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A ROW REMOVAL THEOREM FOR THE EXT^1 QUIVER OF SYMMETRIC GROUPS AND SCHUR ALGEBRAS

DAVID J. HEMMER

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ABSTRACT. In 1981, G. D. James proved two theorems about the decomposition matrices of Schur algebras involving the removal of the first row or column from a Young diagram. He established corresponding results for the symmetric group using the Schur functor. We apply James' techniques to prove that row removal induces an injection on the corresponding Ext^1 between simple modules for the Schur algebra.

We then give a new proof of James' symmetric group result for partitions with the first part less than p. This proof lets us demonstrate that first-row removal induces an injection on Ext^1 spaces between these simple modules for the symmetric group. We conjecture that our theorem holds for arbitrary partitions. This conjecture implies the Kleshchev-Martin conjecture that $\mathrm{Ext}^1_{\Sigma_r}(D_\lambda, D_\lambda) = 0$ for any simple module D_λ in characteristic $p \neq 2$. The proof makes use of an interesting fixed-point functor from Σ_r-modules to Σ_{r-m}-modules about which little seems to be known.

1. INTRODUCTION

We will assume familiarity with representation theory of the symmetric group Σ_r as found in [7] and of the Schur algebra $S(n,r)$ as found in [4]. We write $\lambda \vdash r$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ a partition of r. Let $\mathbf{N} = \{0, 1, 2, \ldots\}$. We do not distinguish between λ and its Young diagram:

$$\lambda = \{(i,j) \in \mathbf{N} \times \mathbf{N} \mid j \leq \lambda_i\}.$$

A partition λ is *p-regular* if there is no i such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$. A partition is *p-restricted* if its conjugate partition, denoted λ', is p-regular.

We write $\lambda \trianglerighteq \mu$ for the usual dominance order on partitions. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we write $\overline{\lambda}$ for λ with its first row removed, i.e.

$$\overline{\lambda} = (\lambda_2, \ldots, \lambda_k) \vdash r - \lambda_1.$$

We write $\hat{\lambda}$ for λ with its first column removed, i.e.

$$\hat{\lambda} = (\lambda_1 - 1, \lambda_2, -1, \ldots, \lambda_k - 1) \vdash r - k.$$

The complex simple Σ_r-modules are the Specht modules $\{S^\lambda \mid \lambda \vdash r\}$. The simple modules in characteristic p can be indexed by p-restricted partitions or by

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p-regular partitions. If we let S_λ denote $(S^\lambda)^*$, then both

$$\{D^\lambda := S^\lambda/\mathrm{rad}(S^\lambda) \mid \lambda \text{ is } p\text{-regular}\}$$

and

$$\{D_\lambda = \mathrm{soc}(S^\lambda) \mid \lambda \text{ is } p\text{-restricted}\}$$

are complete sets of nonisomorphic simple Σ_r-modules in characteristic p. The two indexings are related by $D^\lambda \cong D_{\lambda'} \otimes \mathrm{sgn}$, where sgn denotes the one-dimensional signature representation.

For a module M and a simple module S we let $[M : S]$ denote the composition multiplicity of S in M. James proved the following row removal theorems in [6].

Theorem 1.1 (James). *Let λ and μ be partitions of r with $\lambda_1 = \mu_1 = m$, and let λ be p-restricted. Then $[S_\mu : D_\lambda] = [S_{\overline{\mu}} : D_{\overline{\lambda}}]$.*

Theorem 1.2 (James). *Let λ and μ be partitions of r with $\lambda_1 = \mu_1 = m$, and let λ be p-regular. Then $[S^\mu : D^\lambda] = [S^{\overline{\mu}} : D^{\overline{\lambda}}]$.*

From these results, James deduced corresponding results for first-column removal by tensoring with the sign representation.

We apply James' technique to prove that row removal gives an injection on the corresponding Ext^1 space between simple modules for the Schur algebra. Then we present a new proof of James' theorem for symmetric groups in the case when $\lambda_1 < p$. We apply this proof, together with a theorem of Kleshchev and Sheth, to prove the corresponding Ext^1 result for symmetric groups.

We remark that Theorems 1.1 and 1.2 have been generalized to removing multiple rows and columns by Donkin [1, 2]; however, we will not use these generalizations. We would like to thank Gordon James and Dan Nakano for useful discussions about this paper.

2. An Ext^1-theorem for Schur algebras

Let k be an algebraically closed field of characteristic $p > 0$, and let $n \geq r$. The simple, complex, polynomial representations of GL_n of homogeneous degree r are the Weyl modules $\{V(\lambda) \mid \lambda \vdash r\}$. Over k the Weyl modules may no longer be simple, but each Weyl module has a simple head denoted by $L(\lambda)$.

The category of polynomial representations of $GL_n(k)$ of homogeneous degree r is equivalent to the category mod-$S(n,r)$. The Schur functor is an exact functor from mod-$S(n,r)$ to mod-$k\Sigma_r$ that maps $V(\mu)$ to S_μ and, for μ p-restricted, maps $L(\mu)$ to D_μ. Using this functor (specifically Theorem 6.6g in [4]), it sufficed for James to prove Theorem 1.1 for $S(n,r)$, namely that $[V(\mu) : L(\lambda)] = [V(\overline{\mu}) : L(\overline{\lambda})]$.

