

# $C^*$ -ALGEBRAIC QUANTUM GROMOV-HAUSDORFF DISTANCE

HANFENG LI

ABSTRACT. We introduce a new quantum Gromov-Hausdorff distance between  $C^*$ -algebraic compact quantum metric spaces. Because it is able to distinguish algebraic structures, this new distance fixes a weakness of Rieffel's quantum distance. We show that this new quantum distance has properties analogous to the basic properties of the classical Gromov-Hausdorff distance, and we give criteria for when a parameterized family of  $C^*$ -algebraic compact quantum metric spaces is continuous with respect to this new distance.

## 1. INTRODUCTION

In [9] Connes initiated the study of metric spaces in noncommutative setting in the framework of his spectral triple [10]. The main ingredient of a spectral triple is a Dirac operator  $D$ . On the one hand, it captures the differential structure by setting  $df = [D, f]$ . On the other hand, it enables us to recover the Lipschitz seminorm  $L$ , which is usually defined as

$$(1) \quad L(f) := \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y\right\},$$

where  $\rho$  is the geodesic metric on the Riemannian manifold, instead by means of  $L(f) = \|[D, f]\|$ , and then one recovers the metric  $\rho$  by

$$(2) \quad \rho(x, y) = \sup_{L(f) \leq 1} |f(x) - f(y)|.$$

In Section 2 of [9] Connes went further by considering the (possibly  $+\infty$ -valued) metric on the state space of the algebra defined by (2). Motivated by what happens to ordinary compact metric spaces, in [32, 33, 35] Rieffel introduced “compact quantum metric spaces” which requires the metric on the state space to induce the weak- $*$  topology. Many interesting examples of compact quantum metric spaces have been constructed [32, 34, 29, 28].

Motivated by questions in string theory, Rieffel also introduced a notion of quantum Gromov-Hausdorff distance for compact quantum spaces [35, 36]. This is defined as a modified ordinary Gromov-Hausdorff distance for the state-spaces. The success of this quantum distance is that there is also a quantum version of the Gromov completeness and compactness theorems, which assert that the set of isometry classes of compact metric spaces endowed with the Gromov-Hausdorff distance is complete, and provide a criterion for when a subset of this complete space is precompact. And this quantum distance extends the ordinary Gromov-Hausdorff distance in the sense that sending each compact metric space to the associated compact quantum metric space is a homeomorphism (though not an isometry) from the

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space of isometry classes of compact metric spaces to a closed subspace of the space of isometry classes of compact quantum metric spaces. Since his construction does not involve the multiplicative structure of the algebras but only needs the state-spaces, Rieffel set up everything on more general spaces, namely *order-unit spaces*, based on Kadison's representation theory [23, 1]. As a consequence, his quantum Gromov-Hausdorff distance can not distinguish the multiplicative structures of the algebras, *i.e.* non-isomorphic  $C^*$ -algebras might have distance zero.

In view of the principle of noncommutative geometry, it may be more natural to define the quantum distance as a modified Gromov-Hausdorff distance for the order-unit spaces (or  $C^*$ -algebras) directly. Under this guidance, in [27] we introduced an order-unit quantum Gromov-Hausdorff distance as a modified ordinary Gromov-Hausdorff distance for certain balls in the order-unit spaces. It turns out that this order-unit quantum distance is Lipschitz equivalent to Rieffel's quantum distance [27, Theorem 1.1]. However, as an advantage of our approach, we gave criteria for when a parameterized family of compact quantum metric spaces is continuous with respect to the order-unit quantum distance [27, Theorems 1.2 and 7.1].

Another advantage of our approach is that we can adapt it easily to deal with additional algebraic structures. This is what we shall do in this paper. We introduce a  $C^*$ -algebraic quantum Gromov-Hausdorff distance, and show that it distinguishes the multiplicative structures of the algebras. We also show that it extends the ordinary Gromov-Hausdorff distance in the same way as does Rieffel's quantum distance. There is also a quantum version of the Gromov completeness and compactness theorems, though in order to make sure that the limits are  $C^*$ -algebras we need some compatibility condition between the seminorms and the algebraic structures, which turns out to be some kind of generalized Leibniz rule. We also give similar criteria for when a parameterized family of  $C^*$ -algebraic compact quantum metric spaces is continuous with respect to the  $C^*$ -algebraic quantum distance. These criteria enable us to conclude immediately that the continuity of noncommutative tori [35, Theorem 9.2] and  $\theta$ -deformations [27, Theorem 1.4] and the convergence of matrix algebras to integral coadjoint orbits of compact connected semisimple Lie groups [36, Theorem 3.2] all hold with respect to the  $C^*$ -algebraic quantum distance.

David Kerr [25] has defined a matricial quantum distance which is also able to distinguish the multiplicative structures of algebras. His approach is based on operator systems, very different from ours. In [26] we apply the results and methods of this paper to show that this distance has similar properties as our  $C^*$ -algebraic quantum distance does.

This paper is organized as follows. In Section 2 we review briefly the Gromov-Hausdorff distance for ordinary compact metric spaces and Rieffel's theory of compact quantum metric spaces. The  $C^*$ -algebraic quantum Gromov-Hausdorff distance is introduced in Section 3. We show there that this distance distinguishes multiplicative structures of the algebras. Section 4 is devoted to a quantum version of Gromov completeness and compactness theorems. The criteria for when a parameterized family of  $C^*$ -algebraic compact quantum metric spaces is continuous with respect to the  $C^*$ -algebraic quantum distance are given in Section 5.

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## 2. PRELIMINARIES

In this section we review briefly the Gromov-Hausdorff distance for compact metric spaces [20, 38, 6] and Rieffel's theory of compact quantum metric spaces [33, 35, 37].

Let  $(X, \rho)$  be a metric space, *i.e.*  $\rho$  is a metric on the space  $X$ . For any subset  $Y \subseteq X$  and  $r > 0$ , let

$$\mathcal{B}_r(Y) = \{x \in X : \rho(x, y) < r \text{ for some } y \in Y\}$$

be the set of points with distance less than  $r$  from  $Y$ . When  $Y = \{x\}$ , we also write it as  $\mathcal{B}_r(x)$  and call it the *open ball* of radius  $r$  centered at  $x$ .

For nonempty subsets  $Y, Z \subseteq X$ , we can measure the distance between  $Y$  and  $Z$  inside of  $X$  by the *Hausdorff distance*  $\text{dist}_H^\rho(Y, Z)$  defined by

$$\text{dist}_H^\rho(Y, Z) := \inf\{r > 0 : Y \subseteq \mathcal{B}_r(Z), Z \subseteq \mathcal{B}_r(Y)\}.$$

We will also use the notation  $\text{dist}_H^X(Y, Z)$  when there is no confusion about the metric on  $X$ . When  $X$  is compact, the set  $\text{SUB}(X)$  of nonempty closed subsets of  $X$  is compact equipped with the metric  $\text{dist}_H^X$  [6, Proposition 7.3.7].

For any compact metric spaces  $X$  and  $Y$ , Gromov [19] introduced the *Gromov-Hausdorff distance*,  $\text{dist}_{\text{GH}}(X, Y)$ , which is defined as

$$\text{dist}_{\text{GH}}(X, Y) := \inf\{\text{dist}_H^Z(h_X(X), h_Y(Y)) \mid h_X : X \rightarrow Z, h_Y : Y \rightarrow Z \text{ are isometric embeddings into some metric space } Z\}.$$

It is possible to reduce the space  $Z$  in above to be the disjoint union  $X \amalg Y$ . A distance  $\rho$  on  $X \amalg Y$  is said to be *admissible* if the inclusions  $X, Y \hookrightarrow X \amalg Y$  are isometric embeddings. Then it is not difficult to check that

$$\text{dist}_{\text{GH}}(X, Y) = \inf\{\text{dist}_H^\rho(X, Y) : \rho \text{ is an admissible distance on } X \amalg Y\}.$$

For a compact metric space  $(X, \rho)$ , we shall denote by  $\text{diam}(X) := \max\{\rho(x, y) \mid x, y \in X\}$  the *diameter* of  $X$ . Also let  $r_X = \frac{\text{diam}(X)}{2}$  be the *radius* of  $X$ . For any  $\varepsilon > 0$ , the *covering number*  $\text{Cov}_\rho(X, \varepsilon)$  is defined as the smallest number of open balls of radius  $\varepsilon$  whose union covers  $X$ .

Denote by  $\text{CM}$  the set of isometry classes of compact metric spaces. One important property of Gromov-Hausdorff distance is the completeness and compactness theorem by Gromov [19]:

**Theorem 2.1** (Gromov's Completeness and Compactness Theorems). The space  $(\text{CM}, \text{dist}_{\text{GH}})$  is a complete metric space. A subset  $\mathcal{S} \subseteq \text{CM}$  is totally bounded (*i.e.* has compact closure) if and only if

- (1) there is a constant  $D$  such that  $\text{diam}(X, \rho) \leq D$  for all  $(X, \rho) \in \mathcal{S}$ ;
- (2) for any  $\varepsilon > 0$ , there exists a constant  $K_\varepsilon > 0$  such that  $\text{Cov}_\rho(X, \varepsilon) \leq K_\varepsilon$  for all  $(X, \rho) \in \mathcal{S}$ .

Next we recall Rieffel's definition of ( $C^*$ -algebraic) compact quantum metric spaces. Though Rieffel has set up his theory in the general framework of order-unit spaces, we shall mainly need it only for  $C^*$ -algebras. See the discussion preceding Definition 2.1 in [35] for the reason of requiring the reality condition (3) below.

**Definition 2.2.** [35, Definition 2.1] By a  $C^*$ -algebraic compact quantum metric space we mean a pair  $(\mathcal{A}, L)$  consisting of a unital  $C^*$ -algebra  $\mathcal{A}$  with identify  $e_{\mathcal{A}}$  and a (possibly  $+\infty$ -valued) seminorm  $L$  on  $\mathcal{A}$  satisfying the *reality condition*

$$(3) \quad L(a) = L(a^*),$$

such that  $L$  vanishes on  $\mathbb{C}e_{\mathcal{A}}$  and the metric  $\rho_L$  on the state space  $S(\mathcal{A})$  defined by (2) induces the weak- $*$  topology. The *diameter*, *radius* and *covering number* of  $(\mathcal{A}, L)$  are defined to be the same as those of  $(S(\mathcal{A}), \rho_L)$ . We say that  $L$  is a *Lip-norm*.

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and let  $L$  be a (possibly  $+\infty$ -valued) seminorm  $L$  on  $\mathcal{A}$  satisfying (3). Then one can check easily that the pair  $(\mathcal{A}, L)$  is a  $C^*$ -algebraic compact quantum metric space if and only if the space  $A = \{a \in \mathcal{A}_{\text{sa}} : L(a) < \infty\}$  is dense in  $\mathcal{A}_{\text{sa}}$  containing  $e_{\mathcal{A}}$  and the pair  $(A, L|_A)$  is a compact quantum metric space in the sense of [35, Definition 2.1]. We shall call the latter the associated compact quantum metric space of  $(\mathcal{A}, L)$ . For most parts of this paper the reader may take the compact quantum metric spaces as the ones associated to  $C^*$ -algebraic compact quantum metric spaces.

For any compact metric space  $(X, \rho)$ , define the Lipschitz seminorm (which may take value  $+\infty$ ),  $L_{\rho}$ , on  $C_{\mathbb{C}}(X)$ , the set of all  $\mathbb{C}$ -valued continuous functions over  $X$ , by (1). Then the pair  $(C_{\mathbb{C}}(X), L_{\rho})$  is a  $C^*$ -algebraic compact quantum metric space, and the associated compact quantum metric space is the pair  $(A_X, L_{\rho}|_{A_X})$ , where  $A_X$  is the space of  $\mathbb{R}$ -valued Lipschitz functions on  $X$ . They are called the associated ( $C^*$ -algebraic) compact quantum metric space of  $(X, \rho)$ . Denote by CQM the set of isometry classes of compact quantum metric spaces [35, Definition 6.3]. In [35, Theorem 13.16] Rieffel showed that the map  $(X, \rho) \mapsto (A_X, L_{\rho}|_{A_X})$  from  $(\text{CM}, \text{dist}_{\text{GH}})$  to  $(\text{CQM}, \text{dist}_{\text{q}})$  is continuous, injective, and sends CM to a closed subspace of CQM, where  $\text{dist}_{\text{q}}$  is Rieffel's quantum distance. The argument there actually shows that this map is a homeomorphism from CM to its image.

### 3. DEFINITION OF $C^*$ -ALGEBRAIC QUANTUM DISTANCE

In this section we define a quantum Gromov-Hausdorff distance  $\text{dist}_{\text{cq}}$  which distinguishes the algebraic structures. The main results in this section are Theorems 3.15 and 3.16.

**Notation 3.1.** Let  $(\mathcal{A}, L)$  be a  $C^*$ -algebraic compact quantum metric space. For any  $r \geq 0$ , set

$$\mathcal{D}_r(A) := \{a \in A : L(a) \leq 1, \|a\| \leq r\}, \quad \text{and} \quad \mathcal{D}(A) := \mathcal{D}_{r_{\mathcal{A}}}(A),$$

where  $(A, L|_A)$  is the compact quantum metric space associated to  $(\mathcal{A}, L)$  (see the discussion after Definition 2.2).  $\mathcal{D}_r(\mathcal{A})$  and  $\mathcal{D}(\mathcal{A})$  are defined similarly.

We require  $\text{dist}_{\text{cq}}$  to extend the ordinary Gromov-Hausdorff distance  $\text{dist}_{\text{GH}}$  in the sense that they define the same topology on CM (see the discussion at the end of Section 2). To make  $\text{dist}_{\text{cq}}$  manageable, we also want an analogue of the criteria for continuity of parameterized families of compact quantum metric spaces with respect to the order-unit quantum distance  $\text{dist}_{\text{oq}}$  [27, Theorems 1.2 and 7.1]. In [27, Definition 4.2] the order-unit distance  $\text{dist}_{\text{oq}}$  is defined as a modified Gromov-Hausdorff distance between the balls  $\mathcal{D}(A)$ , based on the observation that these balls catch both the norm and the Lip-norm of the compact quantum metric spaces

[27, Lemma 4.1]. Precisely, for any  $R \geq 0$  and any compact quantum metric spaces  $(A, L_A)$  and  $(B, L_B)$  we have

$$\begin{aligned} \text{dist}_{\text{oq}}(A, B) &= \inf\{\max(\text{dist}_{\mathbb{H}}^V(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B))), \|h_A(r_A e_A) - h_B(r_B e_B)\|)\}, \\ \text{dist}_{\text{oq}}^R(A, B) &= \inf\{\max(\text{dist}_{\mathbb{H}}^V(h_A(\mathcal{D}_R(A)), h_B(\mathcal{D}_R(B))), \|h_A(R e_A) - h_B(R e_B)\|)\}, \end{aligned}$$

where the infima are taken over all triples  $(V, h_A, h_B)$  consisting of a real normed space  $V$  and linear isometric embeddings  $h_A : A \rightarrow V$  and  $h_B : B \rightarrow V$ . The distance  $\text{dist}_{\text{oq}}^R$  was introduced to facilitate certain arguments involving continuity.

If we look at above definition of  $\text{dist}_{\text{oq}}$ , one natural choice is to define  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B})$  as  $\inf\{\text{dist}_{\mathbb{H}}^{\mathcal{C}}(h_{\mathcal{A}}(\mathcal{D}(\mathcal{A})), h_{\mathcal{B}}(\mathcal{D}(\mathcal{B})))\}$ , where the infimum is taken over all faithful  $*$ -homomorphisms  $h_{\mathcal{A}}$  and  $h_{\mathcal{B}}$  of  $\mathcal{A}$  and  $\mathcal{B}$  into some  $C^*$ -algebra  $\mathcal{C}$ . There are two difficulties. The first one is that  $\mathbb{C}$ -valued functions do not behave as well as  $\mathbb{R}$ -valued functions do under Lipschitz seminorm. So if we use the balls  $\mathcal{D}(\mathcal{A})$ , the resulting quantum distance will induce a stronger topology on CM than  $\text{dist}_{\text{GH}}$  does (see Remark 4.11). The second difficulty is that we do not have the  $C^*$ -algebraic analogue of [27, Lemma 7.2] (see [27, Remark 7.5]), which is crucial in the proofs of [27, Theorems 1.2 and 7.1]. Hence if we require  $h_{\mathcal{A}}$  to be a  $*$ -homomorphism, then we do not know how to prove the analogues of [27, Theorems 1.2 and 7.1] (however, see Remark 5.5). Thus we are forced to use the ball  $\mathcal{D}(A)$  and to require  $h_{\mathcal{A}}$  to be a linear isometric embedding into a complex normed space  $V$ . But this will not distinguish the algebraic structures. To get around this difficulty, we will use the graph of the multiplication in  $\mathcal{D}(A)$  instead of  $\mathcal{D}(A)$  itself.

**Notation 3.2.** For a subset  $X$  of a  $C^*$ -algebra  $\mathcal{A}$  we shall denote by

$$X^m := \{(x, y, xy) \in \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A} : x, y \in X\}$$

the graph of the multiplication in  $X$ . For a normed space  $V$  we shall denote by  $V^{(3)}$  the direct sum  $V \oplus V \oplus V$  equipped with the supremum-norm  $\|(v_1, v_2, v_3)\| = \max(\|v_1\|, \|v_2\|, \|v_3\|)$ . Similarly, we define  $V^{(2)}$ . For a linear map  $h : \mathcal{A} \rightarrow V$  we denote by  $h^{(3)}$  the natural extension  $\mathcal{A}^{(3)} \rightarrow V^{(3)}$ .

In the above definition of  $\text{dist}_{\text{oq}}$  we had to consider the term  $\|h_A(r_A e_A) - h_B(r_B e_B)\|$  to keep track of the units. Since the algebraic structures force  $\|h_A(e_A) - h_B(e_B)\|$  to be bounded by the Hausdorff distance between  $h_A^{(3)}(\mathcal{D}(A)^m)$  and  $h_B^{(3)}(\mathcal{D}(B)^m)$  (see Lemma 3.11), we do not need this term any longer as we did in  $\text{dist}_{\text{oq}}$ .

