Comments on completely continuous operators and Fredholm determinants

Hanfeng Li

Completely continuous operators

Let A be a Banach algebra with $|A^m| \supsetneqq \{1\}$, where

$$A^m := \{a \in A^* : |ab| = |a|\,|b| \text{ for all } b \in A\}$$

is the set of all multiplicative units in A (equivalently, $A^m = \{a \in A^* : |a|\,|a^{-1}| = 1\}$). Then for any Banach modules M, N over A, an A-linear map $L : M \to N$ is continuous iff $\sup_{m \neq 0} \frac{|L(m)|}{|m|} < \infty$. Let $\mathscr{B}_A(M, N)$ be the space of such maps. With the norm $|L| := \sup_{m \neq 0} \frac{|L(m)|}{|m|}$, $\mathscr{B}_A(M, N)$ is a Banach A-module.

For a Banach A-module M, let $M^\vee = \mathscr{B}_A(M, A)$ be the continuous dual of M. As in the algebraic case, there is a natural pairing

$$\begin{array}{rcl} (M^\vee \otimes_A N) \times M & \longrightarrow & N \\ (f \otimes d, m) & \mapsto & f(m)d \end{array}$$

It induces an A-linear map $\varphi : M^\vee \otimes_A N \to Hom_A(M, N)$. One checks that the image is actually in $\mathscr{B}_A(M, N)$, and φ is contractive with respect to the projective seminorm on $M^\vee \otimes_A N$. Here for $c \in M^\vee \otimes_A N$, the projective seminorm is defined as $|c| := inf(max_i |f_i|\,|d_i|)$, where inf is taken over all representations $c = \sum_i f_i \otimes d_i$.

Definition 1. Linear maps in $\varphi(M^\vee \otimes_A N)$ are said to be *of finite rank*. Linear maps in the closure of $\varphi(M^\vee \otimes_A N)$ are called *completely continuous operators*.

Let $\mathscr{L}_A(M, N)$ be the set of completely continuous operators from M to N. One checks that $\mathscr{B}_A(N, F)\mathscr{L}_A(M, N)\mathscr{B}_A(E, M) \subseteq \mathscr{L}_A(E, F)$.

Proposition 1. *Suppose M and N are ONable with ON basis $\{e_i\}_I$ and $\{d_j\}_J$ respectively. For $L \in \mathscr{B}_A(M, N)$, suppose $L(e_i) = \sum_{j \in J} a_{j,i} d_j$. Then L is completely continuous iff*

$$\lim_{j \to \infty} \underset{i \in I}{Sup} |a_{j,i}| = 0 \tag{1}$$

Proof. To simplify the notation, let us assume that J is countable. Let P_n be the projection onto the submodule generated by $d_j, 1 \leq j \leq n$.

"$\Leftarrow$": $\lim_{j \to \infty} \underset{i \in I}{Sup} |a_{j,i}| = 0$ implies that $\lim_{n \to \infty} |L - P_n \circ L| = 0$. Since $P_n \circ L \in \sum_{j \leq n} M^\vee \otimes e_j$ is of finite rank, we see that L is completely continuous.

"$\Rightarrow$": Note that the set of continuous A-linear operators satisfying (1) is closed. Then our conclusion follows from the fact that operators of finite rank satisfy (1) trivially. □

Proposition 2. *Suppose A is Noetherian, and both M and N are ONable, then $L \in \mathscr{B}_A(M,N)$ is completely continuous iff L is a limit of operators L_n in $\mathscr{B}_A(M,N)$ such that each L_n ranges in a finitely generated submodule of N.*

Proposition 2 follows directly from:

Lemma 1. *Under the condition of Proposition 2, $L \in \mathscr{B}_A(M,N)$ is of finite rank iff L ranges in a finitely generated submodule of N.*

Proof. We only need to show "$\Leftarrow$". Still let $\{e_i\}_I$ and $\{d_j\}_j$ be ONbasis of M and N respectively. Say $h_1, \cdots, h_n \in N$ and $L(M) \subseteq \sum_{j=1}^n Ah_j$. Then $L(e_i) = \sum_{j=1}^n b_{j,i}h_j$ for some $b_{j,i} \in A$. Notice that $b_{j,i}$ may not be unique since we don't know whether $\sum_{j=1}^n Ah_j$ is free or not. We'll show that we can actually choose $b_{j,i}$ in a way such that $\sup_{j,i}|b_{j,i}| < \infty$. Then $f_j(e_i) = b_{j,i}$ determines an element in $M^\vee$ and $L = \varphi(\sum_{j=1}^n f_j \otimes d_j)$ is of finite rank.