To do this, James defined [6, p. 117] an idempotent $\eta \in S(n,r)$ such that $\eta S(n,r)\eta$ contains a subalgebra isomorphic to $S(n-1, r-m)$. Let

$$\mathcal{F}_m : \text{mod-}S(n,r) \to \text{mod-}S(n-1, r-m)$$

be defined by $\mathcal{F}_m(U) = \mathrm{Res}^{\eta S(n,r)\eta}_{S(n-1,r-m)}(\eta U)$. James showed:

Theorem 2.1 ([6, pp. 117-120]). *Let $\mu \vdash r$ with $\mu_1 = m$. Then:*

(i) $\mathcal{F}_m(V(\mu)) \cong V(\overline{\mu})$.
(ii) $\mathcal{F}_m(L(\mu)) \cong L(\overline{\mu})$.
(iii) $\mathcal{F}_m(\mathrm{rad}(V(\mu)) \cong \mathrm{rad}(V(\overline{\mu}))$.

The only other tool we need for our first theorem is the following result.

Lemma 2.2 ([8, II 2.14])**.** *Suppose λ and μ are partitions of r and $\mu \not\trianglerighteq \lambda$. Then*

$$\mathrm{Ext}^1_{S(n,r)}(L(\lambda), L(\mu)) \cong \mathrm{Hom}_{S(n,r)}(\mathrm{rad}(V(\lambda), L(\mu))).$$

Although Lemma 2.2 is actually stated in Jantzen's book for Ext^1 in the category of rational $GL_n(k)$-modules, this is known to agree with Ext^1 in mod-$S(n,r)$ by [3, 2.2d]. We can now prove:

Theorem 2.3. *Let λ and μ be partitions of r with $\lambda_1 = \mu_1 = m$. Then there is an injection*

$$0 \to \mathrm{Ext}^1_{S(n,r)}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{S(n-1,r-m)}(L(\overline{\lambda}), L(\overline{\mu})).$$

Proof. Since the modules $L(\tau)$ are self-dual we can assume $\mu \not\trianglerighteq \lambda$. The functor $\mathcal{F}_m$ is exact, so any copies of $L(\mu)$ in the second radical layer of $V(\lambda)$ will map to copies of $L(\overline{\mu})$ in the second radical layer of $V(\overline{\lambda})$, by Theorem 2.1. The injection then follows from Lemma 2.2. It is an injection rather than an isomorphism because other copies of $L(\overline{\mu})$ may "float up" to the second radical layer of $V(\overline{\lambda})$. There is no assurance that $\mathcal{F}_m$ preserves the radical layers of $V(\lambda)$. We know only that it preserves the radical. □

We remark that the situation for column removal is much simpler. Namely, if λ and μ have m parts, then

$$\mathrm{Ext}^1_{S(n,r)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{S(n-m,r-m)}(L(\hat{\lambda}), L(\hat{\mu}))$$

is clear by tensoring with the determinant representation.

3. Symmetric group preliminary results

We desire a result like Theorem 2.3 for the symmetric group. To do so it is necessary to first reprove James' results without using the Schur functor. Then we can use a theorem of Kleshchev and Sheth to play the role of Lemma 2.2.

We begin by gathering information on the modules S^λ, D^λ, S_λ and D_λ, and establish our notation. For more details on results presented in this section see [7]. For $\lambda \vdash r$, a *λ-tableau* is one of the $r!$ arrays of integers obtained by replacing each node of λ bijectively with the integers $1, 2, \ldots, r$. There is a natural action of Σ_r on the set of tableaux. For a tableau t, let $R(t)$ denote the set of permutations in Σ_r keeping the rows of t fixed setwise; and similarly let $C(t)$ denote the column stabilizer. A tableau is *standard* if its rows and columns are increasing.

For a λ-tableau t, define the signed column sum

$$\kappa_t = \sum_{\sigma \in C(t)} \mathrm{sgn}(\sigma)\sigma$$

and the row sum

$$\rho_t = \sum_{\sigma \in R(t)} \sigma.$$

There is an equivalence relation on λ-tableaux given by $t_1 \sim t_2$ if $t_2 = \pi t_1$ for some $\pi \in R(t_1)$. The equivalence classes are called *λ-tabloids* and are denoted by $\{t\}$. There is a natural action of Σ_r on the set of λ-tabloids, and this permutation module is denoted by M^λ. For a tableau t, the corresponding *polytabloid* is defined as $e_t := \kappa_t\{t\} \in M^\lambda$. The following theorem is fundamental.

Theorem 3.1 ([7, Thm. 8.4])**.** *$\{e_t \mid t$ is a standard λ-tableau$\}$ is a basis for S^λ.*

In particular, S^λ is a submodule of M^λ.