**Definition 3.3.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. We define the  $C^*$ -algebraic quantum Gromov-Hausdorff distance between them, denoted by  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B})$ , by

$$\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) := \inf\{\text{dist}_{\mathbb{H}}^{V^{(3)}}(h_{\mathcal{A}}^{(3)}(\mathcal{D}(A)^m), h_{\mathcal{B}}^{(3)}(\mathcal{D}(B)^m))\},$$

and, for  $R \geq 0$ , the  $R$ - $C^*$ -algebraic quantum Gromov-Hausdorff distance between them, denoted by  $\text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B})$ , by

$$\text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) := \inf\{\text{dist}_{\mathbb{H}}^{V^{(3)}}(h_{\mathcal{A}}^{(3)}(\mathcal{D}_R(A)^m), h_{\mathcal{B}}^{(3)}(\mathcal{D}_R(B)^m))\},$$

where the infima are taken over all triples  $(V, h_{\mathcal{A}}, h_{\mathcal{B}})$  consisting of a complex normed space  $V$  and linear isometric embeddings  $h_{\mathcal{A}} : \mathcal{A} \rightarrow V$  and  $h_{\mathcal{B}} : \mathcal{B} \rightarrow V$ .

**Remark 3.4.** (1) To simplify the notation, usually we shall identify  $\mathcal{A}$  and  $\mathcal{B}$  with their images  $h_{\mathcal{A}}(\mathcal{A})$  and  $h_{\mathcal{B}}(\mathcal{B})$  respectively, and just say that  $V$  is a normed space containing both  $\mathcal{A}$  and  $\mathcal{B}$ .

(2) Initiated by Kadison and Kastler [24] there has been a lot of work on perturbation of  $C^*$ -algebras [8, 30, 22]. In this theory people consider  $\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathbb{D}(\mathcal{A}), \mathbb{D}(\mathcal{B}))$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -subalgebras of a  $C^*$ -algebra  $\mathcal{C}$ , and  $\mathbb{D}(\mathcal{A})$  is the unit ball of  $\mathcal{A}$ . Using Blackadar's result [2, Theorem 3.1] on amalgamation of  $C^*$ -algebras one sees easily that  $\inf\{\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathbb{D}(\mathcal{A}), \mathbb{D}(\mathcal{B}))\}$  defines a distance  $\text{dist}_{\mathfrak{u}}$ , where  $\mathcal{C}$  runs over all  $C^*$ -algebras containing both  $\mathcal{A}$  and  $\mathcal{B}$  as  $C^*$ -subalgebras. We do not know when  $\text{dist}_{\mathfrak{u}}(\mathcal{A}, \mathcal{B}) = 0$ . This looks very similar to the distance  $\text{dist}_{\text{nu}}$  we shall discuss in Remark 5.5. However, they are quite different. If  $\mathcal{A}$  is a separable unital continuous trace  $C^*$ -algebra, then when  $\text{dist}_{\mathfrak{u}}(\mathcal{A}, \mathcal{B})$  is small,  $\mathcal{A}$  and  $\mathcal{B}$  must be isomorphic as  $C^*$ -algebras [30, Theorem 4.22].

For any normed spaces  $V$  and  $W$  we call a norm  $\|\cdot\|_{V \oplus W}$  on  $V \oplus W$  *admissible* if it extends the norms on  $V$  and  $W$ . By an argument similar to that in the proof of [27, Proposition 4.4] we have:

**Proposition 3.5.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. Then

$$\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) = \inf\{\text{dist}_{\mathbb{H}}^{(\mathcal{A} \oplus \mathcal{B})^{(3)}}(\mathcal{D}(\mathcal{A})^m, \mathcal{D}(\mathcal{B})^m)\},$$

and, for any  $R \geq 0$ ,

$$\text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) = \inf\{\text{dist}_{\mathbb{H}}^{(\mathcal{A} \oplus \mathcal{B})^{(3)}}(\mathcal{D}_R(\mathcal{A})^m, \mathcal{D}_R(\mathcal{B})^m)\},$$

where the infima are taken over all admissible norms on  $\mathcal{A} \oplus \mathcal{B}$ .

Using the amalgamation of normed spaces [27, Lemma 4.5] we have the triangle inequalities for  $\text{dist}_{\text{cq}}$  and  $\text{dist}_{\text{cq}}^R$ :

**Lemma 3.6.** For any  $C^*$ -algebraic compact quantum metric spaces  $(\mathcal{A}, L_{\mathcal{A}})$ ,  $(\mathcal{B}, L_{\mathcal{B}})$  and  $(\mathcal{C}, L_{\mathcal{C}})$  we have

$$\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{C}) \leq \text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) + \text{dist}_{\text{cq}}(\mathcal{B}, \mathcal{C}).$$

For  $R \geq 0$  we also have

$$\text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{C}) \leq \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) + \text{dist}_{\text{cq}}^R(\mathcal{B}, \mathcal{C}).$$

One can take closures for compact quantum metric spaces [33, Definition 4.5] (see also [27, Section 2]). Similarly, we can take closure for  $C^*$ -algebraic compact quantum metric spaces:

**Lemma 3.7.** Let  $(\mathcal{A}, L)$  be a  $C^*$ -algebraic compact quantum metric space. Define a seminorm  $L^c$  on  $\mathcal{A}$  by

$$L^c(a) := \inf\{\liminf_{n \rightarrow \infty} L(a_n) : a_n \in \mathcal{A}, \lim_{n \rightarrow \infty} a_n = a\}.$$

Then  $(\mathcal{A}, L^c)$  is also a  $C^*$ -algebraic compact quantum metric space, and the associated compact quantum metric space is exactly the closure of  $(\mathcal{A}, L|_{\mathcal{A}})$ .

*Proof.* Since  $L(a^*) = L(a)$  for all  $a \in \mathcal{A}$ , clearly  $L^c(a^*) = L^c(a)$  for all  $a \in \mathcal{A}$  too.

For any  $a \in \mathcal{A}_{\text{sa}}$ , if  $a_n \rightarrow a$  when  $n \rightarrow \infty$ , then also  $(a_n)_{\text{sa}} \rightarrow a$ . Notice that  $L((a_n)_{\text{sa}}) \leq \frac{L(a_n) + L(a_n^*)}{2} = L(a_n)$ . So  $\liminf_{n \rightarrow \infty} L((a_n)_{\text{sa}}) \leq \liminf_{n \rightarrow \infty} L(a_n)$ . Hence

$$L^c(a) = \inf\{\liminf_{n \rightarrow \infty} L(a_n) : a_n \in \mathcal{A}_{\text{sa}}, \lim_{n \rightarrow \infty} a_n = a\}$$

for  $a \in \mathcal{A}_{\text{sa}}$ . It follows immediately that  $L^c$  restricted to  $\mathcal{A}_{\text{sa}}$  is exactly the closure of  $L|_{\mathcal{A}}$ . So the set  $\{a \in \mathcal{A}_{\text{sa}} : L^c(a) < \infty\}$  equipped with the restriction of  $L^c$  is a compact quantum metric space and is actually the closure of  $(\mathcal{A}, L|_{\mathcal{A}})$ .  $\square$

**Definition 3.8.** Let  $(\mathcal{A}, L)$  be a  $C^*$ -algebraic compact quantum metric space. We call  $(\mathcal{A}, L^c)$  the *closure* of  $(\mathcal{A}, L)$ . We say that  $(\mathcal{A}, L)$  and  $L$  are *closed* if  $L^c = L$ .

Clearly we have  $\text{dist}_{\text{cq}}((\mathcal{A}, L), (\mathcal{A}, L^c)) = \text{dist}_{\text{cq}}^R((\mathcal{A}, L), (\mathcal{A}, L^c)) = 0$ .

Next we establish some basic inequalities about  $\text{dist}_{\text{cq}}$  and  $\text{dist}_{\text{cq}}^R$ :

**Proposition 3.9.** For any  $C^*$ -algebraic compact quantum metric spaces  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  we have

$$(4) \quad |r_{\mathcal{A}} - r_{\mathcal{B}}| \leq \text{dist}_{\text{GH}}(\mathcal{D}(\mathcal{A}), \mathcal{D}(\mathcal{B})) \leq \text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) \leq \max(r_{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{A}}^2, r_{\mathcal{B}}^2),$$

$$(5) \quad |\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) - \text{dist}_{\text{cq}}^{r_{\mathcal{A}}, r_{\mathcal{B}}}(\mathcal{A}, \mathcal{B})| \leq \max(|r_{\mathcal{A}} - r_{\mathcal{B}}|, |r_{\mathcal{A}}^2 - r_{\mathcal{B}}^2|).$$

For any  $R \geq r \geq 0$  we also have

$$(6) \quad \text{dist}_{\text{cq}}^r(\mathcal{A}, \mathcal{B}) \leq \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) \max(2, \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) + 2r + 1).$$

*Proof.* For any compact metric spaces  $X$  and  $Y$ , one has

$$(7) \quad |r_X - r_Y| \leq \text{dist}_{\text{GH}}(X, Y).$$

The inequality (4) follows from (7) and the fact that for any normed space  $V$  containing  $\mathcal{A}$  and  $\mathcal{B}$  we have

$$\text{dist}_{\text{H}}^V(\mathcal{D}(\mathcal{A}), \mathcal{D}(\mathcal{B})) \leq \text{dist}_{\text{H}}^{V(3)}(\mathcal{D}(\mathcal{A})^m, \mathcal{D}(\mathcal{B})^m) \leq \max(r_{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{A}}^2, r_{\mathcal{B}}^2).$$

To show (5) it suffices to show that  $\text{dist}_{\text{H}}^{A(3)}(\mathcal{D}(\mathcal{A})^m, \mathcal{D}_{r_{\mathcal{B}}}(\mathcal{A})^m) \leq \max(|r_{\mathcal{A}} - r_{\mathcal{B}}|, |r_{\mathcal{A}}^2 - r_{\mathcal{B}}^2|)$ . In fact we have:

**Lemma 3.10.** For any  $C^*$ -algebraic compact quantum metric space  $(\mathcal{A}, L_{\mathcal{A}})$  and any  $R > r \geq 0$  we have

$$\text{dist}_{\text{H}}^{A(3)}(\mathcal{D}_R(\mathcal{A})^m, \mathcal{D}_r(\mathcal{A})^m) \leq \max(R - r, R^2 - r^2).$$

*Proof.* Notice that  $\mathcal{D}_r(\mathcal{A})^m$  is a subset of  $\mathcal{D}_R(\mathcal{A})^m$ . For any  $a_1, a_2 \in \mathcal{D}_R(\mathcal{A})$ , let  $a'_j = \frac{r}{R}a_j$ . Then  $a'_j \in \mathcal{D}_r(\mathcal{A})$  and  $\|(a_1, a_2, a_1a_2) - (a'_1, a'_2, a'_1a'_2)\| \leq \max(R - r, R^2 - r^2)$ . This yields the desired inequality.  $\square$

We proceed to show (6). We may assume that  $r > 0$  and both  $L_{\mathcal{A}}$  and  $L_{\mathcal{B}}$  are closed. Let  $V$  be a complex normed space containing  $\mathcal{A}$  and  $\mathcal{B}$ , and set  $d = \text{dist}_{\text{H}}^{V(3)}(\mathcal{D}_R(\mathcal{A})^m, \mathcal{D}_R(\mathcal{B})^m)$ . For any  $a_1, a_2 \in \mathcal{D}_r(\mathcal{A})$  pick  $b_1, b_2 \in \mathcal{D}_R(\mathcal{B})$  such that  $\|a_1 - b_1\|, \|a_2 - b_2\|, \|a_1a_2 - b_1b_2\| \leq d$ . Then  $\|b_j\| \leq d + r$ . Let  $\eta = r/(d + r)$ , and let  $b'_j = \eta b_j$ . One checks easily that  $b'_j \in \mathcal{D}_r(\mathcal{B})$ , and  $\|a_j - b'_j\|, \|a_1a_2 - b'_1b'_2\| \leq d \cdot \max(2, d + 2r + 1)$ . Similarly, one can deal with pairs in  $\mathcal{D}_r(\mathcal{B})$ . Therefore  $\text{dist}_{\text{H}}^{V(3)}(\mathcal{D}_r(\mathcal{A})^m, \mathcal{D}_r(\mathcal{B})^m) \leq d \cdot \max(2, d + 2r + 1)$ . Then (6) follows.  $\square$

**Lemma 3.11.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. Let  $V$  be a complex normed space containing  $\mathcal{A}$  and  $\mathcal{B}$ . Then for any  $R \geq 0$  we have

$$R^2 \|e_{\mathcal{A}} - e_{\mathcal{B}}\| \leq (4R + 1) \text{dist}_{\text{H}}^{V(3)}(\mathcal{D}_R(\mathcal{A})^m, \mathcal{D}_R(\mathcal{B})^m).$$

*Proof.* Set  $d = \text{dist}_{\mathbb{H}}^{V(3)}(\mathcal{D}_R(A)^m, \mathcal{D}_R(B)^m)$ . Denote  $\mathcal{D}_R(A)$  and  $\mathcal{D}_R(B)$  by  $X$  and  $Y$  respectively. We may assume that both  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are closed. Then  $X$  and  $Y$  are compact. Thus for  $(Re_{\mathcal{A}}, Re_{\mathcal{A}}, R^2e_{\mathcal{A}}) \in X^m$  we can find  $(b, b', bb') \in Y^m$  such that  $\|Re_{\mathcal{A}} - b\|, \|Re_{\mathcal{A}} - b'\|, \|R^2e_{\mathcal{A}} - bb'\| \leq d$ . Also for  $(Re_{\mathcal{B}}, b, Rb) \in Y^m$  we can find  $(a, a', aa') \in X^m$  such that  $\|Re_{\mathcal{B}} - a\|, \|b - a'\|, \|Rb - aa'\| \leq d$ . Then

$$\begin{aligned} \|a' - Re_{\mathcal{A}}\| &\leq \|a' - b\| + \|b - Re_{\mathcal{A}}\| \leq 2d, \\ \|Rb - R^2e_{\mathcal{B}}\| &\leq \|Rb - aa'\| + \|aa' - Ra\| + \|Ra - R^2e_{\mathcal{B}}\| \\ &\leq d + \|a\| \cdot (\|a' - b\| + \|b - Re_{\mathcal{A}}\|) + Rd \leq d + 3Rd, \\ \|R^2e_{\mathcal{A}} - R^2e_{\mathcal{B}}\| &\leq \|R^2e_{\mathcal{A}} - Rb\| + \|Rb - R^2e_{\mathcal{B}}\| \leq Rd + d + 3Rd = 4Rd + d, \end{aligned}$$

as desired.  $\square$

**Proposition 3.12.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. For any  $R > 0$  we have

$$(8) \quad \text{dist}_{\text{oq}}^R(A, B) \leq \frac{4R+1}{R} \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}).$$

If furthermore  $R \geq \max(r_{\mathcal{A}}, r_{\mathcal{B}})$ , then

$$(9) \quad \frac{2R}{5(4R+1)} |r_{\mathcal{A}} - r_{\mathcal{B}}| \leq \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}).$$

*Proof.* (8) follows from Lemma 3.11 and the definitions of  $\text{dist}_{\text{oq}}^R$  and  $\text{dist}_{\text{cq}}^R$ . Also (9) follows from (7), (8), [27, Theorem 1.1], and the fact  $\text{dist}_{\text{GH}}(S(A), S(B)) \leq \text{dist}_{\text{q}}(A, B)$ .  $\square$

**Proposition 3.13.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. For any  $R \geq r \geq \max(r_{\mathcal{A}}, r_{\mathcal{B}}) > 0$  set  $\tau_1 = (4r+1)(R+r)/(r^2) + 1$ , and set  $\tau_2 = (\tau_1+1)(R+r) + 2\tau_1R + 1$ . Then we have

$$(10) \quad \text{dist}_{\text{cq}}^R(\mathcal{A}, \mathcal{B}) \leq \text{dist}_{\text{cq}}^r(\mathcal{A}, \mathcal{B}) \max(2\tau_1, \tau_1 \text{dist}_{\text{cq}}^r(\mathcal{A}, \mathcal{B}) + \tau_2).$$

*Proof.* Set  $X_R = \mathcal{D}_R(A)$ , and set  $Y_R = \mathcal{D}_R(B)$ . Similarly define  $X_r$  and  $Y_r$ . Let  $V$  be a complex normed space containing  $\mathcal{A}$  and  $\mathcal{B}$ . Denote  $\text{dist}_{\mathbb{H}}^{V(3)}((X_r)^m, (Y_r)^m)$  by  $d$ . We may assume that  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are both closed. By Lemma 3.11 we have  $\|e_{\mathcal{A}} - e_{\mathcal{B}}\| \leq ((4r+1)d)/(r^2)$ . Let  $a_1, a_2 \in X_R$ . By [27, Lemma 4.1] we can write  $a_j$  as  $\lambda_j e_{\mathcal{A}} + a'_j$  with  $\lambda_j \in \mathbb{R}$  and  $a'_j \in X_r$ . Then  $|\lambda_j| \leq R+r$ . Pick  $b'_1, b'_2 \in Y_r$  such that  $\|a'_1 - a'_2\|, \|a'_2 - b'_2\|, \|a'_1 a'_2 - b'_1 b'_2\| \leq d$ . Set  $b_j = \lambda_j e_{\mathcal{B}} + b'_j$ . A routine calculation yields

$$\|a_j - b_j\| \leq \tau_1 d, \quad \|b_j\| \leq \tau_1 d + R, \quad \|a_1 a_2 - b_1 b_2\| \leq ((\tau_1+1)(R+r) + 1)d.$$

Set  $\eta = R/(\tau_1 d + R)$ , and set  $b''_j = \eta b_j$ . Then  $b''_j \in Y_R$ , and

$$\|b_j - b''_j\| \leq \tau_1 d, \quad \|b_1 b_2 - b''_1 b''_2\| \leq \tau_1^2 d^2 + 2\tau_1 d R.$$

Consequently,

$$\|a_j - b''_j\| \leq 2\tau_1 d, \quad \|a_1 a_2 - b''_1 b''_2\| \leq \tau_1^2 d^2 + \tau_2 d.$$

Similarly, we can deal with pairs in  $Y_R$ . Thus

$$\text{dist}_{\mathbb{H}}^{V(3)}((X_R)^m, (Y_R)^m) \leq d \cdot \max(2, \tau_1^2 d + \tau_2).$$

Letting  $V$  run over all complex normed spaces containing  $\mathcal{A}$  and  $\mathcal{B}$ , we get (10).  $\square$

To show that  $\text{dist}_{\text{cq}}$  is a metric, we need to see what happens when  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) = 0$ . As promised at the beginning of this section, isometric  $C^*$ -algebraic compact quantum metric spaces are also isomorphic as  $C^*$ -algebras:

**Definition 3.14.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  be  $C^*$ -algebraic compact quantum metric spaces. By an *isometry* from  $(\mathcal{A}, L_{\mathcal{A}})$  to  $(\mathcal{B}, L_{\mathcal{B}})$  we mean a  $*$ -isomorphism  $\varphi$  from  $\mathcal{A}$  onto  $\mathcal{B}$  such that  $L_{\mathcal{A}}^c = L_{\mathcal{B}}^c \circ \varphi$  on  $\mathcal{A}_{\text{sa}}$ . When there exists an isometry from  $(\mathcal{A}, L_{\mathcal{A}})$  to  $(\mathcal{B}, L_{\mathcal{B}})$ , we say that  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are *isometric*. We denote by  $\text{C}^*\text{CQM}$  the set of all isometry classes of  $C^*$ -algebraic compact quantum metric spaces, and denote by  $\text{C}^*\text{CQM}^R$  the subset consisting of isometry classes of  $C^*$ -algebraic compact quantum metric spaces with radii no bigger than  $R$ .