Let A^n be the free rank-n A-module with ONbasis $g_1, \cdots, g_n$. Let $\rho : A^n \to \sum_{j=1}^n Ah_j$ be the A-linear map sending g_j to h_j. Then ρ is continuous and $ker(\rho)$ is closed. So $E := A^n/ker(\rho)$ is a Banach A-module equipped with the quotient norm, and ρ induces a bijective map $\tilde{\rho} : E \to \sum_{j=1}^n Ah_j$. We'll show that $(\tilde{\rho})^{-1}$ is continuous. Then for each i, we have $|(\tilde{\rho})^{-1}(L(e_i))| \leq |(\tilde{\rho})^{-1}|\,|L|$. Hence we can find $\sum_{j=1}^n b_{j,i}g_j \in A^n$ with

$$\sum_{j=1}^n b_{j,i}g_j = (\tilde{\rho})^{-1}(L(e_i)) \text{ mod } ker(\rho)$$

and

$$max_{1\leq j\leq n}|b_{j,i}| = |\sum_{j=1}^n b_{j,i}g_j| \leq |(\tilde{\rho})^{-1}(L(e_i))| + 1 \leq |(\tilde{\rho})^{-1}|\,|L| + 1$$

This gives us $L(e_i) = \sum_{j=1}^n b_{j,i}h_j$ as wanted.

Notice $\tilde{\rho}$ is continuous. By Banach's open mapping theorem, the continuity of $(\tilde{\rho})^{-1}$ is equivalent to the completness of $\sum_{j=1}^n Ah_j$, which is equivalent to the closedness of $\sum_{j=1}^n Ah_j$ in N.

Say $h_j = \sum_k c_{j,k}d_k$. Let F_m be the ideal of A generated by $c_{j,k}, 1 \leq j \leq n, 1 \leq k \leq m$. Then $F_1 \subseteq F_2 \subseteq \cdots$ and hence there exists m' such that $F_{m'} = F_{m'+1} = \cdots$. Notice that $ker(P_m \circ \rho)$ is the annihilator of F_m, and $ker(\rho)$ is the annihilator of $\cup_m F_m$. So $ker(P_{m'} \circ \rho) = ker(\rho)$ and hence $P_{m'} \circ \rho$ also induces a bijectice map $\tilde{\rho_{m'}} : E \to \sum_{j=1}^n AP_{m'}(h_j)$. Since A is Noetherian, any submodule of a finitely generated Banach module is closed. So $\sum_{j=1}^n AP_{m'}(h_j) \subseteq \sum_{k=1}^{m'} Ad_k$ is closed and hence complete. By Banach's open mapping theorem, $(\tilde{\rho_{m'}})^{-1}$ is continuous. Then $(\tilde{\rho})^{-1} = (\tilde{\rho_{m'}})^{-1} \circ P_{m'}$ is also continuous. □

Proposition 2 says that under its assumption, our definition of completely continuous operators coincides with that in BMF ("P-adic Banach spaces and families of modular forms"). One also checks that the proposition in Lecture 11 of Coleman, which is Proposition A5.2 in BMF, still holds in our definition.

One can also check easily that Lemma A1.4 in BMF holds: if $A \to B$ is a contractive homomorphism of Banach algebras, and $L \in \mathscr{L}_A(M,N)$, then $1\hat{\otimes}L \in \mathscr{L}_B(B\hat{\otimes}M, B\hat{\otimes}N)$. We don't know whether it holds in the definition of BMF.

Notice that in above proof, $\tilde{\rho}$ is continuous. By Banach's open mapping theorem, the continuity of $(\tilde{\rho})^{-1}$ is equivalent to the completeness of $\sum_{j=1}^{n} Ah_j$, which is equivalent to the closeness of $\sum_{j=1}^{n} Ah_j$ in N. When A is not Noetherian, free finite-rank submodules of N need not be closed.