We will find it useful to identify the various $k\Sigma_r$-modules as left ideals in the group algebra $k\Sigma_r$. In particular (see for example [10, Thm. 4.2.2]):

Lemma 3.2. *Choose any λ-tableau T. Then*
- (i) $M^\lambda \cong k\Sigma_r\rho_T$;
- (ii) $S^\lambda \cong k\Sigma_r\kappa_T\rho_T$;
- (iii) $S_\lambda \cong k\Sigma_r\rho_T\kappa_T$;
- (iv) $D^\lambda \cong k\Sigma_r\kappa_T\rho_T\kappa_T$;
- (v) $D_\lambda \cong k\Sigma_r\rho_T\kappa_T\rho_T$;
- (vi) *a basis for S_λ is given by $\{\pi\rho_T\kappa_T \mid \pi T \textit{is standard}\}$.*

If λ' denotes the conjugate partition to λ, the following is well known.

Lemma 3.3.
- (i) *For λ p-restricted,* $D_\lambda \cong \mathrm{head}(S_\lambda) \cong \mathrm{soc}(S^\lambda)$.
- (ii) *For λ p-regular,* $D^\lambda \cong \mathrm{head}(S^\lambda) \cong \mathrm{soc}(S_\lambda)$.

To obtain his results on decomposition numbers from Theorem 2.1, James applied the following lemma.

Lemma 3.4 ([4, Lemma 6.6b])**.** *Let S be an algebra and $\eta \in S$ an idempotent. Suppose V is an S-module and F is a simple S-module such that $\eta F \neq 0$. Then ηF is a simple $\eta S\eta$-module and $[V : F] = [\eta V : \eta F]$.*

As in James' proof for $S(n,r)$, we will find an idempotent η in $k\Sigma_r$ such that $\eta k\Sigma_r\eta$ has a subalgebra isomorphic to $k\Sigma_{r-m}$, and such that $\lambda_1 = m < p$ implies $\eta S^\lambda \cong S^{\overline{\lambda}}$ and $\eta D_\lambda \cong D_{\overline{\lambda}}$ as $k\Sigma_{r-m}$-modules.

Our idempotent exists only when $m < p$, which (coincidentally?) is the only case where the symmetric group result corresponding to Lemma 2.2 is known. Thus we can obtain a result on Ext^1 for symmetric groups in this case.

4. Determining the row removal functor on S^λ

Our main result in the next two sections is a new proof of a weaker version (Theorems 4.1, 4.2) of James' theorems, which is entirely contained in symmetric group theory. This proof will lead to new results in Section 6. Henceforth we assume $p > 2$. This eliminates problems with semistandard homomorphisms (see [7, 13.14]) and is not relevant to our results for $k\Sigma_r$, because for $p = 2$ the only $\lambda \vdash r$ with $\lambda_1 < 2$ is (1^r).

Theorem 4.1. *Let $\lambda, \mu \vdash r$ with $\lambda_1 = \mu_1 = m < p$. Then* $[S_\mu : D_\lambda] = [S_{\overline{\mu}} : D_{\overline{\lambda}}]$.

Tensoring with the sign representation gives:

Theorem 4.2. *Let $\lambda, \mu \vdash r$ have m parts, for $m < p$. Then* $[S^\mu : D^\lambda] = [S^{\hat{\mu}} : D^{\hat{\lambda}}]$.

Henceforth when we write Σ_m it will be acting on $\{1, 2, \ldots, m\}$. When we write Σ_{r-m} it will be acting on $\{m+1, m+2, \ldots, r\}$ and will be embedded in Σ_r in the natural way. Similarly, when $\overline{\lambda} \vdash r - m$, a $\overline{\lambda}$-tableau will be labelled with the numbers $\{m+1, m+2, \ldots, r\}$ rather than by $\{1, 2, \ldots, r-m\}$.

We begin by defining

$$\eta = \frac{1}{m!}\sum_{\sigma\in\Sigma_m} \sigma.$$

Notice that η is a nonzero idempotent (and η exists only because $m < p$). So for U a $k\Sigma_r$-module, ηU is a $\eta k\Sigma_r\eta$-module. The group algebra $k\Sigma_{r-m}$ sits naturally inside $\eta k\Sigma_r\eta$ (namely as $\eta k\Sigma_{r-m}\eta$) and η commutes with Σ_{r-m}. So we can regard ηU as a $k\Sigma_{r-m}$-module by restriction. Left multiplication by η then restriction to $k\Sigma_{r-m}$ gives an exact functor

$$\mathcal{F}_m : \text{mod-}k\Sigma_r \to \text{mod-}k\Sigma_{r-m}$$

defined by $\mathcal{F}_m := \mathrm{Res}^{\eta k\Sigma_r\eta}_{k\Sigma_{r-m}}(\eta U)$.

We would like to see how this functor behaves on the Specht, dual Specht, and simple modules. In this section we determine how it acts on Specht modules and use the information to determine which simple modules it annihilates. In the next section we consider dual Specht modules and use the information to determine $\mathcal{F}_m$ on simple modules. We begin with an easy lemma.

Lemma 4.3. *If $\lambda_1 < m$, then $\mathcal{F}_m(S^\lambda) = 0$.*

Proof. Since $\lambda_1 < m$, any standard tableau t must have a column with more than one entry from $\{1, 2, \ldots, m\}$. This implies $\eta\kappa_t = 0$; so $\eta e_t = 0$. Thus, by Theorem 3.1, η annihilates a basis for S^λ; hence $\eta S^\lambda = 0$. □

Next we determine which simple modules are annihilated by $\mathcal{F}_m$.

