REALIZING LARGE GAPS IN COHOMOLOGY FOR SYMMETRIC GROUP MODULES

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Abstract. Using results of the author with Cohen and Nakano, we find examples of Young modules Y^λ for the symmetric group Σ_d for which the Tate cohomology $\widehat{\mathrm{H}}^i(\Sigma_d, Y^\lambda)$ is not identically zero, but vanishes for approximately $\frac{1}{3}d^{3/2}$ consecutive degrees. We conjecture these vanishing ranges are maximal among all Σ_d-modules with nonvanishing cohomology. The best known upper bound on such vanishing ranges stands at $(d-1)^2$, due to work of Benson, Carlson and Robinson. Particularly striking, and perhaps counterintuitive, is that these Young modules have maximum possible complexity.

1. Introduction:

Let G be a finite group and k an algebraically closed field of characteristic p. If G contains an element x of order p such that the centralizer $C_G(x)$ is not p-nilpotent, then a result of Benson [1] guarantees the existence of a nonprojective kG-module M in the principal block such that the cohomology $\mathrm{H}^*(G, M)$ is identically zero. For the remaining principal block modules, those with nonvanishing cohomology, one might ask for the smallest degree that is nonzero, or the number of consecutive degrees in which the cohomology vanishes. In [3], Benson, Carlson and Robinson gave an upper bound $r = r(G)$ on the number of consecutive i for which the cohomology $\mathrm{H}^i(G, M)$ can vanish, without being identically zero:

Theorem 1.1. [3, Thm. 2.4] *Given a finite group G, there exists a positive integer r such that for any commutative ring R of coefficients and any RG-module M, if $\hat{\mathrm{H}}^i(G, M) = 0$ for $r+1$ consecutive values of i then $\hat{\mathrm{H}}^i(G, M) = 0$ for all i positive and negative.*

The $\hat{\mathrm{H}}$ above denotes Tate cohomology, which agrees with the ordinary cohomology in positive degrees. The proof of Theorem 1.1 expresses r in terms of the degrees of a set of homogenous generators for the cohomology ring of G. However, there is no expectation that this r should be the best possible bound.

There do not appear to be any examples in the literature demonstrating large gaps in cohomology, or determining the smallest possible value of r for particular groups. This is not surprising, as calculating $\mathrm{H}^*(G, M)$ is generally difficult. In [4] the author, with Cohen and Nakano, obtained some very general results when M is a Young module Y^λ for the symmetric group Σ_d. The goal of this paper is to use these results to find very large gaps in Young module cohomology. For certain partitions $\lambda \vdash d$ in characteristic two, we find the minimal $i \geq 0$ with $\mathrm{H}^i(\Sigma_d, Y^\lambda) \neq 0$. These gaps

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turn out to be the largest possible among all Young modules, and come "close" to realizing the value of r arising from Theorem 1.1. Remarkably the Young modules with the largest vanishing ranges also have maximum possible complexity. That is the dimensions in a minimal projective resolution grow as quickly as possible. See [2, p.153] for the precise definition of complexity.

2. Computing Young module cohomology

In this section we recall results from [4] on computing Young module cohomology. Let $V \cong k^d$ be the natural module for the general linear group $G := GL_d(k)$. For a partition $\lambda \vdash d$, let $L(\lambda)$ denote the simple G-module of highest weight λ, and let Y^λ denote the Young module for Σ_d. We let $\trianglerighteq$ denote the usual dominance order on partitions of d, and let λ' denote the transpose or conjugate partition. Definitions and information on all these modules can be found in [9].

The commuting actions of G and Σ_d on $V^{\otimes d}$ give the homology $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$ the structure of a G-module. The composition multiplicities of this G-module are related to the dimensions of Young module cohomology in the following way. Let $[M : S]$ denotes the multiplicity of a simple module S in a composition series of M. Then:

Theorem 2.1. [6, Prop 2.6B]

$$\dim_k \mathrm{H}^i(\Sigma_d, Y^\lambda) = [\mathrm{H}_i(\Sigma_d, V^{\otimes d}) : L(\lambda)],\ i \geq 0.$$

Theorem 2.1 indicates that determining the simple constituents of $\mathrm{H}_*(\Sigma_d, V^{\otimes d})$ as a graded G-module allows one to calculate Young module cohomology in all degrees. It turned out to be easier to study this for all d simultaneously, using methods from algebraic topology. In [4, Thm. 8.1.4] the algebra $\oplus_{d\geq 0} \mathrm{H}_*(\Sigma_d, V^{\otimes d})$ is described as a G-module. It is a polynomial algebra, tensored with an exterior algebra if p is odd. Each generator belongs to a certain G-module direct summand, and this summand belongs to $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$ for a particular i and d.

