{B} = B$ , equipped with the natural inclusions. Then the map  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B} = E$  is injective if and only if  $A *_D B$  is exactly the  $C^*$ -subalgebra of  $E$  generated by  $A$  and  $B$ . This doesn't hold in general. Notice that in these examples,  $B \cap \tilde{D} = B \not\subseteq D$ .

**Proposition 3.2.** *Suppose*

$$\begin{array}{ccccc} \tilde{A} & \longleftarrow & \tilde{D} & \longrightarrow & \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & D & \longrightarrow & B \end{array}$$

is a commuting diagram of inclusions of  $C^*$ -algebras and let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the resulting  $*$ -homomorphism of full free product  $C^*$ -algebras. Suppose there are conditional expectations  $E_D^A : A \rightarrow D$  and  $E_D^B : B \rightarrow D$  with  $E_D^B$  faithful. Suppose there are  $\tilde{d} \in \tilde{D}$ ,  $a \in A$  and  $b \in B$  satisfying  $a\tilde{d} \in A$ ,  $\tilde{d}b \in B$ ,

$$D(\tilde{d}b) \cap Db = \{0\} \quad (11)$$

$$E_D^A(\tilde{d}^* a^* a d)b \neq 0. \quad (12)$$

Then  $\lambda$  is not injective.

*Proof.* Letting

$$\begin{aligned} \sigma_A : A &\hookrightarrow A *_D B, & \sigma_B : B &\hookrightarrow A *_D B, \\ \sigma_{\tilde{A}} : \tilde{A} &\hookrightarrow \tilde{A} *_D \tilde{B}, & \sigma_{\tilde{B}} : \tilde{B} &\hookrightarrow \tilde{A} *_D \tilde{B} \end{aligned} \quad (13)$$

be the embeddings as in (1), we have

$$\lambda(\sigma_A(a\tilde{d})\sigma_B(b)) = \sigma_{\tilde{A}}(a\tilde{d})\sigma_{\tilde{B}}(b) = \sigma_{\tilde{A}}(a)\sigma_{\tilde{B}}(\tilde{d}b) = \lambda(\sigma_A(a)\sigma_B(\tilde{d}b)).$$

Thus we need only show

$$\sigma_A(a\tilde{d})\sigma_B(b) \neq \sigma_A(a)\sigma_B(\tilde{d}b). \quad (14)$$

We consider the reduced amalgamated free product of  $C^*$ -algebras (see [11] or [12]),

$$(A *_D^{\text{red}} B, E_D) = (A, E_D^A) *_D (B, E_D^B)$$

and the natural quotient  $*$ -homomorphism  $A *_D B \rightarrow A *_D^{\text{red}} B$ . Let  $L^2(A *_D^{\text{red}} B, E_D)$  be the right Hilbert  $D$ -module obtained by separation and completion from  $A *_D^{\text{red}} B$  with respect to the  $D$ -valued inner product  $\langle x, y \rangle = E_D(x^*y)$ , and given  $x \in A *_D^{\text{red}} B$ , let  $\hat{x}$  denote the corresponding element in  $L^2(A *_D^{\text{red}} B, E_D)$ . Let  $\mathcal{H}_A = L^2(A, E_D^A)$  and  $\mathcal{H}_B = L^2(B, E_D^B)$  be similarly defined. Then in  $L^2(A *_D^{\text{red}} B, E_D)$ , the closure of the subspace spanned by elements of the form  $(ab)^\wedge$  for  $a \in A$  and  $b \in B$  is isomorphic to the tensor product  $\mathcal{H}_A \otimes_D \mathcal{H}_B$  of Hilbert  $D$ -modules. In order to show (14), it will suffice to show

$$(a\tilde{d})^\wedge \otimes \hat{b} \neq \hat{a} \otimes (\tilde{d}b)^\wedge$$

in  $\mathcal{H}_A \otimes_D \mathcal{H}_B$ . Let  $\zeta_B \in \mathcal{H}_B$ . Then

$$\langle (a\tilde{d})^\wedge \otimes \zeta_B, (a\tilde{d})^\wedge \otimes \hat{b} \rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a \tilde{d})b)^\wedge \rangle \quad (15)$$

$$\langle (a\tilde{d})^\wedge \otimes \zeta_B, \hat{a} \otimes (\tilde{d}b)^\wedge \rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a) \tilde{d}b)^\wedge \rangle. \quad (16)$$