Lemma 4.4. *$\mathcal{F}_m(D_\lambda) = 0$ if and only if $\lambda_1 < m$.*

Proof. Suppose $\lambda_1 < m$. Then by Lemma 4.3, $\eta S^\lambda = 0$. Since $D_\lambda = \mathrm{soc}(S^\lambda)$, we know $\eta D_\lambda = 0$. Conversely suppose $\lambda_1 \geq m$. Choose a λ-tableau T with $\{1, 2, \ldots, m\}$ in its first row, so that $\eta\rho_T = \rho_T$. We have

$$D_\lambda = k\Sigma_r\rho_T\kappa_T\rho_T$$

by Lemma 3.2(v), so

$$0 \neq \rho_T\kappa_T\rho_T = \eta\rho_T\kappa_T\rho_T \in \eta D_\lambda.$$

Thus $\eta D_\lambda \neq 0$. □

We will now determine how $\mathcal{F}_m$ acts on the Specht modules S^λ when $\lambda_1 = m$.

Theorem 4.5. *Let $\lambda_1 = m$, and let $t_1, \ldots, t_s$ be the standard λ -tableaux with first row $1\ 2\ \cdots\ m$. Then for any standard λ-tableau t,*

$$\eta e_t \neq 0 \text{ iff } t = t_i \text{ for some } i.$$

The set $\{\eta e_{t_i}\}_{i=1}^s$ is linearly independent. Furthermore, $\eta S^\lambda \cong S^{\overline{\lambda}}$ as Σ_{r-m}-modules, i.e. $\mathcal{F}_m(S^\lambda) \cong S^{\overline{\lambda}}$.

Proof. Suppose t is a standard λ-tableau but $t \notin \{t_i\}_{i=1}^s$. Then the first column of t must contain one, plus at least one other number $\leq m$. Thus $\eta\kappa_t = 0$, so $\eta e_t = 0$ as desired. To prove the linear independence of the set $\{\eta e_{t_i}\}_{i=1}^s$ we first recall the total order on λ-tabloids from [7, p. 10]:

Definition: $\{t_1\} < \{t_2\}$ if $\exists i$ such that:

(i) $\{i+1, \ldots, r\}$ are in the same row of $\{t_1\}$ and $\{t_2\}$.
(ii) i is higher in $\{t_1\}$ than in $\{t_2\}$.

By [7, Lemma 8.2], to prove the linear independence of the set $\{\eta e_{t_i}\}_{i=1}^s$, it is sufficient to prove the following.

Lemma 4.6. *The tabloid $\{t_i\}$ is the greatest (in the total order) tabloid that occurs in ηe_{t_i}, and it occurs with nonzero coefficient.*

Proof. It is easy to check that for any tableau T,

$$\kappa_{\pi T} = \pi \kappa_T \pi^{-1}. \tag{4.1}$$

Set $\{t\} = \{t_i\}$. Since $1\ 2\ \cdots\ m$ is the first row of $\{t\}$, we know $\Sigma_m \leq R(t)$, and hence

$$\sigma\{t\} = \{t\} = \{\sigma t\}\, \forall \sigma \in \Sigma_m; \tag{4.2}$$

so $\eta\{t\} = \{t\}$. Using this plus Equation (4.1) we determine

$$\begin{aligned}
\eta e_t &= \eta \kappa_t \{t\} \\
&= \left(\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sigma \right) \kappa_t \{t\} \\
&= \frac{1}{m!} \left(\kappa_t\{t\} + \sum_{1 \neq \sigma \in \Sigma_m} \kappa_{\sigma t} \{\sigma t\} \right) \\
&= \frac{1}{m!} \left(\sum_{\pi \in C(t)} \operatorname{sgn}(\pi) \pi \{t\} + \sum_{1 \neq \sigma \in \Sigma_m} \kappa_{\sigma t}\{\sigma t\} \right) \\
&= \frac{1}{m!}\{t\} + \frac{1}{m!} \sum_{1 \neq \pi \in C(t)} \operatorname{sgn}(\pi)\pi\{t\} + \frac{1}{m!} \sum_{1 \neq \sigma \in \Sigma_m} \kappa_{\sigma t}\{\sigma t\}. \qquad (4.3)
\end{aligned}$$

Now we will show that all the tabloids that occur in Equation (4.3) except $\{t\}$ are tabloids smaller than $\{t\}$, and that $\{t\}$ occurs with coefficient one. First we recall that by [7, 3.15]:

$$t \text{ standard and } 1 \neq \pi \in C(t) \Longrightarrow \{t\} > \{\pi t\}.$$

This implies that the tabloids in the second summand of (4.3) are all $< \{t\}$.