Example 1. *Let $B := \mathbb{Q}_p < X_1, X_2, \cdots, Y_1, Y_2, \cdots >$ be the infinite Tate algebra. Let I be the closed ideal generated by $X_iY_j, i \leq j$. Let $A = B/I$ with the quotient norm. Then A has OB basis over $\mathbb{Q}_p$ consisting of monomials. Let $h = \sum_{j=1}^{\infty} p^j X_j d_j \in N$. Then Ah is a free A-module. Notice $|p^{-k}Y_k h| = |\sum_{j=k+1}^{\infty} p^{j-k} Y_k X_j d_j| = p^{-1}$, while $|p^{-k}Y_k| = p^k \to \infty$. So the inverse of the map $a \mapsto ah$ is not continuous and hence Ah is not closed.*

Since $\varphi : M^\vee \otimes_A N \to \mathscr{B}_A(M,N)$ is contractive with respect to the projective seminorm, it induces a contractive A-linear map $\hat{\varphi} : M^\vee \hat{\otimes}_A N \to \mathscr{B}_A(M,N)$. When N is ONable, one can check that $\hat{\varphi}$ is isometric. So in this case, we can identify $M^\vee \hat{\otimes}_A N$ and $\mathscr{L}_A(M,N)$.

Fredholm determinant and the topology

Proposition A2.6 and Corollary A2.6.1 of BMF hold without assuming A to be semi-simple:

Proposition 3. *Suppose M is orthonormizable and L is a complete continuous linear operator on M whose image is contained in a free submodule of finite-rank F such that there is a continuous projector from M onto F. Then $P_L(T) = det(1 - TL|F)$.*

Proposition 4. *Suppose M is orthonormizable and L is a complete continuous linear operator on M, then the Fredholm determinant $P_L(T)$ depends only on the topology of M, not the norm.*

Consequently, Corollary A2.6.2 of BMF holds without assuming A to be semi-simple.

Proposition 4 follows directly from continuity of Fredholm determinant and the following proposition, which is a generalization of part (ii) of Theorem A2.1 of BMF:

Proposition 5. *Suppose M is ONable and L is a complete continuous linear operator on M. Also suppose the image of L in M is contained in a free submodule F of finite rank over A of M such that the norm on F is equivalent to a norm with an A-basis of F as an ON basis, then*

$$P_L(T) = det(1 - TL|F) \tag{2}$$

Proof. The proof is merely to write down the matrices and do routine computations. Let $e_1, \cdots, e_n$ be the orthonormal basis of F in the above assumption. For simplicity of notation, we assume that E has a countable ON basis. Let $\vec{b_i} = (b_{i1}, b_{i_2}, \cdots)^T$ be the coordinates of e_i. Then the matrix of L is of the form

$$(\sum_{i=1}^{n} a^{1i}\vec{b_i}, \sum_{i=1}^{n} a^{2i}\vec{b_i}, \cdots) \tag{3}$$

where $a^{ji} \in A$. Since the norm on F is equivalent to the one with $e_1, \cdots, e_n$ as an ON basis, $L(e_k)$ has coordinates $(\sum_{j=1}^{\infty} a^{j1} b_{kj})\vec{b_1} + \cdots + (\sum_{j=1}^{\infty} a^{jn} b_{kj})\vec{b_n}$. Hence the matrix of $L|F$ with respect to the basis $e_1, \cdots, e_n$ is (c_{ik}) with

$$c_{ik} = \sum_{j=1}^{\infty} a^{ji} b_{kj} \tag{4}$$

Using (3) and (4), one can compute $P_L(T)$ and $det(1 - TL|F)$ directly. The results will be algebraic formulas in a^{ji} and b_{kj}. Since (2) is known to be true when A is a field, we see that it holds for any Banach algebra A. □

Proposition 3 follows from Proposition 5 and the following two lemmas:

Lemma 2. *Suppose a Banach A-module M is a free A-module of finite rank. Then for any basis $e_1, \cdots, e_n$, the norm on M is equivalent to the norm with $e_1, \cdots, e_n$ as an orthonormal basis.*

Proof. Let $\| \cdot \|_1$ be the norm on M. Let $\| \cdot \|_2$ be the norm with $e_1, \cdots, e_n$ as an ON basis. Then for any $a = \sum a_i e_i$,

$$\| a \|_1 \leq \max_i (\| a_i \| \cdot \| e_i \|_1) \leq (\max_i \| e_i \|_1) \| a \|_2$$

So $\| \cdot \|_1 \leq (\max_i \| e_i \|_1) \| \cdot \|_2$. Hence the identity map $(M, \| \cdot \|_2) \to (M, \| \cdot \|_1)$ is continuous. Clearly it's also bijective. Since both $\| \cdot \|_1$ and $\| \cdot \|_2$ are Banach norms on M, by Banach's open mapping theorem, $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent to each other. □

Lemma 3. *Let F be an A-submodule of M such that there is a continuous projector from M onto F. Then F is closed in M.*

Proof. Let P be the projector. Just notice that F is the kernel of the continuous endomorphism $P - id$ of M. □