Description for undergraduates of Prof. Hemmer's research area.

1. Introduction

My research is in an area of mathematics known as group representation theory. This is a mixture of group theory and linear algebra. In an undergraduate abstract algebra course one encounters many types of groups, for example cyclic groups, dihedral groups, symmetric groups, matrix groups and various subgroups of these. Sometimes the same group can be thought of in several different ways. For example the symmetric group S_3 is all permutations of the set $\{1, 2, 3\}$. It has six elements. So we can think of it as acting on this set by permutations. However this group is isomorphic to the dihedral group D_3, the symmetries of an equilateral triangle, so we can instead think of the group as acting on this triangle by rotations and reflections. Below we will encounter this group in a third setting, as a collection of 2×2 matrices. These matrices can be thought of as linear transformations of a two-dimensional vector space, so the group S_3 can be thought of as acting on this vector space by linear transformations. It is this last setting which is the subject of representation theory. Given a group G, how can it act on a vector space by linear transformations?

Students wishing to write senior honors theses with me should come see me as early as possible, certainly during your junior year if possible, so we can begin considering potential topics. Doctoral students interested in working we me should feel free to stop in anytime to discuss their options.

2. What is a representation?

Perhaps the most important group of all is the general linear group:

$$GL_n(F) = \{\text{invertible } n \times n \text{ matrices with entries in the field } F\}.$$

Observe that this is not a single group, but an infinite family of different groups! One can change the dimension n. Also one can vary the field F, perhaps using the complex numbers $\mathbb{C}$, the real numbers $\mathbb{R}$ or even a finite field, in which case $GL_n(F)$ is a finite group. Representation theory is the study of how one can realize a given group as a group of matrices.

Suppose G is a group. To each element $g \in G$, we try to assign a matrix $A_g \in GL_n(F)$ in such a way that the multiplication in G agrees with the multiplication of the corresponding matrices in $GL_n(F)$. By "agrees" we mean the following:

$$\text{If } g_1, g_2, g_3 \in G \text{ and } g_1 g_2 = g_3, \text{ then } A_{g_1} A_{g_2} = A_{g_3} \text{ in } GL_n(F). \tag{1}$$

Equation 1 says that the map $\psi : G \to GL_n(F)$ defined by $\psi(g) = A_g$ is a *group homomorphism* from G to $GL_n(F)$. Recall a group homomorphism means $\psi(g_1 g_2) = \psi(g_1)\psi(g_2)$ for all $g_1, g_2 \in G$. We call n the *dimension* of the representation ψ. Different choices of groups G and different choices for n and F lead to very different theories. Below I will focus on a concrete example, namely the symmetric group S_3. This group has six elements and is the smallest nonabelian group.

3. Some representations of S_3

Consider the symmetric group S_3. It is generated by the transpositions $(1, 2)$ and $(2, 3)$. Let's construct a two-dimensional representation $\phi : S_3 \to GL_2(F)$ by :

$$\phi((1,2)) = A_{(1,2)} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \; \phi((2,3)) = A_{(2,3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What about ϕ on the other four elements of S_3? We already know it! Equation 1 is all we need. Every permutation in S_3 can be obtained from $(1, 2)$ and $(2, 3)$. For example $(1, 2)(2, 3) = (1, 2, 3)$, so applying equation 1, the element $(1,2,3)$ must map to

$$\phi((1,2,3)) = A_{(1,2,3)} = A_{(1,2)}A_{(2,3)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

In this way we also obtain:

$$A_{(1,3)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \ A_{(1,3,2)} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \ A_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have the complete representation. One can show that the identity $e \in G$ always maps to the $n \times n$ identity matrix. Notice that in this example the homomorphism is one to one, thus the six matrices above give another way to write the group S_3, in this case as a subgroup of $GL_2(F)$. Thus we have realized this group in 3 ways! (permutations, symmetries, linear transformations)

Some representations are not one to one. For example we have the:

Trivial representation: Maps every element of S_3 to the 1×1 identity matrix (1). Notice that equation 1 is satisfied trivially.

Sign representation: Maps each even permutation in S_3, i.e. $\{e, (1,2,3), (1,3,2)\}$, to (1) and each odd permutation in S_3, i.e. $\{(1,2), (1,3), (2,3)\}$, to (-1).

Observe that every group G has a one-dimensional trivial representation. Also observe that the one-dimensional sign representation is defined for every symmetric group S_n, not just for S_3.

4. **Irreducible Representations**

In linear algebra one learns that matrices can be used to represent linear transformations. Let F^n denote the vector space of n-long column vectors with entries in F. Then an $n \times n$ matrix acts on a column vector by multiplication on the left. Thus we can think of a representation $\psi : G \to GL_n(F)$ as assigning to each element of G a linear transformation from F^n to itself. This gives us an alternate way to think of a representation, that the group is *acting* on a vector space by linear transformations. Let's consider our two-dimensional representation of S_3 again:

$$\phi((1,2)) = A_{(1,2)} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \ \phi((2,3)) = A_{(2,3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Because these matrices act on the vector space F^2 then S_3 also acts on F^2 via ϕ. For example, let's compute how the generator $(1,2)$ acts on the column vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$:

$$(1,2) \circ \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \phi((1,2)) \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

In general, given an n-dimensional representation $\psi : G \to GL_n(F)$, we want to determine the structure of F^n under the action of G. We define:

Definition: Let $\psi : G \to GL_n(F)$ be an n-dimensional representation of G. Say ψ is *irreducible* if there does not exist a subspace $0 \subsetneq V \subsetneq F^n$ which is preserved by G, i.e. such that $g \circ v \in V \, \forall \, g \in G, \, \forall \, v \in V$.