Next we consider the third summand in (4.3). These are tabloids of the form $\{s\} := \pi\{\sigma t\}$, for $\pi \in C(\sigma t)$. For $\pi = 1$ we get $\{s\} = \{\sigma t\} = \{t\}$. This yields another $(m! - 1)/m$ copies of $\{t\}$, bringing the coefficient of $\{t\}$ in (4.3) to one. Now suppose $\pi \neq 1$, and $\pi \in C(\sigma t)$. Then

$$\begin{aligned}
\{t\} &= \{\sigma t\} \\
&> \pi\{\sigma t\} \text{ since } 1 \neq \pi \in C(\sigma t).
\end{aligned}$$

Hence the remaining tabloids in (4.3) are also smaller than $\{t\}$, completing the proof of Lemma 4.6. □

To complete the proof of Theorem 4.5, it remains to show that $\eta S^\lambda \cong S^{\overline{\lambda}}$ as $k\Sigma_{r-m}$-modules. The linear independence of the set $\{\eta e_{t_i}\}$ proves that both modules have dimension equal to the number of standard $\overline{\lambda}$-tableaux. For a λ-tableau T, we let $\overline{T}$ denote T with its first row removed. Notice that the $\overline{\lambda}$-tableaux are all of the form $\overline{T}$ where T is a λ-tableau with first row $1\ 2 \cdots m$. We use Lemma 3.2 to deduce:

Lemma 4.7.

(i) $\{e_T \mid T$ *is a* λ*-tableau*$\}$ *is a spanning set for* S^λ.

(ii) $\{e_{\overline{T}} \mid T \in \{t_i\}_{i=1}^s\}$ *is a spanning set for* $S^{\overline{\lambda}}$.

We remark that $\{\overline{t}_i\}_{i=1}^s$ is a complete set of standard $\overline{\lambda}$-tableaux.

Now, following James [6], we define a $k\Sigma_{r-m}$-homomorphism $\Theta : M^\lambda \to M^{\overline{\lambda}}$ by

$$\Theta(\{t\}) = \begin{cases} 0 & \text{if the first row of } \{t\} \text{ is not } 1\ 2\ \cdots\ m, \\ \{\overline{t}\} & \text{otherwise.} \end{cases} \tag{4.4}$$

Notice that by Equation (4.2),

$$\Theta(\{\eta t\}) = \Theta(\{t\}). \tag{4.5}$$

It is a simple computation that for a λ-tableau t,

$$\Theta(e_t) = \begin{cases} \operatorname{sgn}(\pi) e_{\overline{\pi t}} & \text{if } \{1, 2, \ldots, m\} \text{ are in distinct columns of } t \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

where π is any element of $C(t)$ such that πt has $\{1, 2, \ldots, m\}$ in the first row.

By Lemma 4.7, the set $\{\eta e_T \mid T \text{ a } \lambda\text{-tableau}\}$ spans ηS^λ and $\{e_{\overline{T}} \mid T \in \{t_i\}_{i=1}^s\}$ spans $S^{\overline{\lambda}}$. Thus, Equations (4.5) and (4.6), plus the known equality of the dimensions, prove that restricting Θ to ηS^λ gives an isomorphism onto $S^{\overline{\lambda}}$. This completes the proof of Theorem 4.5. □

5. Determining $\mathcal{F}_m(S_\lambda)$ and $\mathcal{F}_m(D_\lambda)$

In order to determine $\mathcal{F}_m(D_\lambda)$ we will need to understand $\mathcal{F}_m(S_\lambda)$. The analysis will be similar to the last section, but the proof is subtly different. Namely, we will use other means to determine the dimension of ηS_λ before we determine its module structure. In Section 7 we will say more about why we believe the two cases are fundamentally different.

To begin, observe that for a $k\Sigma_r$-module U, the subspace ηU is exactly the space of fixed points U^{Σ_m} under Σ_m. Since Σ_m commutes with Σ_{r-m}, this space carries the structure of a $k\Sigma_{r-m}$-module. We will say more about this in Section 7, but for now we use it to prove:

Lemma 5.1. $\dim_k(\eta S_\lambda) = \dim_k S_{\overline{\lambda}}$.

Proof. As we remarked above,

$$\begin{aligned} \dim_k(\eta S_\lambda) &= \dim_k(S_\lambda)^{\Sigma_m} \\ &= \dim_k \operatorname{Hom}_{k\Sigma_m}(k, S_\lambda) \\ &= \dim_k \operatorname{Hom}_{k\Sigma_r}(S^\lambda, M^{(m,1^{r-m})}). \end{aligned}$$

But since $p > 2$, this is just the number of semistandard λ-tableaux of type $(m, 1^{r-m})$ by [7, 13.14], which (since $\lambda_1 = m$) is the number of standard $\overline{\lambda}$-tableaux, i.e. the dimension of $S_{\overline{\lambda}}$. □

Recall that since $S_\lambda \cong S^{\lambda'} \otimes \operatorname{sgn}$, we can consider S_λ as sitting inside $M^{\lambda'} \otimes \operatorname{sgn}$. So S_λ has a basis of the form $\{e_t \otimes \epsilon\}$ where t is a standard λ'-tableaux and ϵ is such that $\sigma\epsilon = \operatorname{sgn}(\sigma)\epsilon$. As in Theorem 4.5 we have:

Theorem 5.2. *Let $t_1, t_2, \ldots, t_s$ be the standard λ'-tableaux that have $1, 2, \ldots, m$ as their first column. Then:*

(i) $\eta(e_{t_i} \otimes \epsilon) = e_{t_i} \otimes \epsilon$.
(ii) $\{e_{t_i} \otimes \epsilon\}_{i=1}^s$ *is a basis for* ηS_λ.
(iii) $\eta S_\lambda \cong S_{\overline{\lambda}}$ *as* $k\Sigma_{r-m}$*-modules.*

Proof. Part (i) follows from the fact that for $\sigma \in \Sigma_m$ we have $\sigma(e_{t_i}) = \text{sgn}(\sigma)e_{t_i}$. Part (ii) follows from the fact that the e_{t_i} are linearly independent and the number of them is the same as the dimension of ηS_λ determined in Lemma 5.1. Part (iii) is then clear from looking at the action of Σ_{r-m} on the $\{e_{t_i}\}$. (See also Lemma 5.4(iii).) □

We remark that this situation is different from Theorem 4.5, because for t not in the set $\{t_1, t_2, \ldots, t_s\}$, we may still have $\eta e_t \neq 0$.

