Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 224 (2010) 1419–1461
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

On the cohomology of Young modules for the symmetric group

Frederick R. Cohen [a,1], David J. Hemmer [b,2], Daniel K. Nakano [c,*,3]

[a] *Department of Mathematics, University of Rochester, Hylan Building, Rochester, NY 14627, USA*
[b] *Department of Mathematics, University at Buffalo, SUNY, 244 Mathematics Building, Buffalo, NY 14260, USA*
[c] *Department of Mathematics, University of Georgia, Athens, GA 30602, USA*

Received 9 August 2008; accepted 21 January 2010

Available online 12 February 2010

Communicated by David J. Benson

Abstract

The main result of this paper is an application of the topology of the space $Q(X)$ to obtain results for the cohomology of the symmetric group on d letters, Σ_d, with 'twisted' coefficients in various choices of Young modules and to show that these computations reduce to certain natural questions in representation theory. The authors extend classical methods for analyzing the homology of certain spaces $Q(X)$ with mod-p coefficients to describe the homology $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a module for the general linear group $GL(V)$ over an algebraically closed field k of characteristic p. As a direct application, these results provide a method of reducing the computation of $\mathrm{Ext}^\bullet_{\Sigma_d}(Y^\lambda, Y^\mu)$ (where Y^λ, Y^μ are Young modules) to a representation theoretic problem involving the determination of tensor products and decomposition numbers. In particular, in characteristic two, for many d, a complete determination of $\mathrm{H}^\bullet(\Sigma_d Y^\lambda)$ can be found. This is the first nontrivial class of symmetric group modules where a complete description of the cohomology in all degrees can be given.

For arbitrary d the authors determine $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for $i = 0, 1, 2$. An interesting phenomenon is uncovered-namely a stability result reminiscent of generic cohomology for algebraic groups. For each i the cohomology $\mathrm{H}^i(\Sigma_{p^a d}, Y^{p^a\lambda})$ stabilizes as a increases. The methods in this paper are also powerful enough to determine, for any p and λ, precisely when $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda) = 0$. Such modules with vanishing cohomology

* Corresponding author.
E-mail addresses: cohf@math.rochester.edu (F.R. Cohen), dhemmer@math.buffalo.edu (D.J. Hemmer), nakano@math.uga.edu (D.K. Nakano).

[1] Research of the first author was supported in part by NSF grant DMS-0340575 and DARPA grant 2006-06918-01.
[2] Research of the second author was supported in part by NSF grant DMS-0556260.
[3] Research of the third author was supported in part by NSF grant DMS-0654169.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.01.004

are of great interest in representation theory because their support varieties constitute the representation theoretic nucleus.

MSC: 20C30; 55S12; 55P47

Keywords: Cohomology; Young modules; Symmetric group

1. Introduction

1.1. The representation theory of the symmetric group and its connections with the general linear group have been studied for more than 100 years. Yet there exist very few calculations of cohomology groups $\mathrm{Ext}^i_{k\Sigma_d}(M, N)$ for natural modules M and N for the symmetric group Σ_d over an algebraically closed field k of characteristic p. The cohomology ring $\mathrm{H}^\bullet(\Sigma_d, k) \cong \mathrm{Ext}^\bullet_{k\Sigma_d}(k, k)$ was originally computed (as a vector space) by Nakaoka [25]. Feshbach provided a combinatorial formulation of the ring structure over $\mathbb{F}_2$ [15]. However for odd primes the ring structure is only fully understood for small d. Using the results of Araki–Kudo and Dyer–Lashof one can compute the homology $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ over a field with p elements for p prime [2,14]. The aforementioned computations all employ constructions and techniques from algebraic topology.