Thus one problem is to determine all the irreducible representations of a group G. There is an important point to be made first. In mathematics when one defines a new object, for example a group representation, one also needs to define when two such objects are "the same". The group G is acting on a vector space by linear transformations. In order to express these transformations as matrices we must first choose a basis. Changing the basis gives a different collection of matrices $\{A_g\}$. However the action of G is fundamentally the same, just looked at from a different perspective. Thus we define:

Definition: Say two representations $\psi, \rho : G \to GL_n(F)$ are *isomorphic* if the matrices one gets from ψ are the same as those from ρ, but written in a different basis of F^n.

So really we want to know the isomorphism classes of irreducible representations. In general this is a very difficult problem.

5. **Changing the field**

So far in this discussion we have completely ignored the role played by the choice of field F. Notice that our matrices above contained only $0, 1, -1$. Perhaps they are in the field of integers mod p? Perhaps in the rational, real or complex numbers? The choice makes an enormous difference.

Theorem 1. *Let ϕ be the two-dimensional representation of S_3 given in (2). If $F = \mathbb{C}$ then ϕ is irreducible.*

Proof. This is a simple linear algebra exercise. Since we have a two-dimensional representation, the only possibilities for an invariant subspace V are one-dimensional. Thus you need to check there is no one-dimensional subspace which is left invariant by $A_{(1,2)}$ and $A_{(2,3)}$. A fancier way to say this is that $A_{(1,2)}$ and $A_{(2,3)}$ have no common eigenvector. □

Now let's consider ϕ but suppose the field has 3 elements, thus $F = \{0, 1, 2\}$ where the operations are all modulo 3. We say F has *characteristic three*. Let's consider the one-dimensional subspace of F^2 spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We can calculate the action of $(1, 2)$ and $(2, 3)$ on this vector:

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It doesn't appear that this subspace is preserved but look again! In the field F we have $1+1+1 = 3 = 0$! Thus $-2 = 1$ so the subspace really is invariant. Our representation which was irreducible over the complex numbers is no longer irreducible in characteristic three! If we change to the basis

$$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

then (keeping in mind that $0 = 3$) our matrices become:

$$(1, 2) \to \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad (2, 3) \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

In this basis it is clear that there is a one-dimensional subspace left invariant by both matrices. If you are familiar with quotient vector spaces you will notice that both $(1, 2)$ and $(2, 3)$ act as -1 on the quotient. Thus we have:

Theorem 2. *Over a field of characteristic three, the representation ϕ is no longer irreducible. Instead it is built up out of two one-dimensional representations, namely the sign and the trivial representation. It has a one-dimensional invariant subspace on which it acts trivially. It acts on the* *quotient by the one-dimensional sign representation. It does not, however, have a one-dimensional invariant subspace in which it acts by the sign representation.*

In joint work with Nakano we recently proved a surprising theorem about the representation theory of the symmetric group, it was surprising because there were many known counterexamples! However they were all over fields of characteristic two or three. In characteristics $p > 3$ the theorem holds! The obstruction in characteristic three was (after much work!) determined to be precisely the representation ϕ. In any larger characteristic, one cannot build a two-dimensional representation out of the trivial and the sign representations which has the properties discussed in Theorem 2.

6. **What questions does a representation theorist ask?**

My favorite group to consider is the symmetric group. Even for such an important group, there are many many unsolved problems in its representation theory, and this is a very active field of research. Here are some typical problems in representation theory, along with some comments on the current status of these problems for the symmetric group.

Question 1: Can we classify all the irreducible representations of a given group G when the field F is the complex numbers $\mathbb{C}$?

When G is a symmetric group S_n the answer is known and closely related to the field of combinatorics. We actually have done so for S_3 already! There are three irreducible representations, the two-dimensional ϕ constructed above and the two one-dimensional representations; the trivial and the sign. It is no coincidence that the S_3 has six elements and $6 = 1^2 + 1^2 + 2^2$!

Question 2: Given an irreducible representation for G over $\mathbb{C}$, what does it look like if we consider it over a field of finite characteristic p?

Of course not every representation will have integer entries in the matrix, but there is still a procedure for "reducing" the entries mod p. For example the two-dimensional irreducible representation ϕ for S_3 was no longer irreducible in characteristic 3. Only in the past year has a complete answer been obtained to when this phenomenon can occur in the case $G = S_n$.

Question 3: What are the irreducible representations over a field F of characteristic p?

Here the answer is much much harder. For the symmetric group S_n, not even the dimensions of these irreducibles are known.

Question 4: In what ways can we build up larger dimensional representations out of the irreducible representations?

When $F = \mathbb{C}$ this question is easy, is a certain sense we can't, every representation breaks up nicely into irreducibles. When F has finite characteristic this question becomes very difficult.