Let $\langle\ ,\ \rangle$ denote the bilinear form on M^λ defined by setting the basis of tabloids to be orthonormal. This immediately gives a form on $M^\lambda \otimes \text{sgn}$ as well. Let $\langle\langle\ ,\ \rangle\rangle$ denote the form similarly defined on $M^{\overline{\lambda}'} \otimes \text{sgn}$. Then $S_\lambda \subseteq M^{\lambda'} \otimes \text{sgn}$ and from [7] we have:

Lemma 5.3. *Let λ be p-restricted.*

(i) *For $S_\lambda \subseteq M^{\lambda'} \otimes \text{sgn}$ we have $\text{rad}(S_\lambda) = S_\lambda \cap (S_\lambda)^\perp$.*
(ii) *For $S_{\overline{\lambda}} \subseteq M^{\overline{\lambda}'} \otimes \text{sgn}$ we have $\text{rad}(S_{\overline{\lambda}}) = S_{\overline{\lambda}} \cap (S_{\overline{\lambda}})^\perp$.*

As in the proof of Theorem 4.5 we define $\Psi : M^{\lambda'} \otimes \text{sgn} \to M^{\overline{\lambda}'} \otimes \text{sgn}$ by

$$\Psi(\{t\} \otimes \epsilon) = \begin{cases} \{\hat{t}\} \otimes \epsilon & \text{if } 1, 2, \ldots, m \text{ are in distinct rows of } t, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\hat{t}\}$ is the $\overline{\lambda}'$-tableau obtained by removing the numbers $1, 2, \ldots, m$ from $\{t\}$. We let $\overline{\Psi}$ denote the restriction of Ψ to ηS_λ. The following lemma is a straightforward calculation. For part (iv) it suffices to check on the basis of ηS_λ given in Theorem 5.2(ii).

Lemma 5.4. *Let $t_1, t_2, \ldots, t_s$ and Ψ be as above. Let $\hat{t_i}$ be the standard $\overline{\lambda}'$-tableau given by removing the first column of t_i. Then:*

(i) *Ψ is a $k\Sigma_{r-m}$-homomorphism.*
(ii) $\Psi(\eta x) = \Psi(x) \quad \forall x \in M^{\lambda'}$.
(iii) $\Psi(e_{t_i} \otimes \epsilon) = m!(e_{\hat{t_i}} \otimes \epsilon)$. *In particular $\overline{\Psi}$ is an isomorphism from ηS_λ to $S_{\overline{\lambda}}$.*
(iv) *For any $x, y \in \eta S_\lambda$ we have $\langle x, y\rangle = m!(\langle\langle \overline{\Psi}(x), \overline{\Psi}(y)\rangle\rangle)$.*
(v) *For any $u, v \in S_\lambda$ we have $\langle u, v\rangle = \langle \eta u, \eta v\rangle$.*

Finally we can determine $\mathcal{F}_m(D_\lambda)$ as a $k\Sigma_{r-m}$-module:

Theorem 5.5. *$\eta D_\lambda \cong D_{\overline{\lambda}}$ as Σ_{r-m}-modules, i.e. $\mathcal{F}_m(D_\lambda) \cong D_{\overline{\lambda}}$.*

Proof. We know $\eta D_\lambda = \eta(S_\lambda/\text{rad}(S_\lambda))$. Since $D_{\overline{\lambda}} = S_{\overline{\lambda}}/\text{rad}(S_{\overline{\lambda}})$, it is enough to show that $\overline{\Psi}$ maps $\eta(\text{rad}(S_\lambda))$ onto $\text{rad}(S_{\overline{\lambda}})$.

So choose an arbitrary $x \in \text{rad}(S_{\overline{\lambda}})$. Then $x = \overline{\Psi}(\eta u)$ for some u in S_λ. We must show that ηu is in $\eta(\text{rad}(S_\lambda))$, so we prove $u \in \text{rad}(S_\lambda)$. To do this, choose any $v \in S_\lambda$. Then Lemma 5.4 gives

$$\begin{aligned} \langle u, v\rangle &= \langle \eta u, \eta v\rangle \\ &= m!(\langle\langle \overline{\Psi}(\eta u), \overline{\Psi}(\eta v)\rangle\rangle) \\ &= m!(\langle\langle x, \overline{\Psi}(\eta v)\rangle\rangle) \\ &= 0 \text{ since } x \in \text{rad}(S_{\overline{\lambda}}). \end{aligned}$$

Thus $u \in \text{rad}(S_\lambda)$ and so $\eta u \in \eta(\text{rad}(S_\lambda))$ as desired. □

All the pieces are now in place to prove Theorem 4.1. It is well known that any composition factor D_μ of S^λ has $\lambda \trianglerighteq \mu$. In particular, S^λ has no composition factors D_μ with $\mu_1 > \lambda_1 = m$. Thus η annihilates all the composition factors of S^λ except those D_μ with $\mu_1 = \lambda_1 = m$. We have proved that $\eta S^\lambda \cong S^{\overline{\lambda}}$ and that $\eta D_\mu \cong D_{\overline{\mu}}$. In particular ηD_μ, which is guaranteed by Lemma 3.4 to be a simple $\eta\Sigma_r\eta$-module, remains simple as a $\eta k\Sigma_{r-m}\eta \cong k\Sigma_{r-m}$-module.