In the late 1990's, Doty, Erdmann and Nakano [12] introduced a first quadrant spectral sequence which related the cohomology for the symmetric group to that of the general linear group (cf. (2.1.1)). Let $V \cong k^n$ be the natural module for the general linear group $GL(V) \cong GL_n(k)$. The construction utilizes the commuting actions of $GL(V)$ and Σ_d on $V^{\otimes d}$ in a functorial way. The relationship between the two cohomology theories relies heavily on understanding the structure of $\mathrm{Ext}^j_{k\Sigma_d}(V^{\otimes d}, N)$ as a $GL(V)$-module. Vanishing ranges for these cohomology groups were obtained in [21], for N a dual Specht or Young module, which enabled the proofs of stability results between the cohomology of certain $GL(V)$-modules and Σ_d-modules. Beside these examples, the only known computations are for small d. For Specht modules S^λ, even $\mathrm{Ext}^1_{\Sigma_d}(k, S^\lambda)$ is unknown. When D_λ and D_μ are simple modules, $\mathrm{Ext}^1_{\Sigma_d}(D_\lambda, D_\mu)$ is known in some very particular examples.

1.2. In this paper we aim to effectively combine both techniques mentioned above to calculate extension groups and cohomology for the symmetric group. This idea is quite natural because the structure of these Ext groups arises from interconnections of topology and algebra as described above. An example of these connections is provided in a beautiful recent paper of Benson [4], where he shows that the homology of the loop space of the p-completion of BG, denoted $\Omega(BG^\wedge_p)$, depends only on representation theoretic information. One crude overlap with the results here is that the homology groups $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ are also given in terms of the homology of certain loop spaces.

A common theme that runs throughout this paper is that symmetric group cohomology is reduced to $GL_n(k)$-representation theory, i.e., a homological version of Schur–Weyl duality. Our first step involves applying the aforementioned spectral sequence to calculate cohomology of Young modules. In this case the spectral sequence collapses, and one needs to only understand the simple constituents of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module. This demonstrates that homological information can be reduced to questions where purely representation theoretic information arises.

Our Theorem 8.1.4 gives a complete description of this module as a tensor product of twisted symmetric and exterior powers of V.

The computation of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module leads to a description of Young module cohomology $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ in terms of the simple constituents of these modules, which can often be completely calculated. One feature of the result is that computing symmetric group cohomology in arbitrarily high degree with coefficients in a Young module is reduced to determining the composition factors of certain natural modules for the general linear group.

For example, this method can be used easily and directly to determine $\mathrm{H}^i(\Sigma_{16}, Y^\lambda)$ in characteristic two for any $i \geqslant 0$ and any $\lambda \vdash 16$. A useful fact that we prove is when the decomposition matrices are known, a complete description of $\mathrm{Ext}^i_{\Sigma_d}(Y^\lambda, Y^\mu)$ can be given.

With our description of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module in terms of twisted symmetric and exterior powers, we invoke the work of Doty on the composition factors of symmetric powers to determine precisely, in any characteristic, which λ have the property that $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ is identically zero. Such modules in the principal block with no cohomology can be used to describe the *representation theoretic nucleus*. We can also compute $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for small degrees i and arbitrary d and λ.

1.3. It has always been a mystery how the Frobenius morphism on the group scheme GL_n plays a role in the cohomology theory of the symmetric group. Certainly, our description of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module is a good start to understanding this phenomenon. We prove a stability result relating cohomology of Young modules $Y^{p^a\lambda}$ and $Y^{p^{a+1}\lambda}$. This appears to be the first result for the representation theory of the symmetric group which involves multiplying a partition by p. This is further evidence that the Frobenius will play an integral role in our understanding of symmetric group representations, even though there is no obvious interpretation of twisting representations on the symmetric group side (as opposed to twisting GL_n-representations).

Remark 1.3.1. The main result, Theorem 8.1.4, interpreting Young module cohomology in terms of composition factors of certain $GL_n(k)$-modules, holds in any characteristic. However some of the subsequent results and computations are only for characteristic two while others are general, so we will be careful to state in which characteristic the various results apply.

1.4. Organization

The paper is organized in the following manner. Section 2 is an exposition of the relationship between $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module and the cohomology of Young modules based on results from [12]. In Sections 3–8 a natural construction from algebraic topology is presented which gives a complete description of $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module.

In Section 9 we provide a list of sufficient conditions on $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ to determine $\mathrm{Ext}^\bullet_{\Sigma_d}(Y^\lambda, Y^\mu)$. Section 10 gives an application of this machinery to provide complete answers for Σ_6 in characteristic two and explains how one can easily replicate this for all Σ_d, $d \leqslant 16$.