Notice that we require  $L_{\mathcal{A}}^c = L_{\mathcal{B}}^c \circ \varphi$  only on  $\mathcal{A}_{\text{sa}}$ , not on the whole of  $\mathcal{A}$ .

**Theorem 3.15.**  $\text{dist}_{\text{cq}}$  and  $\text{dist}_{\text{cq}}^R$  are metrics on  $\text{C}^*\text{CQM}$  and  $\text{C}^*\text{CQM}^R$  respectively. They define the same topology on  $\text{C}^*\text{CQM}^R$ . The forgetful map  $(\mathcal{A}, L) \mapsto (\mathcal{A}, L|_{\mathcal{A}})$  from  $(\text{C}^*\text{CQM}, \text{dist}_{\text{cq}})$  to  $(\text{CQM}, \text{dist}_{\text{oq}})$  is continuous.

*Proof.* It suffices to show that  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are isometric if and only if  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) = 0$ . Then all the other assertions follow from [27, Theorem 1.1], Lemma 3.6, and Propositions 3.9, 3.12, and 3.13.

If  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are isometric, then clearly  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) = 0$ . Suppose now  $\text{dist}_{\text{cq}}(\mathcal{A}, \mathcal{B}) = 0$ . By (4) we have  $r_{\mathcal{A}} = r_{\mathcal{B}}$ . The case  $r_{\mathcal{A}} = 0$  is trivial. So we assume that  $r_{\mathcal{A}} > 0$ . We may also assume that both  $(\mathcal{A}, L_{\mathcal{A}})$  and  $(\mathcal{B}, L_{\mathcal{B}})$  are closed. Set  $X = \mathcal{D}(A)$  and  $Y = \mathcal{D}(B)$ . For each  $n \in \mathbb{N}$  we can find a complex normed space  $V_n$  containing  $\mathcal{B}$  and a linear isometric embedding  $h_n : \mathcal{A} \hookrightarrow V_n$  such that  $\text{dist}_{\text{H}}^{V^{(3)}}(h_n^{(3)}(X^m), Y^m) < \frac{1}{n}$ . By [27, Lemma 4.5] we can find a complex normed space  $V$  containing all these  $V_n$ 's with the copies of  $\mathcal{B}$  identified. Then for any  $n$  the union  $\cup_{k=n+1}^{\infty} h_k^{(3)}(X^m)$  is contained in the open  $\frac{1}{n}$ -neighborhood of  $Y^m$ , and hence  $\cup_{k=1}^{\infty} h_k^{(3)}(X^m)$  is contained in the open  $\frac{1}{n}$ -neighborhood of  $Y^m \cup (\cup_{k=1}^n h_k^{(3)}(X^m))$ . It follows immediately that  $\cup_{k=1}^{\infty} h_k^{(3)}(X^m)$  is totally bounded. Notice that  $\{h_n^{(3)}|_{X^m}\}_{n \in \mathbb{N}}$  is an equicontinuous sequence of mappings from  $X^m$  to  $V^{(3)}$ . By the Arzela-Ascoli theorem [12] there is a subsequence  $\{h_{n_k}^{(3)}|_{X^m}\}_{n \in \mathbb{N}}$  converging uniformly to a continuous map  $\varphi : X^m \rightarrow V^{(3)}$ . By abuse of notation, we may assume that this subsequence is  $\{h_n^{(3)}|_{X^m}\}_{n \in \mathbb{N}}$  itself. In particular, the sequence  $\{h_n|_X\}_{n \in \mathbb{N}}$  converges uniformly. Clearly for every  $a \in A + iA$  the sequence  $\{h_n(a)\}_{n \in \mathbb{N}}$  also converges. Since  $A + iA$  is dense in  $\mathcal{A}$  and each  $h_n$  is isometric, it is easy to see that  $\{h_n(a)\}_{n \in \mathbb{N}}$  converges for every  $a \in \mathcal{A}$ . Let  $h(a)$  be the limit. Clearly  $h : \mathcal{A} \rightarrow V$  is a linear isometric embedding. Since  $h_n|_X$  converges uniformly to  $h|_X$ , we have  $\text{dist}_{\text{H}}^V(h_n(X), h(X)) \rightarrow 0$ . But  $\text{dist}_{\text{H}}^V(h_n(X), Y) \leq \text{dist}_{\text{H}}^{V^{(3)}}(h_n^{(3)}(X^m), Y^m) \rightarrow 0$ . So  $\text{dist}_{\text{H}}^V(Y, h(X)) = 0$ , and hence  $Y = h(X)$ . Consequently  $h(\mathcal{A}) = \mathcal{B}$ . Clearly  $\varphi = h^{(3)}|_{X^m}$ . Thus  $h(aa') = h(a)h(a')$  for any  $a, a' \in X$ . Consequently  $h$  is an algebra homomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ . Since  $h(\mathcal{A}) \subseteq \mathcal{B}$ , we have  $h(\mathcal{A}_{\text{sa}}) \subseteq \mathcal{B}_{\text{sa}}$ . Therefore  $h : \mathcal{A} \rightarrow \mathcal{B}$  is a  $*$ -isomorphism. In particular,  $h(e_{\mathcal{A}}) = e_{\mathcal{B}}$ . Then [27, Lemma 4.1] tells us that  $h(\mathcal{D}_{\infty}(A)) = \mathcal{D}_{\infty}(B)$ . Hence  $L_{\mathcal{A}} = L_{\mathcal{A}} \circ h$  on  $\mathcal{A}_{\text{sa}}$ . So  $h$  is an isometry from  $(\mathcal{A}, L_{\mathcal{A}})$  to  $(\mathcal{B}, L_{\mathcal{B}})$ .  $\square$

Next we show that our  $C^*$ -algebraic quantum distance  $\text{dist}_{\text{cq}}$  extends the ordinary Gromov-Hausdorff distance  $\text{dist}_{\text{GH}}$  in the way Rieffel's quantum distance  $\text{dist}_{\text{q}}$  does (see the discussion at the end of Section 2).

**Theorem 3.16.** The map  $(X, \rho) \mapsto (C_{\mathbb{C}}(X), L_{\rho})$  is a homeomorphism from  $(\text{CM}, \text{dist}_{\text{GH}})$  onto a closed subspace of  $(C^*\text{CQM}, \text{dist}_{\text{cq}})$ .

**Proposition 3.17.** Let  $(X, \rho_X)$  and  $(Y, \rho_Y)$  be compact metric spaces. For any  $R \geq 0$  we have

$$(11) \quad \text{dist}_{\text{cq}}^R(C_{\mathbb{C}}(X), C_{\mathbb{C}}(Y)) \leq \text{dist}_{\text{GH}}(X, Y) \max(1, 2R).$$

*Proof.* The proof is similar to that of [27, Proposition 4.7]. Let  $\rho$  be an admissible metric on  $X \amalg Y$ . Denote  $\text{dist}_{\mathbb{H}}^{\rho}(X, Y)$  by  $d$ . Denote by  $Z$  the set of elements  $(x, y)$  in  $X \times Y$  with  $\rho(x, y) \leq d$ . Since  $X$  and  $Y$  are compact, the projections  $Z \rightarrow X$  and  $Z \rightarrow Y$  are surjective. Thus the induced  $*$ -homomorphisms  $C_{\mathbb{C}}(X) \rightarrow C_{\mathbb{C}}(Z)$  and  $C_{\mathbb{C}}(Y) \rightarrow C_{\mathbb{C}}(Z)$  are faithful. Notice that for any  $f \in C_{\mathbb{C}}(X)$  and  $g \in C_{\mathbb{C}}(Y)$  we have

$$\|f - g\| = \sup\{|f(x) - g(y)| : (x, y) \in Z\} \leq L_{\rho}(f, g)d,$$

where  $L_{\rho}$  is the Lipschitz seminorm on  $C_{\mathbb{C}}(X \amalg Y)$ . Let  $f_1, f_2 \in \mathcal{D}_R(C_{\mathbb{R}}(X))$ . Then we can extend  $f_j$  to an  $F_j \in C_{\mathbb{R}}(X \amalg Y) = C_{\mathbb{R}}(X) \oplus C_{\mathbb{R}}(Y)$  such that  $L_{\rho}(F_j) = L_{\rho_X}(f_j)$  and  $\|F_j\|_{\infty} = \|f_j\|$ , where  $\|\cdot\|_{\infty}$  is the supremum-norm on  $C_{\mathbb{C}}(X \amalg Y)$ . For example, let  $F_j(w) = \min(\|f_j\|, \inf_{x \in X}(f_j(x) + L_{\rho_X}(f_j)\rho(x, w)))$  for all  $w \in X \amalg Y$ . Say  $F_j = (f_j, g_j)$  with  $g_j \in C_{\mathbb{R}}(Y)$ . Then  $L_{\rho_Y}(g_j) \leq L_{\rho}(F_j) \leq 1$  and  $\|g_j\| \leq \|F_j\|_{\infty} = \|f_j\| \leq R$ . So  $g_j \in \mathcal{D}_R(C_{\mathbb{R}}(Y))$ . Then

$$\begin{aligned} \|f_j - g_j\| &\leq L_{\rho}(F_j)d \leq d, \\ \|f_1 f_2 - g_1 g_2\| &\leq \|f_1 - g_1\| \cdot \|f_2\| + \|g_1\| \cdot \|f_2 - g_2\| \leq 2dR. \end{aligned}$$

Similarly, we can deal with pairs in  $\mathcal{D}_R(C_{\mathbb{R}}(Y))$ . So  $\text{dist}_{\text{cq}}^R(C_{\mathbb{C}}(X), C_{\mathbb{C}}(Y)) \leq \max(d, 2dR) = d \max(1, 2R)$ . Letting  $\rho$  run over the admissible metrics on  $X \amalg Y$ , we get (11).  $\square$

*Proof of Theorem 3.16.* The inequality (11) and Theorem 3.15 tell us that this map is continuous. Composing with the forgetful map  $(C^*\text{CQM}, \text{dist}_{\text{cq}}) \rightarrow (\text{CQM}, \text{dist}_{\text{oq}})$  in Theorem 3.15 we get also a map  $(\text{CM}, \text{dist}_{\text{GH}}) \rightarrow (\text{CQM}, \text{dist}_{\text{oq}})$ . Denote  $(\text{CM}, \text{dist}_{\text{GH}})$  by  $W_1$ , and denote its images in  $(C^*\text{CQM}, \text{dist}_{\text{cq}})$  and  $(\text{CQM}, \text{dist}_{\text{oq}})$  by  $W_2$  and  $W_3$  respectively. Also denote the closure of  $W_2$  in  $(C^*\text{CQM}, \text{dist}_{\text{cq}})$  by  $\overline{W_2}$ . By [27, Theorem 1.1]  $\text{dist}_{\text{q}}$  and  $\text{dist}_{\text{oq}}$  are Lipschitz equivalent. So we may identify  $(\text{CQM}, \text{dist}_{\text{oq}})$  and  $(\text{CQM}, \text{dist}_{\text{q}})$ . As pointed out at the end of Section 2,  $W_3$  is closed in  $(\text{CQM}, \text{dist}_{\text{oq}})$ , and the map  $W_1 \rightarrow W_3$  is a homeomorphism. Hence under the forgetful map  $(C^*\text{CQM}, \text{dist}_{\text{cq}}) \rightarrow (\text{CQM}, \text{dist}_{\text{oq}})$  the image of  $\overline{W_2}$  is contained in  $W_3$ . Now the composition  $\overline{W_2} \rightarrow W_3 \rightarrow W_1 \rightarrow W_2$ , where  $W_3 \rightarrow W_1$  is the inverse of  $W_1 \rightarrow W_3$ , is continuous and restricts to the identity map on  $W_2$ . Thus  $\overline{W_2} = W_2$ , and the map  $W_1 \rightarrow W_2$  is a homeomorphism with inverse  $W_2 \rightarrow W_3 \rightarrow W_1$ .  $\square$

#### 4. $C^*$ -ALGEBRAIC QUANTUM GROMOV COMPACTNESS AND COMPLETENESS THEOREMS

In this section we prove the  $C^*$ -algebraic quantum Gromov compactness and completeness theorems. Unlike in Theorems 2.1 and Rieffel's quantum compactness

and compactness theorems [35, Theorems 12.11 and 13.5] , we have to put on some restriction. Just think about the completeness. Take a Cauchy sequence  $(\mathcal{A}_n, L_n)$  in  $(C^*\text{CQM}^R, \text{dist}_{\text{cq}}^R)$ . By (8) the sequence  $(A_n, L_n)$  is also Cauchy in  $(\text{CQM}^R, \text{dist}_{\text{cq}}^R)$ . By [27, Theorem 1.1] and Rieffel's quantum completeness theorem the space  $(\text{CQM}^R, \text{dist}_{\text{cq}}^R)$  is complete. Denote by  $(A, L)$  the limit of the sequence  $(A_n, L_n)$  in  $(\text{CQM}^R, \text{dist}_{\text{cq}}^R)$ . Then  $\mathcal{A} := \bar{A} + i\bar{A}$  should be the limit of  $(\mathcal{A}_n, L_n)$ , where  $\bar{A}$  is the completion of  $A$ . The trouble is how to get a  $C^*$ -algebraic structure on  $\mathcal{A}$ . For simplicity suppose that  $\mathcal{A}_n$  and  $\mathcal{A}$  are all contained in a normed space  $V$  such that  $\mathcal{D}_R(A_n)$  converges to  $\mathcal{D}_R(A)$  under  $\text{dist}_H^V$ . Then for any  $a, b \in \mathcal{D}_R(A)$  we can find  $a_n, b_n \in \mathcal{D}_R(A_n)$  such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$  as  $n \rightarrow \infty$ . Clearly  $a \cdot b$  should be defined as  $\lim_{n \rightarrow \infty} a_n b_n$ , if that limit exists. Notice that  $V$  is usually infinite-dimensional. The only way we can make sure that  $\{a_n b_n\}_{n \in \mathbb{N}}$  (or some subsequence of  $\{a_n b_n\}_{n \in \mathbb{N}}$ ) converges is that there exists a  $\lambda \in \mathbb{C}$  (not depending on  $n$ ) such that  $a_n b_n \in \lambda \mathcal{D}_R(A_n)$  for all  $n$ . Since  $\|a_n b_n\| \leq \|a_n\| \cdot \|b_n\| \leq R^2$ , this is equivalent to say that  $L_n(a_n b_n)$  is uniformly bounded. In other words,  $L_n(a_n b_n)$  should be controlled by  $L_n(a_n), L_n(b_n), \|a_n\|$ , and  $\|b_n\|$ . We use Kerr's definition [25, page 155]:

**Definition 4.1.** Let  $F : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$  be a continuous nondecreasing function, where  $\mathbb{R}_+^4$  is given the partial order  $(x_1, x_2, x_3, x_4) \leq (y_1, y_2, y_3, y_4)$  if and only if  $x_j \leq y_j$  for all  $j$ . We say that a  $C^*$ -algebraic compact quantum metric space  $(\mathcal{A}, L)$  satisfies the  $F$ -Leibniz property if

$$L(a \cdot b) \leq F(L(a), L(b), \|a\|, \|b\|)$$

for all  $a, b \in \mathcal{A}$ .

**Remark 4.2.** (1) Rescaling  $a$  and  $b$  such that  $\max(\|a\|, L(a)) = \max(\|b\|, L(b)) = 1$ , we see that any  $(\mathcal{A}, L)$  satisfying the  $F$ -Leibniz property for some  $F$  also satisfies that  $L(a \cdot b) \leq F(1, 1, 1, 1)(\|a\| + L(a))(\|b\| + L(b))$  for all  $a, b \in \mathcal{A}$ .

(2) If  $(\mathcal{A}, L)$  satisfies the  $F$ -Leibniz property for some  $F$ , then clearly  $\mathcal{A}_L := \{a \in \mathcal{A} : L(a) < \infty\}$  is a subalgebra of  $\mathcal{A}$ . Conversely, if  $(\mathcal{A}, L)$  is closed and  $\mathcal{A}_L$  is a subalgebra of  $\mathcal{A}$ , applying [15, Proposition 2.2] to the Banach space  $(\mathcal{A}_L, \|\cdot\|_L)$ , where  $\|\cdot\|_L = \|\cdot\| + L(\cdot)$ , we see that there is some constant  $C > 0$  such that  $\|ab\|_L \leq C \cdot \|a\|_L \cdot \|b\|_L$  for all  $a, b \in \mathcal{A}$  (note that the proof of [15, Proposition 2.2] does not need the subalgebra  $A^\infty$  to be closed under the holomorphic functional calculus).