From assumptions (11) and (12), we obtain  $E_D^A(\tilde{d}^* a^* a \tilde{d})b \neq E_D^A(\tilde{d}^* a^* a) \tilde{d}b$ . Since  $E_D^B$  is faithful, there is  $\zeta_B \in \mathcal{H}_B$  such that the right-hand-sides of (15) and (16) are not equal.  $\square$

**Remark 3.3.** From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions (11) and (12) can be dropped, and  $E_D^B$  need not be assumed faithful, but instead one must assume

$$E_D^B(b^*(E_D^A(\tilde{d}^*a^*a\tilde{d}) - E_D^A(\tilde{d}^*a^*a)\tilde{d} - \tilde{d}^*E_D^A(a^*a\tilde{d}) + \tilde{d}^*E_D^A(a^*a)\tilde{d})b^*) \neq 0. \quad (17)$$

Note that the LHS of (17) is nothing other than

$$\langle (a\tilde{d})^\wedge \otimes \hat{b} - \hat{a} \otimes (d\tilde{b})^\wedge, (a\tilde{d})^\wedge \otimes \hat{b} - \hat{a} \otimes (d\tilde{b})^\wedge \rangle.$$

**Corollary 3.4.** *Suppose*

$$\begin{array}{ccccc} \tilde{A} & \longleftarrow & \tilde{D} & \longrightarrow & \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & D & \longrightarrow & B \end{array} \quad (18)$$

is a commuting diagram of inclusions of  $C^*$ -algebras and let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the resulting  $*$ -homomorphism of full free product  $C^*$ -algebras. Suppose one of the following holds:

- (i)  $D = 0$
- (ii)  $D = \mathbf{C}$ ,  $A$  and  $B$  are unital and the inclusions  $D \hookrightarrow A$  and  $D \hookrightarrow B$  are unital.

Suppose there are  $\tilde{d} \in \tilde{D}$ ,  $a \in A$  and  $b \in B$  such that  $a\tilde{d} \in A \setminus \{0\}$ ,  $\tilde{d}b \in B$  and  $\tilde{d}b \notin \mathbf{C}b$ . Then  $\lambda$  is not injective.

*Proof.* We can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume  $A$  and  $B$  are separable. Letting  $E_D^A : A \rightarrow \mathbf{C}$  and  $E_D^B : B \rightarrow \mathbf{C}$  be faithful states, we find the hypotheses of Proposition 3.2 are satisfied.  $\square$

From this corollary, we have the following class of concrete examples, which shows that  $\lambda$  may be non-injective even if

$$B \cap \tilde{D} = D = A \cap \tilde{D}. \quad (19)$$

**Example 3.5.** Let  $\mathcal{H}$  be an infinite dimensional, separable Hilbert space. Inside  $B(\mathcal{H})$ , let  $D = \mathbf{C}1$  and let  $A = B = D + K(\mathcal{H})$ , where  $K(\mathcal{H})$  is the compact operators. Let  $u \in B(\mathcal{H})$  be a unitary operator that does not belong to  $D$  and let  $\tilde{D} = C^*(u)$ ,  $\tilde{A} = \tilde{B} = \tilde{D} + K(\mathcal{H})$ . Let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the  $*$ -homomorphism arising from the inclusions (18). Then  $\lambda$  is not injective.

*Proof.* Take  $\tilde{d} = u$  and  $a \in K(\mathcal{H}) \setminus \{0\}$ . Since  $u \notin \mathbf{C}1$ , there is  $b \in K(\mathcal{H})$  such that  $ub \notin \mathbf{C}b$ . Now apply Corollary 3.4. One can choose  $u$  so that  $C^*(u) \cap (\mathbf{C}1 + K(\mathcal{H})) = \mathbf{C}1$ , in order to get (19).  $\square$

**Proposition 3.6.** *Suppose*

$$\begin{array}{ccccc} \tilde{A} & \longleftarrow & \tilde{D} & \longrightarrow & \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & D & \longrightarrow & B \end{array} \quad (20)$$

is a commuting diagram of inclusions of  $C^*$ -algebras and let  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  be the resulting  $*$ -homomorphism of full free product  $C^*$ -algebras. Suppose one of the following holds:

- (i)  $D = 0$
- (ii)  $D = \mathbf{C}$ ,  $A$  and  $B$  are unital and the inclusions  $D \hookrightarrow A$  and  $D \hookrightarrow B$  are unital.