The subject of Section 11 is a precise determination of which Young modules have vanishing cohomology for arbitrary Σ_d and arbitrary characteristic. In Section 12 a complete calculation of Young module cohomology is given in degrees 0, 1, 2 for arbitrary Σ_d in characteristic two.

The stability theorem discussed above is proven in Section 13. The final section is a description of similar results for cohomology of permutation modules M^λ. In this case the answer is given not in terms of composition factor multiplicities of an explicit $GL_d(k)$-module, but instead in terms of weight–space multiplicities of the same module.

The authors would like to express their appreciation to the referees whose comments were very helpful in improving an earlier version of this manuscript.

2. Homology and commuting actions

2.1. In this section we will explain why cohomology of Young modules is related to the homology groups $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$. We will assume the reader is familiar with the definition of Young modules for the symmetric group, in particular determining which are nonprojective and which lie in the principal block (cf. [23, 4.6] for details).

Let V be the vector space of n-dimensional column vectors over the field k.

Remark 2.1.1. Some of the results below apply to any field k while $\bar{\mathbb{F}}_p$ is required in most of the work below. In addition, there are points where it is useful to use $\mathbb{F}_p$. Thus it will be stated explicitly when the field k will be restricted to either $\bar{\mathbb{F}}_p$ or $\mathbb{F}_p$.

The general linear group $GL_n(k) := GL(V)$ acts naturally on V, and thus on $V^{\otimes d}$. This action commutes with the action of the symmetric group Σ_d, acting by place permutation. Thus there is a map from $kGL_n(k)$ into the *Schur algebra*

$$S(n,d) = \mathrm{End}_{k\Sigma_d}\big(V^{\otimes d}\big).$$

The famous double centralizer theorem [16, 2.6c] states, in part, that this map is a surjection.

We briefly describe the setup from [12]. When $n \geqslant d$ there is an idempotent $e \in S(n,d)$ such that $eS(n,d)e \cong k\Sigma_d$. This gives an exact covariant functor $\mathcal{F}$ going from mod-$S(n,d)$ to mod-$k\Sigma_d$ defined by $\mathcal{F}(M) = eM$ and called the Schur functor. The Schur functor can be realized as both a Hom and a tensor product functor:

$$\mathcal{F}(M) = eM \cong \mathrm{Hom}_{S(n,d)}\big(S(n,d)e, M\big) \cong eS(n,d) \otimes_{S(n,d)} M.$$

Thus it admits two natural (one-sided) adjoint functors from mod-$k\Sigma_d$ to mod-$S(n,d)$:

$$\mathcal{G}_{\mathrm{Hom}}(N) = \mathrm{Hom}_{\Sigma_d}\big(eS(n,d), N\big), \qquad \mathcal{G}_{\otimes}(N) = S(n,d)e \otimes_{eS(n,d)e} N.$$

The functor $\mathcal{G}_{\mathrm{Hom}}$ is left exact, and so admits higher right derived functors $R^\bullet\mathcal{G}_{\mathrm{Hom}}$. The functor $\mathcal{G}_{\otimes}$ is right exact and so admits higher left derived functors $L_\bullet\mathcal{G}_{\otimes}$. These derived functors may be expressed as:

$$R^j\mathcal{G}_{\mathrm{Hom}}(N) = \mathrm{Ext}^j_{k\Sigma_d}\big(V^{\otimes d}, N\big), \qquad L_j\mathcal{G}_{\otimes}(N) = \mathrm{Tor}_j^{k\Sigma_d}\big(V^{\otimes d}, N\big).$$

In [12] two first-quadrant Grothendieck spectral sequences are constructed that relate cohomology for $GL_n(k)$ to that of $k\Sigma_d$. For example, the pair of functors $\mathcal{F}$ and $\mathcal{G}_{\otimes}$ gives rise to a Grothendieck spectral sequence with E_2 page:

$$E_2^{i,j} = \mathrm{Ext}^i_{S(n,d)}\big(\mathrm{Tor}_j^{\Sigma_d}\big(V^{\otimes d}, N\big), M\big) \quad\Rightarrow\quad \mathrm{Ext}^{i+j}_{\Sigma_d}(N, eM). \tag{2.1.1}$$

2.2. In Section 3, we will focus on the computation of

$$\mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big) = \mathrm{Tor}_\bullet^{k\Sigma_d}\big(k, V^{\otimes d}\big).$$

Historically, topologists have viewed this homology computation as more natural than the calculation of the cohomology. This might be due to the fact that one can express the homology in terms of symmetric and exterior powers. On the other hand, algebraists have often viewed cohomology as more natural for applications in representation theory. This section is devoted to keeping track of the various $GL_n(k)$-actions on the homology and cohomology groups.