(3) The function  $F(x_1, x_2, x_3, x_4) = x_1 x_4 + x_2 x_3$  corresponds to the Leibniz rule

$$(12) \quad L(a \cdot b) \leq L(a)\|b\| + \|a\|L(b).$$

The basic examples of  $C^*$ -algebraic compact quantum metric spaces such as  $\theta$ -deformations [28], quantum metric spaces induced by ergodic actions [32], and quantum metric spaces associated to ordinary compact metric spaces (see the discussion at the end of Section 2) all satisfy (12). But examples failing (12) also rise naturally. A nonempty closed subset of a compact metric space is still such a space. The quantum analogue also holds: a nonzero quotient  $\mathcal{B}$  of a  $C^*$ -algebraic quantum compact metric space  $(\mathcal{A}, L)$  is still such a space (see [35, Proposition 3.1] for the corresponding assertion for compact quantum metric spaces). But as the next lemma indicates, even when  $(\mathcal{A}, L)$  satisfies (12), we do not know whether  $\mathcal{B}$  satisfies it or not. See [3] for similar results.

**Lemma 4.3.** Let  $(\mathcal{A}, L_{\mathcal{A}})$  be a  $C^*$ -algebraic compact quantum metric space. For any unital  $C^*$ -algebra  $\mathcal{B}$  and surjective unital positive map  $\pi : \mathcal{A} \rightarrow \mathcal{B}$  let  $L_{\mathcal{B}}$  be the quotient seminorm on  $\mathcal{B}$ :

$$L_{\mathcal{B}}(b) := \inf\{L_{\mathcal{A}}(a) : \pi(a) = b\}.$$

Then  $(\mathcal{B}, L_{\mathcal{B}})$  is a  $C^*$ -algebraic compact quantum metric space. If  $(\mathcal{A}, L_{\mathcal{A}})$  is closed, so is  $(\mathcal{B}, L_{\mathcal{B}})$ . Denote by  $r$  the radius of  $(\mathcal{A}, L_{\mathcal{A}})$ . If  $\pi$  is a  $*$ -homomorphism (*i.e.*  $\mathcal{B}$  is a quotient of  $\mathcal{A}$ ) and  $(\mathcal{A}, L_{\mathcal{A}})$  satisfies the  $F$ -Leibniz property, then  $(\mathcal{B}, L_{\mathcal{B}})$  satisfies the  $F_r$ -Leibniz property, where  $F_r(x_1, x_2, x_3, x_4) = F(x_1, x_2, 2x_3 + 4rx_1, 2x_4 + 4rx_2)$ .

*Proof.* Clearly  $L_{\mathcal{B}}$  satisfies the reality condition (3), and  $L_{\mathcal{B}}|_{\mathcal{B}_{\text{sa}}}$  is the quotient of  $L_{\mathcal{A}}|_{\mathcal{A}_{\text{sa}}}$ . Since the subspace of  $\mathcal{A}_{\text{sa}}$  with finite  $L_{\mathcal{A}}$  is a compact quantum metric space and is dense in  $\mathcal{A}_{\text{sa}}$ , by [35, Proposition 3.1] the subspace of  $\mathcal{B}_{\text{sa}}$  with finite  $L_{\mathcal{B}}$  is a compact quantum metric space and is dense in  $\mathcal{B}_{\text{sa}}$ . Thus  $(\mathcal{B}, L_{\mathcal{B}})$  is a  $C^*$ -algebraic compact quantum metric space. When  $(\mathcal{A}, L_{\mathcal{A}})$  is closed, an argument similar to that in the proof of [35, Proposition 3.3] shows that  $(\mathcal{B}, L_{\mathcal{B}})$  is also closed.

Now suppose that  $\pi$  is a  $*$ -homomorphism and that  $(\mathcal{A}, L_{\mathcal{A}})$  satisfies the  $F$ -Leibniz property. For any  $b_1, b_2 \in \mathcal{B}$  and  $\varepsilon > 0$ , take  $a_1, a_2 \in \mathcal{A}$  such that  $\pi(a_j) = b_j$  and  $L_{\mathcal{A}}(a_j) \leq L_{\mathcal{B}}(b_j) + \varepsilon$ . Denote by  $(a_j)_{\text{sa}}$  and  $(a_j)_{\text{im}}$  the self-adjoint part and the imaginary part of  $a_j$  respectively. From (3) we have  $L_{\mathcal{A}}((a_j)_{\text{sa}}), L_{\mathcal{A}}((a_j)_{\text{im}}) \leq L_{\mathcal{A}}(a_j)$ . By [35, Lemma 3.4] we get  $\|(a_j)_{\text{sa}}\|, \|(a_j)_{\text{im}}\| \leq \|b_j\| + 2rL_{\mathcal{A}}(a_j)$ . So  $\|a_j\| \leq 2\|b_j\| + 4r(L_{\mathcal{B}}(b_j) + \varepsilon)$ . Notice that  $\pi(a_1 a_2) = b_1 b_2$ . Therefore

$$\begin{aligned} & L_{\mathcal{B}}(b_1 b_2) \\ & \leq L_{\mathcal{A}}(a_1 a_2) \leq F(L_{\mathcal{A}}(a_1), L_{\mathcal{A}}(a_2), \|a_1\|, \|a_2\|) \\ & \leq F(L_{\mathcal{B}}(b_1) + \varepsilon, L_{\mathcal{B}}(b_2) + \varepsilon, 2\|b_1\| + 4r(L_{\mathcal{B}}(b_1) + \varepsilon), 2\|b_2\| + 4r(L_{\mathcal{B}}(b_2) + \varepsilon)). \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we get

$$L_{\mathcal{B}}(b_1 b_2) \leq F_r(L_{\mathcal{B}}(b_1), L_{\mathcal{B}}(b_2), \|b_1\|, \|b_2\|).$$

□

Clearly if  $(\mathcal{A}, L)$  satisfies the  $F$ -Leibniz property, then so does its closure. Denote by  $C^*\text{CQM}_F$  the set of all isometry classes of  $C^*$ -algebraic compact quantum metric spaces satisfying the  $F$ -Leibniz property.

**Theorem 4.4** ( $C^*$ -algebraic Quantum Gromov Completeness and Compactness Theorems). Let  $F : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$  be a continuous nondecreasing function. Then  $(C^*\text{CQM}_F, \text{dist}_{\text{cq}})$  is a complete metric space. A subset  $\mathcal{S} \subseteq C^*\text{CQM}_F$  is totally bounded if and only if

- (1) there is a constant  $D$  such that  $\text{diam}(\mathcal{A}, L) \leq D$  for all  $(\mathcal{A}, L) \in \mathcal{S}$ ;
- (2) For any  $\varepsilon > 0$ , there exists a constant  $K_{\varepsilon} > 0$  such that  $\text{Cov}(\mathcal{A}, \varepsilon) \leq K_{\varepsilon}$  for all  $(\mathcal{A}, L) \in \mathcal{S}$ ,

and if and only if

- (1') there is a constant  $D'$  such that  $\text{diam}(\mathcal{D}(\mathcal{A})) \leq D'$  for all  $(\mathcal{A}, L) \in \mathcal{S}$ ;
- (2') For any  $\varepsilon > 0$ , there exists a constant  $K'_{\varepsilon} > 0$  such that  $\text{Cov}(\mathcal{D}(\mathcal{A}), \varepsilon) \leq K'_{\varepsilon}$  for all  $(\mathcal{A}, L) \in \mathcal{S}$ .

**Notation 4.5.** Let  $R \geq 0$ . For any compact metric space  $(X, \rho)$  let  $C(X)_R := \{f \in C_{\mathbb{C}}(X) : L_{\rho}(X) \leq 1, \|f\| \leq R\}$ , equipped with the metric induced from the supremum-norm in  $C_{\mathbb{C}}(X)$ , where  $L_{\rho}$  is the Lipschitz seminorm defined by (1).

**Lemma 4.6.** Let  $\mathcal{S}$  be a subset of C\*QCM satisfying the conditions (1) and (2) in Theorem 4.4. Let  $R \geq \sup\{r_{\mathcal{A}} : (\mathcal{A}, L) \in \mathcal{S}\}$ . Then there exist a complex Banach space  $V$  and a compact convex subset  $Z \subseteq V$  such that for every  $(\mathcal{A}, L) \in \mathcal{S}$  there is a linear isometric embedding  $h_{\mathcal{A}} : \mathcal{A} \hookrightarrow V$  with  $h_{\mathcal{A}}(\mathcal{D}_R(\mathcal{A})) \subseteq Z$ .

*Proof.* We may assume that every  $(\mathcal{A}, L)$  is closed, and that  $R > 0$ . By [27, Lemma 5.4] the set  $\{\mathcal{D}_R(\mathcal{A}) : (\mathcal{A}, L) \in \mathcal{S}\}$  satisfies the condition (2) in Theorem 2.1. Let  $(\mathcal{A}, L) \in \mathcal{S}$ . Denote by  $\mathbb{D}(\mathcal{A}')$  the unit ball of  $\mathcal{A}'$ . Then  $\mathbb{C}\mathcal{D}_R(\mathcal{A}) = A + iA$  is dense in  $\mathcal{A}$ , and hence the natural map  $\psi_{\mathcal{A}} : \mathbb{D}(\mathcal{A}') \rightarrow C_{\mathbb{C}}(\mathcal{D}_R(\mathcal{A}))$  induced by the pairing between  $\mathcal{A}$  and  $\mathcal{A}'$  is injective. Clearly  $\psi_{\mathcal{A}}$  maps  $\mathbb{D}(\mathcal{A}')$  into  $C(\mathcal{D}_R(\mathcal{A}))_R$  (see Notation 4.5). Endow  $\mathbb{D}(\mathcal{A}')$  with the metric  $\rho$  induced by  $\psi_{\mathcal{A}}$  and the supremum-norm in  $C_{\mathbb{C}}(\mathcal{D}_R(\mathcal{A}))$ . Then [27, Corollary 5.3, Lemma 5.1] tell us that the set  $\{\mathbb{D}(\mathcal{A}') : (\mathcal{A}, L) \in \mathcal{S}\}$  satisfies the condition (2) in Theorem 2.1. By [27, Proposition 5.2] we can find a complex Banach space  $V$  and a compact convex subset  $Z \subseteq V$  such that for every  $(\mathcal{A}, L) \in \mathcal{S}$  there is a linear isometric embedding  $\phi_{\mathcal{A}} : C_{\mathbb{C}}(\mathbb{D}(\mathcal{A}')) \hookrightarrow V$  with the image of  $C(\mathbb{D}(\mathcal{A}'))_R$  contained in  $Z$ . Notice that  $\rho(f, g) = \sup\{|f(a) - g(a)| : a \in \mathcal{D}_R(\mathcal{A})\}$  for any  $f, g \in \mathbb{D}(\mathcal{A}')$ . It is easy to see that the topology defined by  $\rho$  on  $\mathbb{D}(\mathcal{A}')$  is exactly the weak-\* topology. Then the natural pairing between  $\mathcal{A}$  and  $\mathcal{A}'$  also gives a linear isometric embedding  $\varphi_{\mathcal{A}} : \mathcal{A} \hookrightarrow C_{\mathbb{C}}(\mathbb{D}(\mathcal{A}'))$ . Clearly  $\varphi_{\mathcal{A}}$  maps  $\mathcal{D}_R(\mathcal{A})$  into  $C(\mathbb{D}(\mathcal{A}'))_R$ . Thus we may just set  $h_{\mathcal{A}} = \phi_{\mathcal{A}} \circ \varphi_{\mathcal{A}}$ .  $\square$

*Proof of Theorem 4.4.* We prove the compactness part first. The equivalence between (1)+(2) and (1')+(2') follows from [27, Lemma 5.4]. Suppose that  $\mathcal{S}$  is totally bounded. By (4) the sets  $\{r_{\mathcal{A}} : (\mathcal{A}, L) \in \mathcal{S}\}$  and  $\{\mathcal{D}(\mathcal{A}) : (\mathcal{A}, L) \in \mathcal{S}\}$  are totally bounded in  $\mathbb{R}$  and  $(\text{CM}, \text{dist}_{\text{GH}})$  respectively. Then the condition (1) follows immediately. Also the condition (2) follows from Theorem 2.1.

Now suppose that  $\mathcal{S}$  satisfies the conditions (1) and (2). We shall show that  $\mathcal{S}$  is pre-compact, which implies that  $\mathcal{S}$  is totally bounded. Set  $R = 1 + \sup\{r_{\mathcal{A}} : (\mathcal{A}, L) \in \mathcal{S}\}$ . By Lemma 4.6 we can find a complex Banach space  $V$  and a compact convex subset  $Z_1 \subseteq V$  such that  $\mathcal{A}$  is a linear subspace of  $V$  with  $\mathcal{D}_R(\mathcal{A}) \subseteq Z_1$  for all  $(\mathcal{A}, L) \in \mathcal{S}$ . Set  $R' = \max(R, F(1, 1, R, R))$ , and set  $Z = R'Z_1 + iR'Z_1$ . Then  $Z \supseteq Z_1$ . Let  $(\mathcal{A}, L) \in \mathcal{S}$ , and let  $a, b \in \mathcal{D}_R(\mathcal{A})$ . Then  $L(ab) \leq F(L(a), L(b), \|a\|, \|b\|) \leq F(1, 1, R, R) \leq R'$ . Also  $\|ab\| \leq R^2 \leq R \cdot R'$ . Thus the self-adjoint and imaginary parts of  $ab$  are both in  $R'\mathcal{D}_R(\mathcal{A}) \subseteq R'Z_1$ . Then  $ab \in Z$ . Therefore  $\mathcal{D}_R(\mathcal{A})^m$  is contained in  $Z^{(3)} := Z \oplus Z \oplus Z \subseteq V^{(3)}$ . Notice that  $Z^{(3)}$  is compact. Then the set of nonempty closed subsets of  $Z^{(3)}$  is compact equipped with the Hausdorff distance  $\text{dist}_{\mathbb{H}}^{Z^{(3)}}$ . Thus we can find a sequence  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  in  $\mathcal{S}$  such that  $(\mathcal{D}_R(\mathcal{A}_n))^m$  converges under  $\text{dist}_{\mathbb{H}}^{V^{(3)}}$ . Denote by  $Y$  the limit. Set  $\pi_j : V^{(3)} \rightarrow V$  to be the projection of  $V^{(3)}$  to the  $j$ -th coordinate. Also set  $\pi_{12} : V^{(3)} \rightarrow V^{(2)}$  to be the projection of  $V^{(3)}$  to the first two coordinates. Clearly  $\text{dist}_{\mathbb{H}}^V(\mathcal{D}_R(\mathcal{A}_n), \pi_j(Y)) = \text{dist}_{\mathbb{H}}^V(\pi_j(\mathcal{D}_R(\mathcal{A}_n)^m), \pi_j(Y)) \leq \text{dist}_{\mathbb{H}}^{V^{(3)}}(\mathcal{D}_R(\mathcal{A}_n)^m, Y)$  for  $j = 1, 2$ . Thus  $\mathcal{D}_R(\mathcal{A}_n)$  converges to  $\pi_1(Y)$  and  $\pi_2(Y)$  under  $\text{dist}_{\mathbb{H}}^V$ . Set  $X = \pi_1(Y) = \pi_2(Y)$ . Similarly,  $(\mathcal{D}_R(\mathcal{A}_n))^{(2)} = \pi_{12}((\mathcal{D}_R(\mathcal{A}_n))^m)$  converges to  $\pi_{12}(Y)$  under  $\text{dist}_{\mathbb{H}}^{V^{(2)}}$ . But clearly  $(\mathcal{D}_R(\mathcal{A}_n))^{(2)}$  converges to  $X^{(2)}$  under  $\text{dist}_{\mathbb{H}}^{V^{(2)}}$ . Thus  $\pi_{12}(Y) = X^{(2)}$ .

Since each  $\mathcal{D}_R(\mathcal{A}_n)$  is  $\mathbb{R}$ -balanced (i.e.  $\lambda a \in \mathcal{D}_R(\mathcal{A}_n)$  for all  $a \in \mathcal{D}_R(\mathcal{A}_n)$  and  $\lambda \in \mathbb{R}$  with  $|\lambda| \leq 1$ ) and convex, has radius  $R$  and contains  $0_V$ , clearly so does  $X$ . Thus the set  $\mathbb{R}_+ \cdot X = \{\lambda x : \lambda \in \mathbb{R}_+, x \in X\}$  is a real linear subspace of  $V$ . Denote it by  $B$ , and denote the closure of  $B + iB$  by  $\mathcal{B}$ . Then  $\mathcal{B}$  is a closed complex linear

subspace of  $V$ . We shall define a  $C^*$ -algebra structure and a Lip-norm on  $\mathcal{B}$ . It is easy to see that for every  $a \in \mathcal{B}$  we can find  $a_n \in \mathcal{A}_n$  for each  $n$  such that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

**Lemma 4.7.** Let  $a, b \in \mathcal{B}$ . Let  $a_n, b_n \in \mathcal{A}_n$  for each  $n$  such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$  as  $n \rightarrow \infty$ . Then the sequence  $\{a_n b_n\}_{n \in \mathbb{N}}$  converges to an element in  $\mathcal{B}$ , and the limit depends only on  $a$  and  $b$ .