Thus Lemma 3.4 implies $[S^\lambda : D_\mu] = [S^{\overline{\lambda}} : D_{\overline{\mu}}]$. Theorem 4.1 then follows because $S_\lambda = (S^\lambda)^*$ and all the D_μ are self-dual; so $[S^\lambda : D_\mu] = [S_\lambda : D_\mu]$. Theorem 4.2 follows from Theorem 4.1 by tensoring with sgn and recalling that $S^\lambda \otimes \mathrm{sgn} \cong S_{\lambda'}$.

6. A row removal theorem for $\mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\mu)$

In this section we combine information from our partial proof of James' result with a theorem of Kleshchev and Sheth to derive a new result about the Ext^1-quiver of the symmetric group.

Given any finite-dimensional algebra S and an idempotent $e \in S$ there is an exact functor $\mathcal{F}$: mod-$S \to$ mod-eSe given by multiplication by e. If $eL(\lambda)$ and $eL(\mu)$ are nonzero, then they are irreducible and there is an injection

$$0 \to \mathrm{Ext}^1_S(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{eSe}(eL(\lambda), eL(\mu)).$$

However both James' proof and our proof involve a restriction functor after multiplication by the idempotent, in our case restricting from $\eta k\Sigma_r\eta$ to $k\Sigma_{r-m}$. But restriction does not in general induce an injection on Ext^1. The following result of Kleshchev and Sheth lets us use our row removal functor to obtain a result on extensions between simple modules. We have translated the theorem to index irreducibles with p-restricted partitions rather than p-regular.

Theorem 6.1 ([9, Theorem 2.10])**.** *Let λ, μ be partitions of r with $\lambda_1, \mu_1 < p$ and assume $\mu \not\trianglerighteq \lambda$. Then*

$$\mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\mu) \cong \mathrm{Hom}_{k\Sigma_r}(D_\mu, S^\lambda/D_\lambda).$$

This is all we need to prove:

Theorem 6.2. *Let $\lambda_1, \mu_1 = m < p$. Then there is an injection*

$$0 \to \mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\mu) \to \mathrm{Ext}^1_{k\Sigma_{r-m}}(D_{\overline{\lambda}}, D_{\overline{\mu}}).$$

Equivalently, if λ and μ both have $m < p$ parts, then there is an injection

$$0 \to \mathrm{Ext}^1_{k\Sigma_r}(D^\lambda, D^\mu) \to \mathrm{Ext}^1_{k\Sigma_{r-m}}(D^{\hat{\lambda}}, D^{\hat{\mu}}).$$

Proof. Since the irreducible modules are self-dual we can assume $\mu \not\trianglerighteq \lambda$ without loss of generality, so of course $\overline{\mu} \not\trianglerighteq \overline{\lambda}$ as well. We have

$$0 \to D_\lambda \to S^\lambda \to S^\lambda/D_\lambda \to 0.$$

Multiplying by η gives

$$0 \to D_{\overline{\lambda}} \to S^{\overline{\lambda}} \to \eta(S^\lambda/D_\lambda) \to 0.$$

Thus,

$$\eta(S^\lambda/D_\lambda) \cong S^{\overline{\lambda}}/D_{\overline{\lambda}}.$$

So each D_μ in the socle of S^λ/D_λ maps to a $D_{\overline{\mu}}$ in the socle of $S^{\overline{\lambda}}/D_{\overline{\lambda}}$. We get

$$0 \to \mathrm{Hom}_{k\Sigma_d}(D_\mu, S^\lambda/D_\lambda) \to \mathrm{Hom}_{k\Sigma_{d-m}}(D_{\overline{\mu}}, S^{\overline{\lambda}}/D_{\overline{\lambda}}),$$

which, together with Theorem 6.1, completes the proof. □

We have verified that Theorem 6.2 holds for all known Ext^1-quivers for the symmetric group, including blocks of small defect and for various small r. This data together with Theorem 6.2 leads us to the following conjecture.

Conjecture 6.3. *Let $p \geq 3$, and let λ and μ be p-restricted partitions of r with $\lambda_1 = \mu_1 = m$. Then there is an injection*

$$0 \to \mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\mu) \to \mathrm{Ext}^1_{k\Sigma_{r-m}}(D_{\overline{\lambda}}, D_{\overline{\mu}}).$$

Equivalently, for τ and ρ p-regular partitions of r with m parts, there is an injection

$$0 \to \mathrm{Ext}^1_{k\Sigma_r}(D^\tau, D^\rho) \to \mathrm{Ext}^1_{k\Sigma_{r-m}}(D^{\hat{\tau}}, D^{\hat{\rho}}).$$

Conjecture 6.3 immediately implies the following conjecture.