Normally, one would consider $V^{\otimes d}$ as a $(\Sigma_d$–$GL_n(k))$-bimodule and view the tensor space as a right $GL_n(k)$-module. However, the action of the symmetric group and the general linear group commute and we can view $V^{\otimes d}$ as a left $(\Sigma_d \times GL_n(k))$-module (the Σ_d-action is by place permutation and the $GL_n(k)$-action is by the diagonal action). This will induce a left $GL_n(k)$-action on $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$.

Next observe that, by twisting by the map on $k\Sigma_d$ given by $\sigma \to \sigma^{-1}$, one can make any left $k\Sigma_d$-module into a right $k\Sigma_d$-module and vice versa. Using this correspondence, one can see that $\mathrm{Tor}_\bullet^{k\Sigma_d}(M, N) \cong \mathrm{Tor}_\bullet^{k\Sigma_d}(N, M)$. In particular,

$$L_\bullet\mathcal{G}_\otimes(k) = \mathrm{Tor}_\bullet^{k\Sigma_d}\big(V^{\otimes d}, k\big) \cong \mathrm{Tor}_\bullet^{k\Sigma_d}\big(k, V^{\otimes d}\big) \cong \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

Moreover, by tracing through the definitions one sees that these are isomorphisms of left $GL_n(k)$-modules.

For an $S(n,d)$-module M, let M^τ denote the contravariant (or transpose) dual of M, as described in [16, 2.7]. The underlying vector space of M^τ is $\mathrm{Hom}_k(M, k)$ and the action of $GL_n(k)$ is [16, 2.7a] $(g.f)(m) = f(g^{\mathrm{tr}}.m)$ for $f \in M^\tau$, $m \in M$, and $g \in GL_n(k)$. Let $*$ be the contragredient dual on $k\Sigma_d$-modules. By applying the change of rings formula, one has

$$\begin{aligned}\big[V^{\otimes d} \otimes_{k\Sigma_d} (-)^*\big]^\tau &\cong \mathrm{Hom}_k\big(V^{\otimes d} \otimes_{k\Sigma_d} (-)^*, k\big)\\ &\cong \mathrm{Hom}_{k\Sigma_d}\big(V^{\otimes d}, \mathrm{Hom}_k\big((-)^*, k\big)\big)\\ &\cong \mathrm{Hom}_{k\Sigma_d}\big(V^{\otimes d}, (-)\big).\end{aligned}$$

In the last line we are using the fact that $\mathrm{Hom}_k((-)^*, k) \cong ((-)^*)^* \cong (-)$. This induces a natural isomorphism of functors: $R^\bullet\mathcal{G}_{\mathrm{Hom}}(-) \cong L_\bullet\mathcal{G}_\otimes((-)^*)^\tau$. In particular as left $GL_n(k)$-modules

$$R^j\mathcal{G}_{\mathrm{Hom}}(k) \cong L_j\mathcal{G}_\otimes\big((k)^*\big)^\tau \cong \mathrm{H}_j\big(\Sigma_d, V^{\otimes d}\big)^\tau \tag{2.2.1}$$

for all $j \geqslant 0$. Since the τ-duality fixes all simple $S(n,d)$-modules, the $GL_n(k)$-composition factors of $R^j\mathcal{G}_{\mathrm{Hom}}(k)$ are the same as in $\mathrm{H}_j(\Sigma_d, V^{\otimes d})$.