*Proof.* We show first that for any  $x, y \in B + iB$  there exist  $x_n, y_n \in \mathcal{A}_n$  for each  $n$  such that  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ , and  $\{x_n y_n\}_{n \in \mathbb{N}}$  converges to an element in  $\mathcal{B}$ . Since  $B = \mathbb{R}_+ \cdot X$ , it suffices to show this for all  $x, y \in X$ . Let  $x, y \in X$ . Since  $\pi_{12}(Y) = X^{(2)}$ , we can find  $z \in V$  with  $(x, y, z) \in Y$ . Then we can pick  $(x_n, y_n, x_n y_n) \in (\mathcal{D}_R(A_n))^m$  for each  $n$  such that  $(x_n, y_n, x_n y_n) \rightarrow (x, y, z)$  in  $V^{(3)}$  as  $n \rightarrow \infty$ . Notice that  $x_n y_n \in R' \mathcal{D}_R(A_n) + iR' \mathcal{D}_R(A_n)$ . Thus  $\lim_{n \rightarrow \infty} (x_n y_n)$  is in  $R'X + iR'X \subseteq \mathcal{B}$ .

Now let  $a$  and  $b$  be arbitrary elements in  $\mathcal{B}$ , and let  $a_n, b_n \in \mathcal{A}_n$  with  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Let  $\varepsilon > 0$ . Pick  $x, y \in B + iB$  with  $\|a - x\|, \|b - y\| < \varepsilon$ . Let  $x_n$  and  $y_n$  be as in the above. Take  $N_1$  such that  $\|a_n - a\|, \|b_n - b\|, \|x_n - x\|, \|y_n - y\| < \varepsilon$  for all  $n > N_1$ . Since  $\|a_n - x_n\| \rightarrow \|a - x\|$  and  $\|b_n - y_n\| \rightarrow \|b - y\|$ , there exists  $N_2 > N_1$  such that  $\|a_n - x_n\|, \|b_n - y_n\| < \varepsilon$  for all  $n > N_2$ . Then

$$\begin{aligned} \|a_n b_n - x_n y_n\| &\leq \|a_n - x_n\| \cdot \|b_n\| + \|x_n\| \cdot \|b_n - y_n\| \\ &\leq \varepsilon(\|b\| + \varepsilon) + (\|x\| + \varepsilon)\varepsilon \leq \varepsilon(\|b\| + \|a\| + 3\varepsilon) \end{aligned}$$

for all  $n > N_2$ . Since  $\{x_n y_n\}_{n \in \mathbb{N}}$  converges, we can find  $N_3 > N_2$  such that  $\|x_n y_n - x_k y_k\| \leq \varepsilon$  for all  $n, k > N_3$ . Then  $\|a_n b_n - a_k b_k\| \leq \varepsilon + 2\varepsilon(\|b\| + \|a\| + 3\varepsilon)$  for all  $n, k > N_3$ . Thus  $\{a_n b_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence, and hence converges. We also get that  $\|\lim_{n \rightarrow \infty} (a_n b_n) - \lim_{n \rightarrow \infty} (x_n y_n)\| \leq \varepsilon(\|b\| + \|a\| + 3\varepsilon)$ . Since  $\lim_{n \rightarrow \infty} (x_n y_n)$  does not depend on the choice of the sequences  $\{a_n\}_{n \in \mathbb{N}}$  and  $\{b_n\}_{n \in \mathbb{N}}$ , neither does  $\lim_{n \rightarrow \infty} (a_n b_n)$ . The above inequality also shows that  $\lim_{n \rightarrow \infty} (a_n b_n)$  is in  $\mathcal{B}$ .  $\square$

For any  $a, b \in \mathcal{B}$  denote by  $a \cdot b$  the limit  $\lim_{n \rightarrow \infty} (a_n b_n)$  in Lemma 4.7. Then clearly this makes  $\mathcal{B}$  into an algebra over  $\mathbb{C}$ . Also notice that

$$\|a \cdot b\| = \left\| \lim_{n \rightarrow \infty} (a_n b_n) \right\| \leq \lim_{n \rightarrow \infty} \|a_n\| \cdot \|b_n\| = \|a\| \cdot \|b\|.$$

Thus  $\mathcal{B}$  is a Banach algebra. It also follows from the proof of Lemma 4.7 that  $Y = X^m$ . By Lemma 3.11 the sequence  $\{e_{\mathcal{A}_n}\}_{n \in \mathbb{N}}$  converges. Denote by  $e_{\mathcal{B}}$  the limit. Clearly  $e_{\mathcal{B}}$  is an identity of  $\mathcal{B}$ .

For any  $a_n \in \mathcal{A}_n$  with  $a_n \rightarrow a \in \mathcal{B}$ , by a similar argument as in Lemma 4.7 we see that  $\{a_n^*\}_{n \in \mathbb{N}}$  converges to an element in  $\mathcal{B}$ . Denote the limit by  $a^*$ . Then this makes  $\mathcal{B}$  into a  $*$ -algebra. Clearly  $\bar{\mathcal{B}} = (\mathcal{B})_{\text{sa}}$ . Notice that

$$\|a^* a\| = \left\| \lim_{n \rightarrow \infty} a_n^* a_n \right\| = \lim_{n \rightarrow \infty} \|a_n^* a_n\| = \lim_{n \rightarrow \infty} \|a_n\|^2 = \|a\|^2.$$

Hence  $\mathcal{B}$  is  $C^*$ -algebra.

We proceed to define a Lip-norm  $L$  on  $\mathcal{B}$ . As hinted by [27, Lemma 4.1], we describe  $L|_{\mathcal{B}}$  first. For this we need:

**Lemma 4.8.** We have  $X = \{b \in (X + \mathbb{R}e_{\mathcal{B}}) : \|b\| \leq R\}$ .

*Proof.* Clearly we have  $X \subseteq \{b \in (X + \mathbb{R}e_{\mathcal{B}}) : \|b\| \leq R\}$ . Let  $x \in X$  and  $\lambda \in \mathbb{R}$  with  $\|x + \lambda e_{\mathcal{B}}\| \leq R$ . Pick  $x_n \in \mathcal{D}_R(A_n)$  for each  $n \in \mathbb{N}$  such that  $x_n \rightarrow x$ . Then  $x_n + \lambda e_{\mathcal{A}_n} \rightarrow x + \lambda e_{\mathcal{B}}$ , and hence  $\lim_{n \rightarrow \infty} \|x_n + \lambda e_{\mathcal{A}_n}\| \leq R$ . Set  $t_n = \max(R, \|x_n + \lambda e_{\mathcal{A}_n}\|)$ , and set  $y_n = \frac{R}{t_n}(x_n + \lambda e_{\mathcal{A}_n})$ . Then  $y_n \in \mathcal{D}_R(A_n)$ , and  $y_n \rightarrow x + \lambda e_{\mathcal{B}}$ . Thus  $x + \lambda e_{\mathcal{B}} \in X$ .  $\square$

By [27, Lemma 4.1] there is a unique closed Lip-norm  $L_*$  on  $B$  satisfying  $\mathcal{D}_R(B) = X$ . And  $(B, L_*)$  has radius no bigger than  $R$ .

Now we describe the whole  $L$ . As hinted by [27, Proposition 6.9], we have:

**Lemma 4.9.** Define a (possibly  $+\infty$ -valued) seminorm,  $L$ , on  $\mathcal{B}$  by

$$(13) \quad L(a) := \inf\{\limsup_{n \rightarrow \infty} L_n(a_n) : a_n \in \mathcal{A}_n \text{ for all } n, \text{ and } a_n \rightarrow a\}.$$

Then  $(\mathcal{B}, L)$  is  $C^*$ -algebraic compact quantum metric space with radius no bigger than  $R$  and satisfying the  $F$ -Leibniz property. Also  $L|_B = L_*$ .

*Proof.* Clearly  $L$  is a seminorm satisfying the reality condition (3). Also clearly  $L(x) \leq 1$  for every  $x \in X$ , and  $L(e_B) = 0$ . Thus  $\mathbb{R}e_B + X \subseteq \{a \in \mathcal{B} : L(a) \leq 1\}$ . It is easy to see that the infimum in the definition of  $L$  can always be achieved. Let  $a \in \mathcal{B}_{\text{sa}}$  with  $L(a) \leq 1$ . Then there exist  $a_n \in \mathcal{A}_n$  such that  $a_n \rightarrow a$  and  $\limsup_{n \rightarrow \infty} L_n(a_n) = L(a) \leq 1$ . Replacing  $a_n$  by the self-adjoint part of  $a_n$ , we may assume that  $a_n \in (\mathcal{A}_n)_{\text{sa}}$ . Rescaling  $a_n$  slightly, we may also assume that  $L_n(a_n) \leq L(a)$  and  $\|a_n\| \leq \|a\|$  for all  $n \in \mathbb{N}$ . By [27, Lemma 4.1] we can write  $a_n$  as  $x_n + \lambda_n e_{\mathcal{A}_n}$  for some  $x_n \in \mathcal{D}_R(\mathcal{A}_n)$  and  $\lambda_n \in \mathbb{R}$ . Then  $|\lambda_n| \leq \|a_n\| + \|x_n\| \leq \|a\| + R$ . Thus some subsequence  $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$  converges to a  $\lambda \in \mathbb{R}$ . Since  $\{x_n + \lambda_n e_{\mathcal{A}_n}\}_{n \in \mathbb{N}}$  converges, the subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  also converges. Denote by  $x$  the limit. Then  $x \in X$  and  $a = x + \lambda e_B$ . Therefore  $\mathbb{R}e_B + X = \{a \in \mathcal{B}_{\text{sa}} : L(a) \leq 1\}$ . By [27, Lemma 4.1] we have  $L|_B = L_*$ . Hence  $(\mathcal{B}, L)$  is  $C^*$ -algebraic compact quantum metric space with radius no bigger than  $R$ . Since each  $(\mathcal{A}_n, L_n)$  satisfies the  $F$ -Leibniz property, clearly so does  $(\mathcal{B}, L)$ .  $\square$

Back to the proof of Theorem 4.4. We have seen that  $\mathcal{D}_R(\mathcal{A}_n)^m$  converges to  $Y = X^m = \mathcal{D}_R(B)^m$  under  $\text{dist}_{\mathbb{H}}^{V(3)}$ . Thus

$$\text{dist}_{\text{cq}}^R(\mathcal{A}_n, \mathcal{B}) \leq \text{dist}_{\mathbb{H}}^{V(3)}(\mathcal{D}_R(\mathcal{A}_n)^m, X^m) \rightarrow 0.$$

By Theorem 3.15 we get  $\text{dist}_{\text{cq}}(\mathcal{A}_n, \mathcal{B}) \rightarrow 0$ . This shows that  $\mathcal{S}$  is pre-compact.

We are left to show that  $(C^*\text{CQM}_F, \text{dist}_{\text{cq}})$  is complete. Let  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $(C^*\text{CQM}_F, \text{dist}_{\text{cq}})$ . From (4) and Theorem 2.1 one sees that  $\mathcal{S} := \{(\mathcal{A}_n, L_n) : n \in \mathbb{N}\}$  satisfies the conditions (1') and (2'). Thus  $\mathcal{S}$  is pre-compact. Then some subsequence of  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  converges under  $\text{dist}_{\text{cq}}$ . Since  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  is Cauchy, it converges itself. This finishes the proof of Theorem 4.4.  $\square$

**Remark 4.10.** Similarly, one can give a direct proof of [27, Theorem 5.5]. One can also give a simple proof of Rieffel's quantum completeness and compactness theorems (as formulated in [27, Theorem 2.4]) along the same line: the proof for the completeness and the "only if" part is essentially the same as in the proof of Theorem 4.4. We indicate briefly how to show that a subset  $\mathcal{S} \subseteq \text{CQM}$  satisfying the conditions (1) and (2) is precompact: Pick  $R > \sup\{r_A : (A, L) \in \mathcal{S}\}$ . Then the natural pairing between  $A$  and  $A'$  gives an affine isometric embedding  $S(A) \hookrightarrow C(\mathcal{D}_R(A))_R$  (see Notation 4.5). By [27, Lemma 5.4, Proposition 5.2] we can find a normed space  $V$  and a compact convex subset  $Z$  such that  $Z \supseteq C(\mathcal{D}_R(A))_R \supseteq S(A)$  for all  $(A, L) \in \mathcal{S}$ . Since the set of convex closed subsets of  $Z$  is compact equipped with the Hausdorff distance  $\text{dist}_{\mathbb{H}}^Z$ , some sequence  $\{S(\mathcal{A}_n)\}_{n \in \mathbb{N}}$  converges to a convex subset  $X \subseteq Z$  under  $\text{dist}_{\mathbb{H}}^V$ . By [27, Proposition 3.1]  $X$  is the state-space of some

compact quantum metric space  $(B, L)$ . Then [27, Proposition 3.2] tells us that  $(A_n, L_n) \rightarrow (B, L)$  under  $\text{dist}_q$ .

**Remark 4.11.** In Definition 3.3 we defined our  $C^*$ -algebraic quantum distance as  $\inf\{\text{dist}_H^{V^{(3)}}(\mathcal{D}(A)^m, \mathcal{D}(B)^m)\}$ , where  $V$  runs over complex normed spaces containing  $\mathcal{A}$  and  $\mathcal{B}$ . From the reality condition (3) one can see easily that the ball  $\mathcal{D}(\mathcal{A})$  is also totally bounded. Then one can try to define a quantum distance,  $\text{dist}'_{\text{cq}}$ , as  $\inf\{\text{dist}_H^{V^{(3)}}(\mathcal{D}_{\text{diam}(\mathcal{A})}(\mathcal{A})^m, \mathcal{D}_{\text{diam}(\mathcal{B})}(\mathcal{B})^m)\}$ . One can show that  $\text{dist}'_{\text{cq}}$  is still a metric on the set of isometry classes of  $C^*$ -algebraic compact quantum metric spaces, though now the isometry in Definition 3.14 should be required to preserve the Lip-norm on the whole algebra  $\mathcal{A}$  instead of only on  $\mathcal{A}_{\text{sa}}$ . At first sight, this sounds more satisfactory. However, the map  $(X, \rho) \mapsto (C_{\mathbb{C}}(M), L_{\rho})$  turns out not to be continuous with respect to  $\text{dist}'_{\text{cq}}$ . This phenomenon is rooted in the different extension behavior of  $\mathbb{C}$ -valued Lipschitz functions and  $\mathbb{R}$ -valued Lipschitz functions on metric spaces: for any metric space  $(X, \rho)$  and a subspace  $Y \subseteq X$ , if  $f$  is an  $\mathbb{R}$ -valued Lipschitz function on  $Y$ , we can always extend  $f$  to a function  $\hat{f}$  (for example,  $\hat{f}(x) = \inf_{y \in Y}(f(y) + L(f)\rho(x, y))$  or  $\hat{f}(x) = \sup_{y \in Y}(f(y) - L(f)\rho(x, y))$ ) on  $X$  with the same Lipschitz seminorm. But when  $f$  is  $\mathbb{C}$ -valued, sometimes one has to increase the Lipschitz seminorm to extend  $f$ . The following example comes from [41]:

**Example 4.12.** [41, Example 1.5.7] Let  $X$  be a four-element set, say  $X = \{x, y_1, y_2, y_3\}$ , with  $\rho(x, y_i) = \frac{1}{2}$  for all  $i$  and  $\rho(y_j, y_k) = 1$  for distinct  $j, k$ . Let  $Y = X \setminus \{x\}$ , and let  $f : Y \rightarrow \mathbb{C}$  be an isometric map. Then  $L(f) = 1$  and every extension  $\hat{f}$  of  $f$  to  $X$  has Lipschitz seminorm at least  $\frac{2}{\sqrt{3}}$ .

The nice behavior of  $\mathbb{R}$ -valued functions with respect to Lipschitz seminorms is also reflected in [27, Proposition 6.11]. In fact, the analogue of [27, Proposition 6.11] for  $\mathbb{C}$ -valued functions holds for convergence under  $\text{dist}'_{\text{cq}}$  (at least for commutative case), but not for convergence under  $\text{dist}_{\text{GH}}$ . This explains the discontinuity of the map  $(X, \rho) \mapsto (C_{\mathbb{C}}(M), L_{\rho})$  with respect to  $\text{dist}'_{\text{cq}}$ . Let us give one example. Suppose that  $\{X_n\}_{n \in \mathbb{N}}$  is a sequence of closed subsets of a compact metric space  $Z$  and that  $X_n$  converges to some  $X_0 \subseteq Z$  under  $\text{dist}_H^Z$ . Then the restrictions of elements in  $C_{\mathbb{C}}(Z)$  generate a continuous field of  $C^*$ -algebras over  $T = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$  with fibres  $C_{\mathbb{C}}(X_n)$  at  $t = \frac{1}{n}$  and  $C_{\mathbb{C}}(X_0)$  at  $t = 0$ . Denote by  $\Gamma$  the space of continuous sections. Set  $L_n = L_{\rho_{X_n}}$ , and set  $L_0 = L_{\rho_{X_0}}$ . As in the proof of Theorem 4.4 embed all the  $C_{\mathbb{C}}(X_n)$ 's isometrically into a common normed space  $V$ . Notice that the proof of Theorem 4.4 can be easily modified to construct the limit  $(\mathcal{B}, L)$  of some subsequence of  $\{(C_{\mathbb{C}}(X_n), L_n)\}_{k \in \mathbb{N}}$  under  $\text{dist}'_{\text{cq}}$ . Then  $\mathcal{B}$  is a subspace of  $V$ , and  $L$  is still given by (13) of Lemma 4.9. Without loss of generality, we may assume that this subsequence is  $\{C_{\mathbb{C}}(X_n)\}_{n \in \mathbb{N}}$  itself. Take a dense sequence  $\{a_m\}_{m \in \mathbb{N}}$  in  $\mathcal{D}_R(C_{\mathbb{C}}(X_0))$ , and take  $f_m \in \Gamma$  with  $(f_m)_0 = a_m$ . By a standard diagonal argument we can find a subsequence of  $\{C_{\mathbb{C}}(X_n)\}_{n \in \mathbb{N}}$  such that the corresponding subsequence of  $\{(f_m)_{1/n}\}_{n \in \mathbb{N}}$  converges in  $V$  for every  $m$ . Without loss of generality, we still assume that this subsequence is  $\{C_{\mathbb{C}}(X_n)\}_{n \in \mathbb{N}}$  itself. Notice that the limit  $b_m = \lim_{n \rightarrow \infty} (f_m)_{1/n}$  is in  $\mathcal{B}$ , and that the map  $a_m \mapsto b_m$  extends to an isometric affine map from  $\mathcal{D}_R(C_{\mathbb{C}}(X_0))$  into  $\mathcal{B}$ . Then one can verify that this map extends to a  $*$ -isomorphism from  $C_{\mathbb{C}}(X_0)$  onto  $\mathcal{B}$ . Identify  $C_{\mathbb{C}}(X_0)$  with  $\mathcal{B}$ . Then one verifies that elements in  $\Gamma$  are exactly continuous

maps  $\psi : T \rightarrow V$  with  $\psi(\frac{1}{n}) \in C_{\mathbb{C}}(X_n)$  and  $\psi(0) \in C_{\mathbb{C}}(X_0)$ . Thus (13) becomes  $L(a) = \inf\{\limsup_{n \rightarrow \infty} L_n(f_{\frac{1}{n}}) : f \in \Gamma \text{ and } f_0 = a\}$ , which is the analogue of [27, Proposition 6.11]. As pointed out in the proof of Lemma 4.9, the infimum here can always be achieved by some  $f \in \Gamma$ .