Suppose there are  $\tilde{d} \in \tilde{D}$ ,  $a_1, a_2 \in A$  and  $b \in B \setminus D$  such that  $a_1 \tilde{d}, \tilde{d} a_2 \in A$ ,  $a_1 \tilde{d} \notin \mathbf{C}$  and  $\tilde{d} b = b \tilde{d}$ . Then  $\lambda$  is not injective.

*Proof.* We can reduce to the case in which (ii) holds by application of Lemma 2.3. We use the same notation as in (13). We have

$$\begin{aligned} \lambda(\sigma_A(a_1 \tilde{d}) \sigma_B(b) \sigma_A(a_2)) &= \sigma_{\tilde{A}}(a_1 \tilde{d}) \sigma_{\tilde{B}}(b) \sigma_{\tilde{A}}(a_2) \\ &= \sigma_{\tilde{A}}(a_1) \sigma_{\tilde{B}}(b) \sigma_{\tilde{A}}(\tilde{d} a_2) = \lambda(\sigma_A(a_1) \sigma_B(b) \sigma_A(\tilde{d} a_2)), \end{aligned}$$

and we must only show

$$\sigma_A(a_1 \tilde{d}) \sigma_B(b) \sigma_A(a_2) \neq \sigma_A(a_1) \sigma_B(b) \sigma_A(\tilde{d} a_2). \quad (21)$$

Without loss of generality, assume  $A$  and  $B$  are separable. Let  $\phi_A : A \rightarrow \mathbf{C}$  and  $\phi_B : B \rightarrow \mathbf{C}$  be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume  $\phi_B(b) = 0$ . Let

$$(A *_C^{\text{red}} B, \phi) = (A, \phi_A) *_C (B, \phi_B)$$

be the reduced free product of  $C^*$ -algebras. Using arguments and notation as in the proof of Proposition 3.2, the closure of the subspace of  $L^2(A *_C^{\text{red}} B, \phi)$  spanned by elements of the form  $(aba')^\wedge$  for  $a, a' \in A$  is isomorphic to  $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$ . To show (21), it will suffice to show

$$(a_1 \tilde{d})^\wedge \otimes \hat{b} \otimes \hat{a}_2 \neq \hat{a}_1 \otimes \hat{b} \otimes (\tilde{d} a_2)^\wedge$$

in  $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$ . However, this follows from the assumptions.  $\square$

From the above proposition, we get the following example, which requires only “bad” relations between  $A$  and  $\tilde{D}$ , not between  $B$  and  $\tilde{D}$ .

**Example 3.7.** Let  $D, \tilde{D}, A$  and  $\tilde{A}$  be as in Example 3.5. Let  $B$  be any unital  $C^*$ -algebra of dimension greater than 1 and let  $\tilde{B} = B \otimes \tilde{D}$ , (for the unique  $C^*$ -tensor norm). Then the  $*$ -homomorphism  $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$  arising from the inclusions (20) is not injective.

**Remark 3.8.** The problem with injectivity of  $\lambda$  in Examples 3.5 and 3.7 arises already at the algebraic level

$$A *_D^{\text{alg}} B \rightarrow \tilde{A} *_D^{\text{alg}} \tilde{B}. \quad (22)$$

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while  $\lambda$  fails to be injective, e.g. by taking  $E$  to be a reduced free product. However, we do not know of an example where  $\lambda$  fails to be injective and where the algebraic map (22) is injective, but where  $A \cap \tilde{D} = D = B \cap \tilde{D}$ .