The G-modules that occur are described below. For a G-module M let $M^{(a)}$ denote the a-th Frobenius twist of M (see [8, p. 132]), and let $S^a(M)$ and $\Lambda^a(M)$ denote respectively the a-th symmetric and exterior power of M.

Theorem 2.2. [4, Cor. 8.2.1] *In characteristic two, the G-module $\mathrm{H}_*(\Sigma_d, V^{\otimes d})$ is a direct sum of modules of the form:*

$$S^{a_0}(V) \otimes S^{a_1}(V^{(c_1)}) \otimes \cdots \otimes S^{a_s}(V^{(c_s)}) \tag{2.1}$$

where each $a_i \geq 0$, $c_i > 0$ and $d = a_0 + \sum_{j=1}^{s} a_j 2^{c_j}$.

In odd characteristic the G-module $\mathrm{H}_*(\Sigma_d, V^{\otimes d})$ is a direct sum of modules of the form

$$S^{a_0}(V) \otimes S^{a_1}(V^{(c_1)}) \otimes \cdots \otimes S^{a_s}(V^{(c_s)}) \otimes \Lambda^{d_1}(V^{(e_1)}) \otimes \cdots \otimes \Lambda^{d_t}(V^{(e_t)}) \tag{2.2}$$

where each $a_i \geq 0$, each $c_i, d_i, e_i > 0$ and where $d = a_0 + \sum_{j=1}^{s} a_j p^{c_j} + \sum_{j=1}^{t} d_j p^{e_j}$.

Each summand in (2.1) or (2.2) occurs in $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$ for a single value of d but for infinitely many different degrees i, for a description see Theorem 8.1.4 in [4] or the special cases below, which are all we will use. To compute a particular $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ one must first determine the (finitely many) summands which contribute to this d and i, and then compute the multiplicities of $L(\lambda)$ in each summand. In the next section we will let $p = 2$ and make a strategic choice for λ. For these λ we can determine precisely the summand (2.1) of smallest degree which contains $L(\lambda)$ as a composition factor, and thus determine the initial vanishing range. In Section 4 we use these computations to produce Young modules with very large gaps in cohomology. In the final section we discuss the situation in odd characteristic, and present a few open problems.

3. Initial vanishing ranges in characteristic two.

In this section assume $p = 2$. Notation such as $(2^3, 1^2)$ will be shorthand for the partition $(2, 2, 2, 1, 1)$, not the partition $(8, 1)$. It is clear from (2.1) that understanding the composition factors of $S^a(V)$ is necessary for computing Young module cohomology (but not sufficient, as one must also decompose the tensor products).

Fortunately, in [5] Doty determined the entire submodule structure for $S^a(V)$. The composition factors all occur with multiplicity at most one, and have a particularly nice form in characteristic two:

Proposition 3.1. [5]*(see also* [4, Prop. 12.2.1]*) Let $\lambda \vdash s$ have a 2-adic expansion*

$$\lambda = \sum_{i=0}^{m} 2^i \lambda_{(i)}$$

where each $\lambda_{(i)}$ is 2-restricted. Then $L(\lambda)$ is a constituent of $S^s(V)$ if and only if each $\lambda_{(i)}$ is of the form (1^{a_i}) for $a_i \geq 0$.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \vdash d$ be 2-restricted. Denote $\mu' = ((\mu')_1, (\mu')_2, \ldots, (\mu')_{\mu_1})$. We will compute the first i such that $\mathrm{H}^i(\Sigma_{2d}, Y^{2\mu})$ is nonzero, and see that a particular such μ will maximize the initial vanishing range.