I am indebted to Nik Weaver showing the next example [42]. Note that this example modifies Example 4.12, and shows that the limit Lip-norm  $L$  so constructed does not coincide with  $L_0$  on the whole of  $C_{\mathbb{C}}(X_0)$ . Consequently, the map  $(X, \rho) \mapsto (C_{\mathbb{C}}(M), L_{\rho})$  is not continuous with respect to  $\text{dist}'_{\text{cq}}$ .

**Example 4.13.** Let  $X_0$  be a solid equilateral triangle in  $\mathbb{C}$  with sides of length 1 and center at the origin. The metric  $\rho_{X_0}$  is just the Euclidean metric on  $\mathbb{C}$ . We define  $X_n$  as follows: subdivide  $X_0$  into  $4^n$  congruent equilateral sub-triangles with sides of length  $1/2^n$ . Let  $V_n$  be the vertices of these sub-triangles, and let  $B_n$  be their barycenters. Then let  $X_n$  be  $V_n \cup B_n$ . For each of the sub-triangles put edges of length  $1/2^n$  between its vertices and put edges of length  $1/2^{n+1}$  between each vertex and the barycenter. Now extend the distance on  $V_n$  to a distance on  $X_n$  via setting distances between points in  $B_n$  and points in  $X_n$  to be the graph distance, *i.e.*  $d(x, y) = \text{length of the shortest path between } x \text{ and } y$  for any  $x \in B_n$  and any  $y \in X_n$ . We construct a metric space  $Z$  containing all these  $X_n$ 's and  $X_0$  such that  $X_n \rightarrow X_0$  under  $\text{dist}_{\mathbb{H}}^Z$ :

**Lemma 4.14.** There exists a compact metric space  $(Z, \rho)$  with isometric embeddings  $\varphi_n : X_n \hookrightarrow Z$  for all  $n \in \mathbb{N}$  and  $\varphi_0 : X_0 \hookrightarrow Z$  such that  $\varphi_m(p) = \varphi_0(p)$  for each  $p \in V_n$  and all  $m \geq n$ .

*Proof.* Notice that for any metric space  $X$  and  $Y$  with a common closed subspace  $W$ , we can “glue”  $X$  and  $Y$  together along  $W$  by setting  $\rho(x, y) = \inf_{w \in W} (\rho_X(x, w) + \rho_Y(w, y))$  for all  $x \in X, y \in Y$ . Gluing  $X_0$  and  $X_1$  together along  $V_1$ , we get a metric on the disjoint union  $X_0 \coprod B_1$ . Gluing  $X_0 \coprod B_1$  and  $X_2$  together along  $V_2$  we get a metric on  $X_0 \coprod B_1 \coprod B_2$ . In this way, we get a metric on  $X_0 \coprod (\coprod_{n \in \mathbb{N}} B_n)$ . Clearly  $X_0 \coprod (\coprod_{n \in \mathbb{N}} B_n)$  is totally bounded. So the completion of  $X_0 \coprod (\coprod_{n \in \mathbb{N}} B_n)$ , denoted by  $Z$ , is compact. The natural embeddings  $\varphi_n : X_n \hookrightarrow Z$  and  $\varphi_0 : X_0 \hookrightarrow Z$  satisfy our requirement.  $\square$

Identify  $X_n$  and  $X_0$  with their images in  $Z$ . Then

$$\begin{aligned} \text{dist}_{\mathbb{H}}^Z(X_0, X_n) &\leq \text{dist}_{\mathbb{H}}^Z(X_0, V_n) + \text{dist}_{\mathbb{H}}^Z(V_n, X_n) \\ &\leq \text{dist}_{\mathbb{H}}^{X_0}(X_0, V_n) + \text{dist}_{\mathbb{H}}^{X_n}(V_n, X_n) \rightarrow 0. \end{aligned}$$

Let  $a$  be the identity map from  $X_0$  into  $\mathbb{C}$ . Then we have:

**Proposition 4.15.** For any  $a_n : X_n \rightarrow \mathbb{C}$  with  $L_n(a_n) \leq 1$ , we have  $\|a_n|_{V_n} - a|_{V_n}\| \geq \frac{\sqrt{3}}{6} - \frac{1}{4}$ .

*Proof.* Suppose that  $\|a_n|_{V_n} - a|_{V_n}\| < \frac{\sqrt{3}}{6} - \frac{1}{4}$ . In particular, for any  $x$  in  $V_n$  that lies on the boundary of  $X_0$ , we have  $|a_n(x) - a(x)| < \frac{\sqrt{3}}{6} - \frac{1}{4}$ . It follows by some elementary geometry that  $\frac{\sqrt{3}}{2}X_0$  is strictly contained in the closed curve formed by joining the points  $a_n(x)$  (for  $x$  in  $V_n \cap \partial X_0$ , where  $\partial X_0$  is the boundary of  $X_0$ ) with line segments.

Extend  $a_n|_{V_n}$  to  $X_0$  by affine extension on sub-triangles. Call the extension  $g$ . Then  $g$  takes the boundary of  $X_0$  to a curve in  $\mathbb{C}$  which contains  $\frac{\sqrt{3}}{2}X_0$ . Since the

boundary of  $X_0$  is contractible in  $X_0$ , it follows that  $g(X_0)$  contains  $\frac{\sqrt{3}}{2}X_0$ , and hence that the area of  $g(X_0)$  is strictly greater than  $\frac{3}{4}$  times the area of  $X_0$ .

However, for any sub-triangle  $S$  with barycenter  $y$ , if  $L_n(a_n) \leq 1$  then  $g(S)$  is contained in the disk of radius  $1/2^{n+1}$  about  $a_n(y)$ . It is easy to see that the area of  $g(S)$  is thus at most  $\frac{3}{4}$  times the area of  $S$ . Summing over  $S$ , we get that the area of  $g(X_0)$  is at most  $\frac{3}{4}$  times the area of  $X_0$ . This contradicts the previous paragraph, so we conclude that  $L_n(a_n) > 1$ .  $\square$

Suppose that  $L_0(a) = L(a) = 1$ . Then there is an  $f \in \Gamma$  with  $\limsup_{n \rightarrow \infty} L_n(f_{\frac{1}{n}}) = L_0(a) = 1$ . Rescaling  $f_{\frac{1}{n}}$  slightly, we may assume that  $L_n(f_{\frac{1}{n}}) \leq 1$  for all  $n$ . Then Proposition 4.15 tells us that  $\|f_{\frac{1}{n}}|_{V_n} - a|_{V_n}\| \geq \frac{\sqrt{3}}{6} - \frac{1}{4}$ . This contradicts the assumption that  $f \in \Gamma$ . Therefore  $\bar{L}$  does not coincide with  $L_0$  on the whole  $C_{\mathbb{C}}(X_0)$ .

## 5. CONTINUOUS FIELDS OF $C^*$ -ALGEBRAIC COMPACT QUANTUM METRIC SPACES

In this section we discuss the continuity of parameterized families of  $C^*$ -algebraic compact quantum metric spaces with respect to  $\text{dist}_{\text{cq}}$ . General criteria are given in Subsection 5.1. In Subsection 5.2 we consider especially families of  $C^*$ -algebraic compact quantum metric space induced by ergodic compact group actions.

**5.1. Criteria of  $C^*$ -algebraic quantum distance convergence.** The criteria we shall give are analogues of [27, Theorems 1.2 and 7.1]. A notion of continuous fields of compact quantum metric spaces was introduced in [27, Definition 6.4] for families of compact quantum metric spaces. In view of Remark 4.11, we give the following definition of continuous fields of  $C^*$ -algebraic compact quantum metric spaces. We refer the reader to [13, Section 10.1 and 10.2] for basic definitions and facts about continuous fields of  $C^*$ -algebras.

**Definition 5.1.** Let  $T$  be a locally compact Hausdorff space, and let  $(\mathcal{A}_t, L_t)$  be a  $C^*$ -algebraic compact quantum metric space for each  $t \in T$ . Let  $\Gamma$  be the set of continuous sections of a continuous field of  $C^*$ -algebras over  $T$  with fibres  $\mathcal{A}_t$ , and let  $\Gamma_{\text{sa}} := \{f \in \Gamma : f^* = f\}$  be the set of continuous sections of the associated continuous field of real Banach spaces with fibres  $(\mathcal{A}_t)_{\text{sa}}$ . We call  $(\{(\mathcal{A}_t, L_t)\}, \Gamma)$  a *continuous field of  $C^*$ -algebraic compact quantum metric spaces* over  $T$  if  $(\{(\mathcal{A}_t, L_t)\}, \Gamma_{\text{sa}})$  is a continuous field of compact quantum metric spaces, *i.e.*, the unit section  $t \mapsto e_{\mathcal{A}_t}$  is in  $\Gamma$ , and for any  $t' \in T$ ,  $a \in \mathcal{A}_{t'}$ , and  $\varepsilon > 0$  there exists  $f \in (\Gamma_{\text{sa}})_{t'}$  such that  $\|f_{t'} - a\| < \varepsilon$  and  $L_{t'}(f_{t'}) < L_{t'}(a) + \varepsilon$ , where

$$(\Gamma_{\text{sa}})_{t'}^L = \{f \in \Gamma_{\text{sa}} : \text{the function } t \mapsto L_t(f_t) \text{ is upper semi-continuous at } t'\}.$$

As an analogue of [27, Theorem 1.2], we have the following criterion for continuous fields of  $C^*$ -algebraic compact quantum metric spaces to be continuous with respect to the  $C^*$ -algebraic quantum distance:

**Theorem 5.2.** Let  $(\{(\mathcal{A}_t, L_t)\}, \Gamma)$  be a continuous field of  $C^*$ -algebraic compact quantum metric spaces over a locally compact Hausdorff space  $T$ . Let  $t_0 \in T$ , and let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence in  $\Gamma_{\text{sa}}$  such that  $(f_n)_{t_0} \in \mathcal{D}(\mathcal{A}_{t_0})$  for each  $n \in \mathbb{N}$  and the set  $\{(f_n)_{t_0} : n \in \mathbb{N}\}$  is dense in  $\mathcal{D}(\mathcal{A}_{t_0})$ . Then the following are equivalent:

- (1)  $\text{dist}_{\text{cq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (2)  $\text{dist}_{\text{oq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;

- (3)  $\text{dist}_{\text{GH}}(\mathcal{D}(A_t), \mathcal{D}(A_{t_0})) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (4) for any  $\varepsilon > 0$ , there is an  $N$  such that the open  $\varepsilon$ -balls in  $A_t$  centered at  $(f_1)_t, \dots, (f_N)_t$  cover  $\mathcal{D}(A_t)$  for all  $t$  in some neighborhood  $\mathcal{U}$  of  $t_0$ .

*Proof.* (1) $\implies$ (2) follows from Theorem 3.15. (2) $\implies$ (3) $\implies$ (4) follows from [27, Theorem 1.2]. Hence we are left to show (4) $\implies$ (1).

As in the proof of [27, Theorem 1.2] we may assume that  $(f_n)_t \in \mathcal{D}(A_t)$  for all  $t \in T$  and  $n \in \mathbb{N}$ . We can also find a normed space  $V$  containing all  $\mathcal{A}_t$ 's such that for every  $f \in \Gamma$  the map  $t \mapsto f_t$  from  $T$  to  $V$  is continuous at  $t_0$ . For any  $n \in \mathbb{N}$  and  $t \in T$  set  $Y_{n,t} = \{(f_1)_t, \dots, (f_n)_t\}$ . Also set  $X_t = \mathcal{D}(A_t)$  for all  $t \in T$ .

It is easy to see from (4) that there are a  $R > 0$  and a neighborhood  $\mathcal{U}_1$  of  $t_0$  such that  $R > r_{\mathcal{A}_t}$  throughout  $\mathcal{U}_1$ . Let  $\varepsilon > 0$  be given. Pick  $N$  and a neighborhood  $\mathcal{U} \subseteq \mathcal{U}_1$  of  $t_0$  for  $\varepsilon$  as in (4). Then  $\text{dist}_{\text{H}}^V(X_t, Y_{N,t}) \leq \varepsilon$  throughout  $\mathcal{U}$ . Let  $t \in \mathcal{U}$ . For any  $a, b \in X_t$  and  $j, k \in \mathbb{N}$  we have

$$\begin{aligned} \|ab - (f_j)_t(f_k)_t\| &\leq \|a - (f_j)_t\| \cdot \|b\| + \|(f_j)_t\| \cdot \|b - (f_k)_t\| \\ &\leq R(\|a - (f_j)_t\| + \|b - (f_k)_t\|). \end{aligned}$$

Consequently

$$\text{dist}_{\text{H}}^{V(3)}((X_t)^m, (Y_{N,t})^m) \leq \max(2R, 1) \text{dist}_{\text{H}}^V(X_t, Y_{N,t}) \leq (2R + 1)\varepsilon.$$

Clearly  $\text{dist}_{\text{H}}^{V(3)}((Y_{N,t})^m, (Y_{N,t_0})^m) \rightarrow 0$  as  $t \rightarrow t_0$ . By shrinking  $\mathcal{U}$  we may assume that  $\text{dist}_{\text{H}}^{V(3)}((Y_{N,t})^m, (Y_{N,t_0})^m) < \varepsilon$  throughout  $\mathcal{U}$ . Then

$$\text{dist}_{\text{cq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \leq \text{dist}_{\text{H}}^{V(3)}((X_t)^m, (X_{t_0})^m) \leq (4R + 3)\varepsilon$$

for all  $t \in \mathcal{U}$ . □

It was pointed out in [27, Section 11] that the  $\theta$ -deformations  $M_\theta$  of Connes and Landi [11] for a compact spin manifold  $M$  form a natural continuous field of  $C^*$ -algebras over the space of  $n \times n$  skew-symmetric matrices  $\theta$ . By [27, Theorem 1.4], when  $M$  is connected the  $M_\theta$ 's form a continuous field of compact quantum metric spaces and are continuous with respect to  $\text{dist}_{\text{oq}}$ . Then Theorem 5.2 tells us that they are also continuous with respect to  $\text{dist}_{\text{cq}}$ .

Combining [27, Theorems 7.1] and Theorems 3.15 and 5.2 together, we get

**Theorem 5.3.** Let  $(\{\mathcal{A}_t, L_t\}, \Gamma)$  be a continuous field of  $C^*$ -algebraic compact quantum metric spaces over a locally compact Hausdorff space  $T$ . Suppose  $R \geq r_{\mathcal{A}_t}$  for all  $t \in T$ . Let  $t_0 \in T$ , and let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence in  $\Gamma_{\text{sa}}$  such that  $(f_n)_{t_0} \in \mathcal{D}_R(A_{t_0})$  for each  $n \in \mathbb{N}$  and the set  $\{(f_n)_{t_0} : n \in \mathbb{N}\}$  is dense in  $\mathcal{D}_R(A_{t_0})$ . Then the following are equivalent:

- (1)  $\text{dist}_{\text{cq}}^R(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (2)  $\text{dist}_{\text{oq}}^R(A_t, A_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (3)  $\text{dist}_{\text{GH}}(\mathcal{D}_R(A_t), \mathcal{D}_R(A_{t_0})) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (4) for any  $\varepsilon > 0$ , there is an  $N$  such that the open  $\varepsilon$ -balls in  $A_t$  centered at  $(f_1)_t, \dots, (f_N)_t$  cover  $\mathcal{D}_R(A_t)$  for all  $t$  in some neighborhood  $\mathcal{U}$  of  $t_0$ ;
- (5)  $\text{dist}_{\text{cq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (6)  $\text{dist}_{\text{oq}}(A_t, A_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ .

In the theory of operator algebras usually continuous fields of  $C^*$ -algebras are used to describe continuous families of algebras qualitatively. Certainly, convergence under quantum distances is more concrete. So one may expect the following:

**Proposition 5.4.** Let  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  be a sequence of  $C^*$ -algebraic compact quantum metric spaces converging to some  $C^*$ -algebraic compact quantum metric space  $(\mathcal{A}_0, L_0)$  under  $\text{dist}_{\text{cq}}$ . Then there is a continuous field of  $C^*$ -algebraic compact quantum metric spaces over  $T = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$  with fibres  $(\mathcal{A}_n, L_n)$  at  $\frac{1}{n}$  and  $(\mathcal{A}_0, L_0)$  at 0.