Conjecture 6.4 (Kleshchev, Martin)**.** *For $p \geq 3$, $\mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\lambda) = 0$.*

The reason for this is that if Conjecture 6.3 holds and if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then we can remove rows one at a time. This eventually gives an injection from $\mathrm{Ext}^1_{k\Sigma_r}(D_\lambda, D_\lambda)$ into $\mathrm{Ext}^1_{k\Sigma_{\lambda_k}}(D_{\lambda_k}, D_{\lambda_k}) \cong \mathrm{Ext}^1_{k\Sigma_{\lambda_k}}(k,k)$, which is known to be zero.

We present an example where the injection in Theorem 6.2 is proper. Let $p = 3$, $m = 2$ and choose $\lambda = (2^3, 1^6)$ and $\mu = (2, 1^{10})$. Then

$$S^{(2^3,1^6)} \cong \begin{matrix} D_{(2,1^{10})} \\ D_{(1^{12})} \\ D_{(2^3,1^6)}, \end{matrix}$$

so $\mathrm{Ext}^1_{k\Sigma_{12}}(D_{(2,1^{10})}, D_{(2^3,1^6)}) = 0$. But

$$\eta S^{(2^3,1^6)} \cong S^{(2^2,1^6)} \cong \begin{matrix} D_{(1^{10})} \\ D_{(2^2,1^6)}. \end{matrix}$$

Notice that η annihilated the $D_{(1^{12})}$ term. So the $D_{(2,1^{10})}$ term dropped down, and

$$\mathrm{Ext}^1_{k\Sigma_{10}}(D_{(2^2,1^6)}, D_{(1^{10})}) \cong k.$$

James also proved a result corresponding to removing the first *column* from D_λ. Using the idempotent

$$\eta' := \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \mathrm{sgn}(\sigma)\sigma$$

and proceeding with a similar analysis we could obtain Theorem 6.2 for first-column removal from D_λ. However, for fixed p there are only finitely many p-restricted partitions with less than p parts, so the corresponding theorem is weaker. We are not aware of any counterexamples to the column removal statement for D_λ corresponding to Conjecture 6.3, so perhaps this holds as well.

7. Remarks on fixed point functors

As above we consider Σ_m and Σ_{r-m} as subgroups of Σ_r, but we drop the assumption that $m < p$. For $U \in \text{mod-}k\Sigma_r$, the fixed points of U under Σ_m are clearly invariant under the action of Σ_{r-m}. So we can define

$$\mathcal{F}_m : \text{mod-}k\Sigma_r \to \text{mod-}k\Sigma_{r-m}$$

by

$$\mathcal{F}_m(U) = U^{\Sigma_m} \cong \text{Hom}_{k\Sigma_m}(k, U) \cong \text{Hom}_{k\Sigma_r}(M^{(m,1^{r-m})}, U).$$

When $m < p$ this functor agrees with the functor $\mathcal{F}_m$ defined previously, namely it is multiplication by an idempotent and then restriction (and hence is exact). When $m \geq p$, the module k is not projective as a $k\Sigma_m$-module, so the functor $\mathcal{F}_m$ is only left exact. Very little seems to be known about this functor. For example what is $\mathcal{F}_m(S^\lambda)$? In Section 4 we determined this in the special case $\lambda_1 = m < p$.

We also determined $\mathcal{F}_m(S_\lambda)$ in this case by a similar but not identical proof. It is clear the two situations are very different. In particular the dimension $\mathcal{F}_m(S_\lambda)$ is independent of the characteristic and is the number of semistandard λ-tableaux of type $(m, 1^{r-m})$. We will study this functor in more detail in [5]. In particular we can show:

Theorem 7.1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then:*

(i) *If $m > \lambda_1$, then $\mathcal{F}_m(S_\lambda) = 0$.*
(ii) *If $m = \lambda_1$, then $\mathcal{F}_m(S_\lambda) \cong S_{\overline{\lambda}}$.*
(iii) *If $m < \lambda_1$, then $\mathcal{F}_m(S_\lambda) \cong S_{\lambda\backslash(m)}$, the dual of a skew Specht module.*

The situation for Specht modules in characteristic p is much more difficult; not even the dimension of $\mathcal{F}_m(S^\lambda)$ is known. Of course $\mathcal{F}_m(S^\lambda)$ is not just a vector space, but has the structure of a $k\Sigma_{r-m}$-module. The author is not aware of any investigation of this module structure.

We remark that part (i) of Theorem 7.1 is not true for Specht modules. For example, when $p = 3$, $\lambda = (7, 2, 2)$ and $m = 8 > \lambda_1$,

$$\dim \mathcal{F}_8(S^\lambda) = 3 > 0.$$

We make the following conjecture.

Conjecture 7.2. *$\mathcal{F}_m(S^\lambda)$ has a filtration by Specht modules for $k\Sigma_{r-m}$.*

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Department of Mathematics, University of Toledo, 2801 W. Bancroft, Toledo, Ohio 43606

E-mail address: `david.hemmer@utoledo.edu`