Since μ is 2-restricted, the 2-adic expansion of 2μ is just 2μ. So Steinberg's Tensor Product Theorem (STPT) [8, II.3.17] implies the summands from (2.1) with any $c_i > 1$ do not have $L(2\mu)$ as a composition factor. So to compute $\mathrm{H}^i(\Sigma_{2d}, Y^{2\mu})$ we must determine the multiplicity of $L(2\mu)$ in summands of the form:

$$S^a(V) \otimes S^{a_1}(V^{(1)}) \otimes \cdots \otimes S^{a_s}(V^{(1)}) \cong S^a(V) \otimes S^{\tau}(V^{(1)}) \tag{3.1}$$

where we can assume without loss that $a_i \geq a_{i+1}$, so $\tau = (a_1, a_2, \ldots, a_s) \vdash d - \frac{a}{2}$.

Analysis just as in Section 10 of [4] shows that a summand of the form (3.1) corresponds to monomials in the polynomial algebra of the form:

$$v^a \cdot Q_{i_1}^{a_1}(v) \cdots Q_{i_s}^{a_s}(v),$$

for distinct i_t. By [4, Thm. 8.1.4(a)], such a summand contributes to the cohomology in degree $a_1 i_1 + a_2 i_2 + \cdots + a_s i_s$. To determine the smallest i with $\mathrm{H}^i(\Sigma_{2d}, Y^{2\mu}) \neq 0$ we must first determine which modules (3.1) contain $L(2\mu)$ as a composition factor. Then for each we must determine the smallest possible corresponding degree where the summand can occur. Our assumption on μ limits how $L(2\mu)$ can arise as a composition factor in (3.1):

Proposition 3.2. *Let $2\mu \vdash 2d$ where μ is 2-restricted. Then $\mathrm{H}^i(\Sigma_{2d}, Y^{2\mu}) \neq 0$ if and only if there exists an integer $a \geq 0$, a partition $\tau = (a_1, a_2, \ldots, a_s) \vdash d - a$ and integers $\{i_t > 0\}$ such that:*

(i) $i = a_1 i_1 + a_2 i_2 + \cdots + a_s i_s$

(ii)
$$[L(2^a)\otimes L(2^{a_1})\otimes\cdots\otimes L(2^{a_s}):L(2\mu)]\neq 0.$$

Proof. By Proposition 3.1 and the STPT, the composition factors of $S^m(V)$ are all of the form:
$$L(1^{c_0})\otimes L(2^{c_1})\otimes L(4^{c_2})\otimes\cdots.$$
But 2μ is its own 2-adic expansion, so any $L(2\mu)$ occurring in (3.1) must arise as in part (2) by the STPT. The corresponding degree i follows from [4, Thm. 8.1.4(a)]. □

Notice that the "if" part of the preceding result did not require μ be 2-restricted, a fact we will need later.

Now we want to find the smallest degree i where the cohomology $\mathrm{H}^i(\Sigma_{2d},Y^{2\mu})$is nonzero. Since $a_1\geq a_2\geq\cdots$, it is clear from Proposition 3.2(1) that we should choose $i_t=t$ to minimize the degree i. The smallest nonzero degree is given in terms of the following function on partitions. Let $\rho=(\rho_1,\rho_2,\ldots\rho_s)\vdash d$. Define:
$$x(\rho)=\sum_{l=1}^{s}(l-1)\rho_l.$$
The following easy lemma is left to the reader:

Lemma 3.3. *Suppose $\lambda\trianglerighteq\mu$. Then $x(\lambda)\leq x(\mu)$. If $\lambda\neq\mu$ the inequality is strict.*

We can now determine the first nonvanishing degree for $\mathrm{H}^*(\Sigma_{2d},Y^{2\mu})$.

Theorem 3.4. *Let $\mu\vdash d$ be arbitrary. Then:*
(i) $\mathrm{H}^{x(\mu')}(\Sigma_{2d},Y^{2\mu})\neq 0$.
(ii) *Moreover, if μ is 2-restricted, then*
$$\dim\mathrm{H}^i(\Sigma_{2d},Y^{2\mu})=\begin{cases}0 & \text{for } 0\leq i<x(\mu').\\ 1 & \text{for } i=x(\mu').\end{cases}$$