*Proof.* We may assume that all the  $L_n$ 's and  $L_0$  are closed. Let  $R > \sup\{r_{\mathcal{A}_n} : n \in \mathbb{N}\}$ . Then  $R \geq r_{\mathcal{A}_0}$ , and  $\text{dist}_{\text{cq}}^R(\mathcal{A}_n, \mathcal{A}_0) \rightarrow 0$  by Theorem 3.15. For each  $n$  let  $V_n$  be a complex normed space containing both  $\mathcal{A}_n$  and  $\mathcal{A}_0$  with  $\text{dist}_{\mathbb{H}}^{V_n(3)}(\mathcal{D}_R(\mathcal{A}_n), \mathcal{D}_R(\mathcal{A}_0)) < \text{dist}_{\text{cq}}^R(\mathcal{A}_n, \mathcal{A}_0) + \frac{1}{n}$ . By [27, Lemma 4.5] we can find a complex normed space  $V$  containing all these  $V_n$ 's with the copies of  $\mathcal{A}_0$  identified. Then

$$\text{dist}_{\mathbb{H}}^{V(3)}(\mathcal{D}_R(\mathcal{A}_n), \mathcal{D}_R(\mathcal{A}_0)) \rightarrow 0$$

as  $n \rightarrow \infty$ . Let  $\Gamma$  be the set of all continuous maps  $f$  from  $T$  to  $V$  such that  $f(\frac{1}{n}) \in \mathcal{A}_n$  for all  $n$  and  $f(0) \in \mathcal{A}_0$ . Clearly  $\Gamma$  is the space of continuous sections of a continuous field of Banach spaces over  $T$  with fibres  $(\mathcal{A}_n, L_n)$  at  $\frac{1}{n}$  and  $(\mathcal{A}_0, L_0)$  at 0. As in Lemma 4.7 it is easy to see that for any  $f, g \in \Gamma$  the sections  $f \cdot g$  and  $f^*$  are also in  $\Gamma$ . Thus this is a continuous field of  $C^*$ -algebras. Also as in Lemma 4.9 the restriction of  $L_0$  on  $(\mathcal{A}_0)_{\text{sa}}$  is given by (13). Thus this is a continuous field of  $C^*$ -algebraic compact quantum metric spaces.  $\square$

**Remark 5.5.** [27, Proposition 7.4] is crucial in the proof of (4) $\implies$ (1) in Theorem 5.2. As pointed out at the beginning of Section 3, one can try to define a quantum distance,  $\text{dist}_{\text{nu}}$ , as  $\inf\{\text{dist}_{\mathbb{H}}^{\mathcal{C}}(h_{\mathcal{A}}(\mathcal{D}(A)), h_{\mathcal{B}}(\mathcal{D}(B)))\}$ , where the infimum is taken over all faithful  $*$ -homomorphisms  $h_{\mathcal{A}}$  and  $h_{\mathcal{B}}$  of  $\mathcal{A}$  and  $\mathcal{B}$  into some  $C^*$ -algebra  $\mathcal{C}$ . Similarly, one defines  $\text{dist}_{\text{nu}}^R$ . One can check that  $\text{dist}_{\text{nu}}$  (resp.  $\text{dist}_{\text{nu}}^R$ ) is also a metric on  $C^*\text{CQM}$  (resp.  $C^*\text{CQM}^R$ ) and that the forgetful map  $(C^*\text{CQM}, \text{dist}_{\text{nu}}) \rightarrow (\text{CQM}, \text{dist}_{\text{q}})$  is continuous (see Theorem 3.15). Also  $\text{dist}_{\text{nu}}$  and  $\text{dist}_{\text{nu}}^R$  define the same topology on  $C^*\text{CQM}^R$ . The proofs of Proposition 3.17 and Theorem 3.16 hold with  $\text{dist}_{\text{cq}}$  replaced by  $\text{dist}_{\text{nu}}$  (actually we get  $\text{dist}_{\text{nu}}^R(C_{\mathbb{C}}(X), C_{\mathbb{C}}(Y)) \leq \text{dist}_{\text{GH}}(X, Y)$  for all  $R \geq 0$  and compact metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$ ). The completeness part of Theorem 4.4 also holds with  $\text{dist}_{\text{cq}}$  replaced by  $\text{dist}_{\text{nu}}$ , that is,  $(C^*\text{CQM}_F, \text{dist}_{\text{cq}})$  is complete. To see this, for any Cauchy sequence  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  in  $C^*\text{CQM}_F^R$ , by [2, Theorem 3.1] we may find a  $C^*$ -algebra  $\mathcal{C}$  containing all the  $\mathcal{A}_n$ 's as  $C^*$ -subalgebras such that  $\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathcal{D}_R(\mathcal{A}_n), \mathcal{D}_R(\mathcal{A}_m)) \rightarrow 0$  as  $n, m \rightarrow \infty$ . If we denote by  $(\mathcal{B}, L)$  the limit constructed in the proof of the compactness part of Theorem 4.4 (with  $V$  replaced by  $\mathcal{C}$ ), then clearly  $(\mathcal{A}_n, L_n) \rightarrow (\mathcal{B}, L)$  under  $\text{dist}_{\text{nu}}^R$ . But without knowing whether we can require  $V$  to be a  $C^*$ -algebra and the embeddings  $h_{\mathcal{A}}$ 's to be  $*$ -homomorphisms in Lemma 4.6, we do not know whether the compactness part of Theorem 4.4 holds with  $\text{dist}_{\text{cq}}$  replaced by  $\text{dist}_{\text{nu}}$  (however, since the topology induced by  $\text{dist}_{\text{nu}}$  on  $C^*\text{CQM}$  is stronger than that induced by  $\text{dist}_{\text{cq}}$ , one sees easily that the compactness part of Theorem 4.4 holds with  $\text{dist}_{\text{cq}}$  replaced by  $\text{dist}_{\text{nu}}$  if and only if  $\text{dist}_{\text{cq}}$  and  $\text{dist}_{\text{nu}}$  induce the same topology on  $C^*\text{CQM}_F$ ). Without the  $C^*$ -algebraic analogue of [27, Proposition 7.4] we do not know either whether (4) still implies (1) in Theorem 5.2 with  $\text{dist}_{\text{cq}}$  replaced by  $\text{dist}_{\text{nu}}$ . However, Blanchard has proved [4] that every separable unital continuous field of nuclear  $C^*$ -algebras over a compact metric space  $T$  can be subtrivialized. Thus  $\text{dist}_{\text{nu}}$  behaves well at least for nuclear

$C^*$ -algebras, and Theorems 5.2 and 5.3 hold with  $\text{dist}_{\text{cq}}$  and  $\text{dist}_{\text{cq}}^R$  replaced by  $\text{dist}_{\text{nu}}$  and  $\text{dist}_{\text{nu}}^R$  when  $T$  is a compact metric space and each fibre  $\mathcal{A}_t$  is nuclear. Limits of exact  $C^*$ -algebras under  $\text{dist}_{\text{nu}}$  are also exact:

**Proposition 5.6.** Let  $\{(\mathcal{A}_n, L_n)\}_{n \in \mathbb{N}}$  be a sequence of  $C^*$ -algebraic compact quantum metric spaces converging to some  $C^*$ -algebraic compact quantum metric space  $(\mathcal{A}, L)$  under  $\text{dist}_{\text{nu}}$ . If each  $\mathcal{A}_n$  is exact, then so is  $\mathcal{A}$ .

*Proof.* Let  $R > \sup\{r_{\mathcal{A}_n} : n \in \mathbb{N}\}$ . Then  $R \geq r_{\mathcal{A}}$ , and  $\text{dist}_{\text{nu}}^R(\mathcal{A}_n, \mathcal{A}_0) \rightarrow 0$ . For each  $n$  let  $\mathcal{C}_n$  be a  $C^*$ -algebra containing both  $\mathcal{A}_n$  and  $\mathcal{A}$  as  $C^*$ -subalgebras with

$$\text{dist}_{\mathbb{H}}^{\mathcal{C}_n}(\mathcal{D}_R(\mathcal{A}_n), \mathcal{D}_R(\mathcal{A})) < \text{dist}_{\text{nu}}^R(\mathcal{A}_n, \mathcal{A}) + \frac{1}{n}.$$

Using amalgamation of  $C^*$ -algebras [2, Theorem 3.1] we can find a  $C^*$ -algebra  $\mathcal{C}$  containing all these  $\mathcal{C}_n$ 's with the copies of  $\mathcal{A}$  identified. Then  $\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathcal{D}_R(\mathcal{A}_n), \mathcal{D}_R(\mathcal{A})) \rightarrow 0$  as  $n \rightarrow \infty$ . Taking a faithful  $*$ -representation of  $\mathcal{C}$ , we may assume that  $\mathcal{C} = B(H)$  for some Hilbert space  $H$ . Denote by  $1_H$  the identity operator on  $H$ . Set  $\tilde{\mathcal{A}}_n = \mathbb{C} \cdot 1_H + \mathcal{A}_n$ , and let  $\tilde{\mathcal{A}} = \mathbb{C} \cdot 1_H + \mathcal{A}$ . Notice that  $\tilde{\mathcal{A}}_n = \mathbb{C}(1_H - e_{\mathcal{A}_n}) \oplus \mathcal{A}_n$ . So  $\tilde{\mathcal{A}}_n$  is also exact.

Recall that a unital completely positive (u.c.p.) map  $\varphi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  between unital  $C^*$ -algebras is *nuclear* if for any finite subset  $\mathcal{J} \subseteq \mathcal{B}_1$  and any  $\varepsilon > 0$  there are an integer  $k$  and u.c.p. maps  $\phi : \mathcal{B}_1 \rightarrow M_k(\mathbb{C})$ ,  $\psi : M_k(\mathbb{C}) \rightarrow \mathcal{B}_2$  such that  $\|(\psi \circ \phi)(x) - \varphi(x)\| < \varepsilon$  for all  $x \in \mathcal{J}$ . (For nonunital  $C^*$ -algebras,  $\phi$  and  $\psi$  are required to be completely positive contractions.) A  $C^*$ -algebra  $\mathcal{B}_1$  is *nuclearly embeddable* if for some  $C^*$ -algebra  $\mathcal{B}_2$  there is a faithful  $*$ -homomorphism  $\varphi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  such that  $\varphi$  is nuclear. It is a theorem of Kirchberg [40] that a  $C^*$ -algebra  $\mathcal{B}_1$  is exact if and only if  $\mathcal{B}_1$  is nuclearly embeddable, and if and only if any faithful  $*$ -homomorphism  $\varphi$  of  $\mathcal{B}_1$  into  $B(H_1)$  of any Hilbert space  $H_1$  is nuclear. We will show that the inclusion  $h_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \rightarrow B(H)$  is nuclear. Then  $\tilde{\mathcal{A}}$  is exact. Since  $\tilde{\mathcal{A}} = \mathbb{C}(1_H - e_{\mathcal{A}}) \oplus \mathcal{A}$ , the algebra  $\mathcal{A}$  is also exact.

Let a finite subset  $\mathcal{J} \subseteq \tilde{\mathcal{A}}$  and  $\varepsilon > 0$  be given. Since  $A + iA + \mathbb{C}1_H$  is dense in  $\tilde{\mathcal{A}}$  and u.c.p. maps are contractions, we may assume that  $\mathcal{J} \subseteq A + \mathbb{R}1_H$ . Notice that  $\mathcal{D}_R(A)$  is absorbing in  $A$ , *i.e.*  $\mathbb{R}_+ \mathcal{D}_R(A) = A$ . Thus we may assume that  $\mathcal{J} \subseteq \mathcal{D}_R(A) \cup \{1_H\}$ . Take  $N$  such that  $\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathcal{D}_R(\mathcal{A}_N), \mathcal{D}_R(\mathcal{A})) < \frac{1}{3}\varepsilon$ . Then we can find a finite subset  $\mathcal{J}_N \subseteq \mathcal{D}_R(\mathcal{A}_N) \cup \{1_H\}$  with  $\text{dist}_{\mathbb{H}}^{\mathcal{C}}(\mathcal{J}_N, \mathcal{J}) < \frac{1}{3}\varepsilon$ . Since  $\tilde{\mathcal{A}}_N$  is exact, the inclusion  $\tilde{\mathcal{A}}_N \rightarrow B(H)$  is nuclear. So there are an integer  $k$  and u.c.p. maps  $\phi : \tilde{\mathcal{A}}_N \rightarrow M_k(\mathbb{C})$ ,  $\psi : M_k(\mathbb{C}) \rightarrow B(H)$  such that  $\|(\psi \circ \phi)(x) - \varphi(x)\| < \frac{1}{3}\varepsilon$  for all  $x \in \mathcal{J}_N$ . By Arveson's extension theorem [14, Theorem 5.1.7] we can extend  $\phi$  to a u.c.p. map from  $B(H)$  to  $M_k(\mathbb{C})$  which we still denote by  $\phi$ . Then it is easy to see that  $\|(\psi \circ \phi)(y) - \varphi(y)\| < \varepsilon$  for all  $y \in \mathcal{J}$ . Therefore the inclusion  $h_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \rightarrow B(H)$  is nuclear.  $\square$

**Example 5.7** (Quotient Field of a  $C^*$ -algebraic Compact Quantum Metric Space). Let  $(\mathcal{A}, L)$  be a closed  $C^*$ -algebraic compact quantum metric space. Denote by  $\mathcal{I}$  the set of all closed two-sided ideals not equal to  $\mathcal{A}$ . Let  $(\{(B_t, L_t)\}, \Gamma)$  be the quotient field over  $T$  for  $(\mathcal{A}, L|_{\mathcal{A}})$  as constructed in [27, Example 6.3], where  $T$  is the set of all nonempty convex closed subsets of  $S(\mathcal{A}) = S(\mathcal{A})$  equipped with the Hausdorff distance  $\text{dist}_{\mathbb{H}}^{S(\mathcal{A})}$  and  $(B_t, L_t)$  is the quotient compact quantum metric space of  $(\mathcal{A}, L|_{\mathcal{A}})$  for each  $t \in T$ . Denote by  $\pi_t$  the quotient map  $\mathcal{A} \rightarrow B_t$  for each

$t \in T$ . For any  $I \in \mathcal{I}$  according to Lemma 4.3 the quotient  $\mathcal{A}/I$  equipped with the quotient Lip-norm is a closed  $C^*$ -algebraic compact quantum metric space. Now  $S(\mathcal{A}/I)$  is a convex closed subset of  $S(\mathcal{A})$ . Clearly  $I$  is determined by  $S(\mathcal{A}/I)$  as

$$I = \{a \in \mathcal{A} : a|_{S(\mathcal{A}/I)} = 0\}.$$

So the map  $I \mapsto S(\mathcal{A}/I)$  from  $\mathcal{I}$  to  $T$  is injective. In fact, the image of  $\mathcal{I}$  is closed in  $(T, \text{dist}_{\mathbb{H}}^{S(\mathcal{A})})$ : let  $I_j$  be a sequence of in  $\mathcal{I}$  such that  $t_j = S(\mathcal{A}/I_j)$  converges to some  $t_0 \in T$ . Then  $\|\pi_{t_j}(a)\| \rightarrow \|\pi_{t_0}(a)\|$  for all  $a \in \mathcal{A}_{\text{sa}}$ . It follows immediately that the function  $\|\cdot\|_{t_0}$  defined by  $\|a\|_{t_0} = (\|\pi_{t_0}(aa^*)\|)^{\frac{1}{2}}$  is a nontrivial  $C^*$ -algebraic seminorm on  $\mathcal{A}$ , i.e.  $\|aa'\|_{t_0} \leq \|a\|_{t_0}\|a'\|_{t_0}$  and  $\|aa^*\|_{t_0} = \|a\|_{t_0}^2$  for all  $a, a' \in \mathcal{A}$ . Denote by  $I_0$  the kernel of  $\|\cdot\|_{t_0}$ . Then  $I_0$  is a closed two-sided ideal of  $\mathcal{A}$ . We claim that  $t_0 = S(\mathcal{A}/I_0)$ . For any  $\mu \in t_0$  clearly the evaluation at  $\mu$  induces a state of  $\mathcal{A}/I_0$ . Under the identification of  $S(\mathcal{A}/I_0)$  with a closed convex subset of  $S(\mathcal{A})$ , this state is just  $\mu$  itself. Hence we see that  $t_0 \subseteq S(\mathcal{A}/I_0)$ . For each  $a \in \mathcal{A}_{\text{sa}}$  notice that  $\|\pi_{t_0}(a)\| = \|a\|_{t_0} = \|\pi_{S(\mathcal{A}/I_0)}(a)\|$ . By Lemma 5.9 below we have  $t_0 = S(\mathcal{A}/I_0)$ . Therefore the image of  $\mathcal{I}$  in  $T$  is closed. Identify  $\mathcal{I}$  with its image, and let  $(\{(B_t, L_t)\}, \Gamma_{\mathcal{I}})$  be the restriction of  $(\{(B_t, L_t)\}, \Gamma)$  on  $\mathcal{I}$ . Then  $(B_{S(\mathcal{A}/I)}, L_{S(\mathcal{A}/I)})$  is the associated quantum metric space of  $(\mathcal{A}/I, L_{\mathcal{A}/I})$ . Notice that for any  $a \in \mathcal{A}$  the map  $I \mapsto \|\pi_{\mathcal{A}/I}(a)\| = (\|\pi_{\mathcal{A}/I}(aa^*)\|)^{\frac{1}{2}}$  is continuous. Then it is easy to see that  $(\{(\mathcal{A}/I, L_{\mathcal{A}/I})\}, \Gamma_{\mathcal{I}} + i\Gamma_{\mathcal{I}})$  is a continuous field of  $C^*$ -algebraic compact quantum metric spaces. We call it the *quotient field of  $(\mathcal{A}, L)$* . It was noticed in [27, Example 6.3] that  $\text{dist}_{\text{q}}(B_{S(\mathcal{A}/I)}, B_{S(\mathcal{A}/I_0)}) \rightarrow 0$  as  $I \rightarrow I_0$ . By [27, Theorems 1.1] and Theorem 5.2 we see that  $\text{dist}_{\text{cq}}(\mathcal{A}/I, \mathcal{A}/I_0) \rightarrow 0$  as  $I \rightarrow I_0$ .