## 4. AN APPLICATION TO RESIDUAL FINITE DIMENSIONALITY

A  $C^*$ -algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite dimensional  $*$ -representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi's proof [6] that the full group  $C^*$ -algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d.  $C^*$ -algebras  $A$  and  $B$  with amalgamation over either the zero  $C^*$ -algebra or over the scalar multiples of the identity (if  $A$  and  $B$  are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras  $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$  over a unital subalgebra  $D$  is r.f.d. provided that the normalized traces on  $M_k(\mathbf{C})$  and  $M_\ell(\mathbf{C})$  restrict to the same trace on  $D$ . In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite dimensional algebras.

**Lemma 4.1.** *Let  $S = \{x \in \mathbf{R}^n \mid Ax = 0\}$ , where  $A$  is an  $m \times n$  matrix having only rational entries. Then vectors having only rational entries are dense in  $S$ .*

*Proof.* By considering the reduced row-echelon form of  $A$ , we see that there is a basis for  $S$  consisting of rational vectors.  $\square$

**Theorem 4.2.** *Consider unital inclusions of  $C^*$ -algebras  $A \supseteq D \subseteq B$  with  $A$  and  $B$  finite dimensional. Let  $A *_D B$  be the corresponding full amalgamated free product. Then  $A *_D B$  is residually finite dimensional if and only if there are faithful tracial states  $\tau_A$  on  $A$  and  $\tau_B$  on  $B$  whose restrictions to  $D$  agree.*

*Proof.* Since every separable r.f.d.  $C^*$ -algebra has a faithful tracial state, the necessity of the existence of  $\tau_A$  and  $\tau_B$  is clear.

Let us recall some well known facts about a unital inclusion  $D \subseteq A$  of finite dimensional  $C^*$ -algebras (see e.g. Chapter 2 of [8]). Let  $p_1, \dots, p_m$  be the minimal central projections of  $A$  and  $q_1, \dots, q_n$  the minimal central projections of  $D$ . Then the inclusion matrix  $\Lambda_D^A$  is a  $m \times n$  integer matrix whose  $(i, j)$ th entry is  $\text{rank}(q_j p_i A q_j) / \text{rank}(q_j D)$ , where the rank of a matrix algebra  $M_k(\mathbf{C})$  is  $k$ . To a trace  $\tau$  on  $A$ , we associate the column vector  $s$  of length  $m$  whose  $i$ th entry is the trace of a minimal projection in  $p_i A$ . Then the restriction of  $\tau$  to  $D$  has associated column vector  $(\Lambda_D^A)^t s$ , where the superscript  $t$  indicates transpose.

Thus, given  $A \supseteq D \subseteq B$  as in the statement of the theorem, the existence of faithful tracial states  $\tau_A$  and  $\tau_B$  agreeing on  $D$  is equivalent to the existence of column vectors  $s_A$  and  $s_B$ , none of whose components are zero, such that  $(\Lambda_D^A)^t s_A = (\Lambda_D^B)^t s_B$ , i.e.

$$\left[ (\Lambda_D^A)^t, -(\Lambda_D^B)^t \right] \begin{bmatrix} s_A \\ s_B \end{bmatrix} = 0. \quad (23)$$

Supposing now that such traces  $\tau_A$  and  $\tau_B$  exist, by Lemma 4.1 there is a solution  $\begin{bmatrix} s_A \\ s_B \end{bmatrix}$  to (23) whose entries are all strictly positive and rational. Therefore, the traces  $\tau_A$  and  $\tau_B$  agreeing on  $D$  can be chosen to take only rational values on minimal projections of  $A$  and, respectively,  $B$ . Hence there are unital inclusions into matrix algebras,

$$M_k(\mathbf{C}) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(\mathbf{C}),$$

so that  $\tau_A$  is the restriction of the tracial state on  $M_k(\mathbf{C})$  to  $A$  and  $\tau_B$  is the restriction of the tracial state on  $M_\ell(\mathbf{C})$  to  $B$ . By Proposition 2.2,  $A *_D B$  is a subalgebra of  $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ . By Theorem 2.3 of [5],  $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$  is r.f.d. Therefore,  $A *_D B$  is r.f.d.  $\square$

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