Proof. For convenience let $\tau=\mu'$. Observe that
$$\mu=(1^{\tau_1})+(1^{\tau_2})+\cdots+(1^{\tau_{\mu_1}}).$$
Then $L(2^{\tau_1})\otimes L(2^{\tau_2})\otimes\cdots\otimes L(2^{\tau_{\mu_1}})$ has highest weight 2μ with multiplicity one so:

$$[L(2^{\tau_1})\otimes L(2^{\tau_2})\otimes\cdots\otimes L(2^{\tau_{\mu_1}}):L(2\mu)]=1 \tag{3.2}$$
and thus:

$$[S^{2\tau_1}(V)\otimes S^{\tau_2}(V^{(1)})\otimes S^{\tau_3}(V^{(1)})\cdots\otimes S^{\tau_{\mu_1}}(V^{(1)}):L(2\mu)]\geq 1. \tag{3.3}$$

Choosing $a=\tau_1$ and $i_t=t$, the proof of Proposition 3.2 tells us $\mathrm{H}^{x(\mu')}(\Sigma_{2d},Y^{2\mu})\neq 0$. (The "if" part did not require μ be 2-restricted.)

Now suppose further that μ is 2-restricted, and consider Proposition 3.2. Suppose
$$[L(2^a)\otimes L(2^{a_1})\otimes\cdots\otimes L(2^{a_s}):L(2\mu)]\neq 0.$$

In order to minimize the degree i it is clear from Proposition 3.2(2) that we may assume $a \geq a_1 \geq a_2 \geq \cdots \geq a_s$. Then $\rho := (a, a_1, a_2, \ldots, a_s) \vdash d$, and by Proposition 3.2(1), the corresponding cohomological degree is $x(\rho)$. Since $L(2^a) \otimes L(2^{a_1}) \otimes \cdots \otimes L(2^{a_s})$ has highest weight $2\rho'$ then $\rho' \trianglerighteq \mu$, and thus $\mu' \trianglerighteq \rho$. When $\rho = \mu'$ we get a single copy of $L(2\mu)$ as above, contributing to degree $x(\mu')$. Otherwise $\mu' \triangleright \rho$. Then Lemma 3.3 implies $x(\mu') < x(\rho)$, so $x(\mu')$ is the smallest degree with nonzero cohomology. So the cohomology is one-dimensional in degree $x(\mu')$ and zero in smaller degrees. □

4. Mind the Gap

In this section we apply Theorem 3.4 to find large gaps in cohomology. For comparison we first compute the smallest currently known $r(\Sigma_d)$ which satisfies Theorem 1.1.

A faithful complex representation of a group G gives rise to an embedding into a compact unitary group $G \hookrightarrow U(n)$. The cohomology of the classifying space $BU(n)$ is a polynomial ring on generators in degrees $2, 4, 6, .., 2n$ (see [2, Sec. 2.6]). The value of r coming from these generators by the construction in [3] is $1 + 3 + 5 + .. + (2n-1) = n^2$. Thus if G has a faithful representation of dimension n, one can take $r = r(G) = n^2$ in Theorem 1.1, see [2, Section 5.14-15] for details.

The smallest faithful irreducible $\mathbb{C}\Sigma_d$ module is $d-1$ dimensional, so one can take $r(\Sigma_d) = (d-1)^2$, and this is the smallest known bound. To find Young modules with large vanishing ranges, Theorem 3.4(2) suggests finding p-restricted μ with $x(\mu')$ as large as possible. In this section we show careful choice of Young module can realize gaps on the order of $\frac{1}{3}d^{3/2}$.

Fix $n \geq 1$ and define $\rho_n = (n, n-1, n-2, \ldots, 2, 1) \vdash \frac{n^2+n}{2}$. Notice that

$$\rho_n = (\rho_n)' = (1^n) + (1^{n-1}) + \cdots + (1^2) + (1).$$

One easily computes:

$$x(\rho_n) = \frac{n^3 - n}{6}. \tag{4.1}$$

We have:

Proposition 4.1. *Let $p = 2$ and $\rho_n \vdash (n^2+n)/2$ be as above. Then:*

$$\dim \hat{\mathrm{H}}^i(\Sigma_{n^2+n}, Y^{2\rho_n}) = \begin{cases} 0 & \text{for } -\frac{n^3-n}{6} < i < \frac{n^3-n}{6} \\ 1 & \text{for } i = \pm\frac{n^3-n}{6} \end{cases}$$

Proof. Since ρ_n is 2-restricted, we can apply Theorem 3.4(2) and (4.1). The extension to negative degrees comes from Tate duality, using the fact that Young modules are self-dual. □

Proposition 4.1 shows that for $d = n^2 + n$, the best possible $r(\Sigma_d)$ is at least $\frac{n^3-n}{3}$.