**Remark 5.8.** The construction of the continuous field of order-unit spaces in [27, Example 6.3] and that of the continuous field of  $C^*$ -algebras in example 5.7 work in general. For any compact Hausdorff space  $X$ , the set  $\text{SUB}(X)$  of closed nonempty subsets of  $X$  is a compact Hausdorff space equipped with the Hausdorff topology, which is defined using the pseudometrics  $\text{dist}_{\mathbb{H}}^{\rho}$  for all the continuous pseudometrics  $\rho$  on  $X$  (see [17] for a generalization). For any order-unit space  $(A, e)$  (see for example [27, Section 2]), it is clear that the set of all nonempty closed convex subsets of the state-space  $S(A)$ , which we still denote by  $T$  as in [27, Example 6.3], is a closed subspace of  $\text{SUB}(S(A))$ . So  $T$  is a compact Hausdorff space with the relative topology. As in [27, Example 6.3] the restrictions of  $a \in A$  to closed convex subsets of  $S(A)$  still generate a continuous field of order-unit spaces over  $T$ , which we denote by  $(\{B_t\}, \Gamma)$ . For a unital  $C^*$ -algebra  $\mathcal{A}$  the argument in Example 5.7 still works to show that the set of all proper closed two-sided ideals  $\mathcal{I}$  is a closed subset of the set  $T$  of all nonempty convex closed subsets of  $S(\mathcal{A}_{\text{sa}}) = S(\mathcal{A})$ , and the quotient images for  $a \in \mathcal{A}$  generate a continuous field of  $C^*$ -algebras over  $\mathcal{I}$ . When  $\mathcal{A}$  is nonunital, let  $\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}e$  be  $\mathcal{A}$  with a unit adjoined. Then  $\mathcal{I}(\tilde{\mathcal{A}}) = \mathcal{I}(\mathcal{A}) \cup \{A\}$ . So  $\mathcal{I}(\mathcal{A})$  becomes a locally compact Hausdorff space with  $\infty$  being the ideal  $\mathcal{A}$  itself, and we still get a continuous field of  $C^*$ -algebras over  $\mathcal{I}(\mathcal{A})$  generated by the quotient images of  $a \in \mathcal{A}$ . In both cases, the natural  $*$ -homomorphism from  $\mathcal{A}$  to the algebras of global sections vanishing at  $\infty$  is injective and the composed homomorphism to each fibre algebra is surjective. This means that  $\mathcal{A}$  is represented as a *full algebra of operator fields* in the sense of [16, page 236]. In fact, one can check easily that our topology on  $\mathcal{I}$  coincides with that on the space of norm-functions (with 0 removed) in [16, page 243]. Now it is clear that the usage of the quantum metric on  $A$  (resp.  $\mathcal{A}$ ) in [27, Example 6.3] (resp. Example 5.7) is

just to make  $T$  (resp.  $\mathcal{T}$ ) into a metric space and to endow each fibre space of the continuous field of order-unit spaces (resp.  $C^*$ -algebras) with a quantum metric (resp.  $C^*$ -algebraic quantum metric).

**Lemma 5.9.** Let  $(A, e)$  be an order-unit space, and let  $\{\mathfrak{X}_n\}$  be a net of closed convex subsets of  $S(A)$ . Let  $\pi_n : A \rightarrow \text{Af}_{\mathbb{R}}(\mathfrak{X}_n)$  be the restriction map, where  $\text{Af}_{\mathbb{R}}(\mathfrak{X}_n)$  is the space of continuous affine  $\mathbb{R}$ -valued functions on  $\mathfrak{X}_n$  and we identify  $A$  with a dense subspace of  $\text{Af}_{\mathbb{R}}(S(A))$  via the pairing between  $A$  and  $S(A)$ . If  $\|a\| = \lim_{n \rightarrow \infty} \|\pi_n(a)\|$  for all  $a \in A$ , then  $\mathfrak{X}_n$  converges to  $S(A)$  under the Hausdorff topology in  $\text{SUB}(S(A))$  when  $n \rightarrow \infty$ .

*Proof.* Suppose that  $\mathfrak{X}_n$  does not converge to  $S(A)$  under the Hausdorff topology when  $n \rightarrow \infty$ . Then there are some  $\varepsilon > 0$  and a subnet of  $\{\mathfrak{X}_n\}$  converging to some closed convex  $\mathfrak{X} \subsetneq S(A)$ . Without loss of generality we may assume that this subnet is  $\{\mathfrak{X}_n\}$  itself. Pick  $\mu \in S(A) \setminus \mathfrak{X}$ . By the Hahn-Banach theorem there exist  $a' \in \bar{A}$  and  $t' \in \mathbb{R}$  with  $\mu(a') > t' \geq \nu(a')$  for all  $\nu \in \mathfrak{X}$ . Since  $A$  is dense in  $\bar{A}$ , we may also find  $a \in A$  and  $t \in \mathbb{R}$  such that  $\mu(a) > t \geq \nu(a)$  for all  $\nu \in \mathfrak{X}$ . Then  $\frac{\mu(a)+t}{2} \geq \nu(a)$  for all  $\nu \in \mathfrak{X}_n$  when  $n$  is big enough. Consequently,  $\mu(a - s \cdot e) > \frac{\mu(a)+t}{2} - s \geq \nu(a - s \cdot e) \geq 0$  for all  $\nu \in \mathfrak{X}_n$ , where  $s = \min\{a(\nu) : \nu \in S(A)\}$ . Therefore  $\|a - s \cdot e\| \geq \mu(a - s \cdot e) > \frac{\mu(a)+t}{2} - s \geq \|\pi_n(a - s \cdot e)\|$ , which contradicts the assumption that  $\lim_{n \rightarrow \infty} \|a - s \cdot e\| = \|\pi_n(a - s \cdot e)\|$ . So  $\mathfrak{X}_n$  does converge to  $S(A)$  under the Hausdorff topology when  $n \rightarrow \infty$ .  $\square$

**Example 5.10** (Berezin-Toeplitz Quantization). Let  $(M, \omega)$  be a Kähler manifold [7, 18], *i.e.*  $M$  is a complex manifold and  $\omega$  is a Kähler form. This means that  $\omega$  is a positive, non-degenerate closed 2-form of type  $(1, 1)$ . Locally, if  $\dim_{\mathbb{C}}(M) = d$  and  $z_1, \dots, z_d$  are local holomorphic coordinates, then  $\omega$  can be written as

$$\omega = i \sum_{j,k=1}^d g_{jk}(z) dz_j \wedge d\bar{z}_k, \quad g_{jk} \in C^\infty(M, \mathbb{C}),$$

where the matrix  $(g_{jk}(z))$  is a positive definite hermitian matrix for each  $z$ . Consider triples  $(E, h, \nabla)$ , with  $E$  a holomorphic line bundle,  $h$  a hermitian metric on  $E$ , and  $\nabla$  a connection which is compatible with the metric and the complex structure. With respect to local holomorphic coordinates and a local holomorphic frame  $s$  of the bundle, this means that  $\nabla = \partial + \partial \log \hat{h} + \bar{\partial}$ , where  $\hat{h}(z) = h_z(s(z), s(z))$ . The curvature of  $\nabla$  is defined as

$$F(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

The Kähler manifold  $(M, \omega)$  is called *quantizable* [39, Section 2] if there is such a triple  $(E, h, \nabla)$  with

$$(14) \quad F = -i\omega.$$

This includes all the compact Riemann surfaces and the complex projective spaces. Now let  $M$  be a quantizable compact Kähler manifold, and let  $\Omega = \frac{1}{d!} \omega^d$  be the volume form on  $M$ . Then the space of holomorphic sections  $\Gamma_{\text{hol}}(M, E)$  is a finite dimensional complex vector space [18, page 152]. On the space of smooth sections  $\Gamma_\infty(M, E)$  we have the inner product

$$\langle \varphi, \psi \rangle := \int_M h(\varphi, \psi) \Omega.$$

Denote by  $L^2(M, E)$  the  $L^2$ -completion of  $\Gamma_\infty(M, E)$ , and denote by  $\Pi$  the orthogonal projection  $L^2(M, E) \rightarrow \Gamma_{\text{hol}}(M, E)$ . For any  $f \in C_{\mathbb{C}}(M)$  the *Toeplitz operator*  $T_f$  is defined as

$$T_f := \Pi(f \cdot) : \Gamma_{\text{hol}}(M, E) \rightarrow \Gamma_{\text{hol}}(M, E).$$

It is easy to see that  $f \mapsto T_f$  is a unital completely positive map from  $C_{\mathbb{C}}(M)$  to  $B(\Gamma_{\text{hol}}(M, E))$ , the space of bounded operators on  $\Gamma_{\text{hol}}(M, E)$ . In particular,  $T_{f^*} = (T_f)^*$  for all  $f \in C_{\mathbb{C}}(M)$ . We may replace  $E$  by the  $n$ -th tensor powers  $E^n := E^{\otimes n}$  and apply above construction for every  $n$ . In this way we get linear maps

$$T^{(n)} : C_{\mathbb{C}}(M) \rightarrow B(\Gamma_{\text{hol}}(M, E^n)), \quad f \mapsto T_f^{(n)}.$$

The condition (14) implies that  $E$  is ample. Replacing  $E$  and  $\omega$  by  $E^{n_0}$  and  $n_0\omega$  respectively for some  $n_0$  we may assume that  $E$  is very ample. Then  $T^{(n)}$  is surjective for all  $n \in \mathbb{N}$  [5, Proposition 4.2]. For any metric  $\rho_M$  on  $M$  inducing the manifold topology, we have the associated  $C^*$ -algebraic compact quantum metric space  $(C_{\mathbb{C}}(M), L)$ . Then for any  $n \in \mathbb{N}$  the  $C^*$ -algebra  $B(\Gamma_{\text{hol}}(M, E^n))$ , equipped with the quotient Lip-norm  $L_{1/n}$  constructed in Lemma 4.3, is a  $C^*$ -algebraic quantum compact metric space. One has  $\|T^{(n)}(f)\| \rightarrow \|f\|$  [5, Theorem 4.1] and  $\|(T^{(n)}(f)) \cdot (T^{(n)}(g)) - T^{(n)}(f \cdot g)\| \rightarrow 0$  [5, page 291] [39, Theorem 3] as  $n \rightarrow \infty$  for any  $f, g \in C_{\mathbb{C}}(M)$ . Consequently, the sections given by the images of  $f$  under  $T^{(n)}$  for  $f \in C_{\mathbb{C}}(M)$  generate a continuous field of  $C^*$ -algebras over  $T' = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ , with fibres  $B(\Gamma_{\text{hol}}(M, E^n))$  at  $1/n$  and  $C_{\mathbb{C}}(M)$  at 0. In fact, it was shown in [5] that this is a strict quantization of the symplectic structure on  $M$ , though we do not need this fact here. For each  $f \in C_{\mathbb{R}}(M)$ , we have  $L_{\frac{1}{n}}(T^{(n)}(f)) \leq L(f)$ . So this is a continuous field of  $C^*$ -algebraic compact quantum metric spaces. If we identify  $S(B(\Gamma_{\text{hol}}(M, E^n)))$  with the corresponding convex closed subset in  $S(C_{\mathbb{C}}(M))$ , then Lemma 5.9 tells us that  $S(B(\Gamma_{\text{hol}}(M, E^n)))$  converges to  $S(C_{\mathbb{C}}(M))$  under  $\text{dist}_{\mathbb{H}}^{S(C_{\mathbb{C}}(M))}$ , and this is a special case of [27, Example 6.10]. By [27, Theorem 1.1] and Theorem 5.2 we get that  $\text{dist}_{\text{cq}}(B(\Gamma_{\text{hol}}(M, E^n)), C_{\mathbb{C}}(M)) \rightarrow 0$  as  $n \rightarrow \infty$ . Let us mention that when  $M$  is an integral coadjoint orbit of a compact connected semisimple Lie group with the standard symplectic form, this becomes the Berezin quantization in [36] (see also [27, Example 10.12]). For such case, the quotient Lip-norms  $L_{1/n}$  and the induced metrics on the state-spaces  $S(B(H_n))$  are also considered in [43]. Note that the quotient Lip-norm here is different from the Lip-norm induced by the ergodic action as in [36].

**5.2. Continuous fields of  $C^*$ -algebraic compact quantum metric spaces induced by ergodic compact group actions.** Let  $G$  be a compact group with a fixed length function  $l$ , *i.e.*, a continuous real-valued function,  $l$ , on  $G$  such that

$$\begin{aligned} l(xy) &\leq l(x) + l(y) \text{ for all } x, y \in G \\ l(x^{-1}) &= l(x) \text{ for all } x \in G \\ l(x) &= 0 \text{ if and only if } x = e_G, \end{aligned}$$

where  $e_G$  is the identity of  $G$ .

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $\alpha$  be a strongly continuous ergodic action of  $G$  on  $\mathcal{A}$  by automorphisms. In [32, Theorem 2.3] Rieffel showed that  $\mathcal{A}$  is a  $C^*$ -algebraic compact quantum metric space equipped with the seminorm  $L$  defined

by

$$(15) \quad L(a) = \sup\left\{\frac{\|\alpha_x(a) - a\|}{l(x)} : x \in G, x \neq e_G\right\}.$$

Let  $(\{\mathcal{A}_t\}, \Gamma)$  be a continuous field of  $C^*$ -algebras over a locally compact Hausdorff space  $T$ , and let  $\alpha_t$  be a strongly continuous action of  $G$  on  $\mathcal{A}_t$  for each  $t \in T$ . Recall that  $\{\alpha_t\}$  is a *continuous field of strongly continuous actions* of  $G$  on  $(\{\mathcal{A}_t\}, \Gamma)$  if the actions  $\{\alpha_t\}$  give rise to a strongly continuous action of  $G$  on  $\Gamma_\infty$  [31, Definition 3.1], where  $\Gamma_\infty$  is the space of continuous sections vanishing at  $\infty$ . If each  $\mathcal{A}_t$  is unital and  $\alpha_t$  is ergodic, we say that this is a *field of ergodic actions*. In such case, one can show easily that the unit section is continuous.

Denote by  $\hat{G}$  the dual group of  $G$ . For any strongly continuous action of  $G$  on a  $C^*$ -algebra  $\mathcal{A}$  and  $\gamma \in \hat{G}$ , denote by  $\text{mul}(\mathcal{A}, \gamma)$  the multiplicity of  $\gamma$  in  $\mathcal{A}$ . Combining [27, Theorem 1.3] and Theorem 5.2 together, we have

**Theorem 5.11.** Let  $\{\alpha_t\}$  be a continuous field of strongly continuous ergodic actions of  $G$  on a continuous field of unital  $C^*$ -algebras  $(\{\mathcal{A}_t\}, \Gamma)$  over a locally compact Hausdorff space  $T$ . Then the induced field  $(\{(\mathcal{A}_t, L_t)\}, \Gamma)$  is a continuous field of  $C^*$ -algebraic compact quantum metric spaces. For any  $t_0 \in T$  the following are equivalent:

- (1)  $\lim_{t \rightarrow t_0} \text{mul}(\mathcal{A}_t, \gamma) = \text{mul}(\mathcal{A}_{t_0}, \gamma)$  for all  $\gamma \in \hat{G}$ ;
- (2)  $\limsup_{t \rightarrow t_0} \text{mul}(\mathcal{A}_t, \gamma) \leq \text{mul}(\mathcal{A}_{t_0}, \gamma)$  for all  $\gamma \in \hat{G}$ ;
- (3)  $\text{dist}_{\text{oq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ;
- (4)  $\text{dist}_{\text{cq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ .

In particular, Rieffel's continuity of noncommutative tori [35, Theorem 9.2] and matrices converging to integral coadjoint orbits of compact connected semisimple Lie groups [36, Theorem 3.2] hold with respect to  $\text{dist}_{\text{cq}}$  (see [27, Examples 10.11 and 10.12]).

As a consequence of Theorem 5.11 we show that when  $T$  is a compact metric space, the map  $t \mapsto (\mathcal{A}_t, L_t)$  from  $T$  to  $C^*$ CQM is continuous at most points of  $T$ :

**Corollary 5.12.** Let  $\{\alpha_t\}$  be a continuous field of strongly continuous ergodic actions of  $G$  on a continuous field of unital  $C^*$ -algebras  $(\{\mathcal{A}_t\}, \Gamma)$  over a compact metric space  $T$ . Then there is a nowhere dense  $F_\sigma$  subset  $Z \subseteq T$  such that for any  $t_0 \in T \setminus Z$ ,  $\text{dist}_{\text{cq}}(\mathcal{A}_t, \mathcal{A}_{t_0}) \rightarrow 0$  as  $t \rightarrow t_0$ ,

*Proof.* By [21, Proposition 2.1], for each  $\gamma \in \hat{G}$  the multiplicity of  $\alpha_t$  in  $\mathcal{A}_t$  is no bigger than  $\dim(\gamma)$ . Set  $X_{\gamma, \dim(\gamma)} = \{t \in T : \text{mul}(\mathcal{A}_t, \gamma) = \dim(\gamma)\}$ . Since the multiplicity function is lower semi-continuous over  $T$  by [27, Lemma 10.6],  $X_{\gamma, \dim(\gamma)}$  is an open subset of  $T$ . Inductively, for  $1 \leq j \leq \dim(\gamma)$  set  $X_{\gamma, \dim(\gamma)-j} = \{t \in T \setminus \bigcup_{k=0}^{j-1} X_{\gamma, \dim(\gamma)-k} : \text{mul}(\mathcal{A}_t, \gamma) = \dim(\gamma) - j\}$ . Then the  $X_{\gamma, \dim(\gamma)-j}$ 's are all open subsets of  $T$ . Set  $X_\gamma = \bigcup_{0 \leq j \leq \dim \gamma} X_{\gamma, \dim(\gamma)-j}$ . Then  $X_\gamma$  is the subset of  $T$  with constant multiplicity with respect to  $\gamma$  locally. Clearly  $X_\gamma$  is open, and its complement  $Z_\gamma$  has no interior points. By [27, Remark 8.1],  $G$  is metrizable and hence  $L^2(G)$  is separable. Since every  $\gamma \in \hat{G}$  appears in the left regular representation,  $\hat{G}$  is countable. Denote by  $Z$  the union of all these  $Z_\gamma$ 's. Then  $Z$  satisfies the requirement according to Theorem 5.11.  $\square$

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DEPARTMENT OF MATHEMATICS, SUNY AT BUFFALO, BUFFALO, NY 14260, USA  
E-mail address: hfli@math.buffalo.edu