Example 4.2. Let $\lambda = (28, 26, 24, \ldots, 6, 4, 2) \vdash 210$. Then

$$\mathrm{H}^i(\Sigma_{210}, Y^\lambda) = \begin{cases} 0 & \text{if } -455 < i < 455 \\ k & \text{if } i = 455. \end{cases}$$

It follows from [7, Thm. 3.3.2] that the Y^λ in Example 4.2 has complexity 105, the maximum possible among Σ_{210}-modules.. This means the dimension of the module P_i in the minimal projective resolution $P_* \to k$ of the trivial module grows like a polynomial of degree 104 in i. However it is not until P_{455} that the projective cover $P(k)$ makes its first appearance!

Remark 4.3. Proposition 4.1 applies to Σ_{2d} where d is a triangular number $T(n) = \frac{n^2+n}{2}$. For arbitrary d one can still choose a 2-restricted μ maximizing $x(\mu')$ in a similar way. Write $d = T(n)+a$ for $0 \le a < n+1$ and choose

$$\mu = (n, n-1, \ldots, a+1, a, a, a-1, a-2, \ldots, 2, 1). \tag{4.2}$$

One still has $x((2\mu)')$ asymptotic to a constant times n^3.

So there is a constant c so that for arbitrary d we can obtain Young modules in characteristic two with cohomology vanishing for for first $cd^{3/2}$ degrees.

5. Odd primes and Further Directions

Since Theorem 1.1 gives a bound $r(G)$ independent of the characteristic, we have focused on $p = 2$ which gives the cleanest results. For an arbitrary prime one can still achieve gaps that are a constant times $d^{3/2}$ in length, using $\mu = p(p-1)\rho$, although the answer is messier, and involves polynomials in p. For example the nice compact form for $x(\rho_n)$ in (4.1) becomes replaced by:

$$(p-1)[n(2p-3) + (n-1)(4p-5) + \cdots + 1(2n(p-1)-1)].$$

The corresponding result, which we state without proof, is:

Proposition 5.1. *Let $d = \frac{p(p-1)(n^2+n)}{2}$. Let $\mu = p(p-1)\rho_n \vdash d$. Then there is a constant $c(p)$ and a polynomial $p(n) = c(p)n^3 + an^2 + bn$ such that:*

$$\mathrm{H}^i(\Sigma_d, Y^\mu) = 0 \text{ if } -p(n) < i < p(n)$$

So once again we have find an $r(\Sigma_d)$ asymptotic to a constant times $d^{3/2}$. The function $c(p)$ is decreasing, so the best estimates for $r(\Sigma_d)$ come from the $p = 2$ case. This might lead one to wildly conjecture:

Conjecture 5.2. Let $d = \frac{p(p-1)(n^2+n)}{2}$. Let $\mu = p(p-1)\rho_n \vdash d$. Among all Σ_d modules in the principal block with nonvanishing cohomology, the Young module Y^μ has the largest gap in cohomology, and thus determines the best possible r in Theorem 1.1. For d not of this form, a similar choice, in the spirit of (4.2), for μ achieves the maximal gap.

There are many problems which remain, although it isn't clear one should expect nice answers to any of them. For example one might find the smallest positive i with $\mathrm{H}^i(\Sigma_d, Y^\lambda) \neq 0$. The corresponding problem for simple modules is a subject of active research, for example for groups of Lie type. A first step would be to generalize Doty's work from the $\mu = (d)$ case to the more general:

Problem 5.3. Given $\lambda \vdash d$, find the maximal $\mu \vdash d$ such that $[S^\mu(V) : L(\lambda)] \neq 0$.

Determining the λ for which $\mu = (d)$ is just Doty's result on the composition factors of $S^d(V)$. At the opposite extreme, such a μ always exists, as $S^{(1^d)}(V) \cong V^{\otimes d}$ and each $L(\mu)$ occurs as a composition factor of $V^{\otimes d}$.

Finally we observe that the partition μ appearing in Proposition 5.1 is just the twist of the Steinberg weight (see [8, p. 199]), but there seems to be no representation-theoretic interpretation of this fact.

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