2.3. The simple $S(n,d)$-modules for $n \geqslant d$ are parameterized by partitions of d. When λ is a partition of d (denoted by $\lambda \vdash d$), let $L(\lambda)$ be the corresponding simple $S(n,d)$-module and $I(\lambda)$ be its injective hull in the category of $S(n,d)$-modules. Note that the simple $S(n,d)$-modules are exactly the simple polynomial $GL_n(k)$-modules of degree d. For a description of the simple $GL_n(k)$-modules and the Steinberg Tensor Product Theorem (which will be used repeatedly), the reader is referred to [19]. If M is a finite-dimensional $GL_n(k)$-module then let $[M : S]$ denote the multiplicity of the simple module S in a composition series of M. Under the Schur functor, $I(\lambda)$

maps to the Young module Y^λ. If one sets $M := I(\lambda)$ then the spectral sequence (2.1.1) collapses to get part (a) of the following theorem. Part (b) is obtained by further specializing $N := k$ and using the fact that both $V^{\otimes d}$ and Y^λ are self-dual $k\Sigma_d$-modules while $L(\lambda) \cong L(\lambda)^\tau$, together with (2.2.1).

Theorem 2.3.1. *(See [12, Prop. 2.6B].)*

(a) $\dim_k \operatorname{Ext}^i_{\Sigma_d}(N, Y^\lambda) = [L_i\mathcal{G}_\otimes(N) : L(\lambda)]$, $i \geqslant 0$.
(b) $\dim_k \mathrm{H}^i(\Sigma_d, Y^\lambda) = [\mathrm{H}_i(\Sigma_d, V^{\otimes d}) : L(\lambda)]$, $i \geqslant 0$.

The theorem above indicates that determining the simple constituents of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a graded $GL_n(k)$-module allows one to calculate Young module cohomology in all degrees. At the time of [12] there seemed to be no way to understand even the dimension of $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$, let alone its $GL_n(k)$-module structure, even in the case $i = 1$. In the following sections some results from algebraic topology are used to give a complete, explicit description of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ as a $GL_n(k)$-module. In particular, Theorem 8.1.4 and Corollary 8.2.1 imply that the module is just a direct sum of tensor products of Frobenius twists of symmetric and (for p odd) exterior powers of the natural module V.

3. On the homology of $Q(X)$ and $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$

3.1. Fix a field k together with a graded vector space V over k where V is concentrated in degrees strictly greater than 0 (i.e., V is $\mathbb{N}$-graded). Such vector spaces are called *connected*. This choice of grading is a formal convenience concerning V which reflects the natural topological setting used below. Properties of the homology groups $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ are addressed below.

The purpose of this section is to describe the functor given by the direct sum $\bigoplus_{d\geqslant 0}\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ in terms of the known singular homology of a topological space with coefficients in a field k. Some of the results described below hold for any field k while some depend on the choice of field.

Most of the results used below hold in case k is either $\mathbb{F}_p$, the field with p elements for p prime or the algebraic closure of $\mathbb{F}_p$, denoted $\bar{\mathbb{F}}_p$. In these cases below, the specific choice of field, either $\mathbb{F}_p$ or $\bar{\mathbb{F}}_p$, will be indicated.

The results presented next are classical. Our intent is to provide a clear self-contained exposition while keeping track of additional new data, in particular the $GL(V)$-module structure on the homology with coefficients in $k = \bar{\mathbb{F}}_p$. With this information, we show that a combination of known algebraic and topological results admits new applications within representation theory. The authors intend, in a future paper, to approach related questions for certain choices of Hecke algebras.

3.2. Assume that V is given in degree s by a vector space V_s having dimension $b_s < \infty$. Thus V is a graded vector space which may be nontrivial in arbitrarily many degrees, but is required to be of finite dimension in any fixed degree. To coincide with standard topological constructions, say that such a V has finite type. It is not necessary to make this assumption in what follows below concerning homology groups. However, this assumption is useful in passage to cohomology groups.

In addition as remarked above, V is assumed to be connected. Notice that by natural degree shift arguments (cf. Section 8), it suffices to consider $\bigoplus_{d\geqslant 0}\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ for connected vector spaces V (assuming that a given vector space is totally finite-dimensional as a graded vector space).

Let sgn denote the one-dimensional sign representation of the symmetric group. One feature which arises by carrying out the work below in the context of graded vector spaces, rather than ungraded vector spaces, is that the methods also apply directly to the case of the Σ_d-module $V^{\otimes d} \otimes \mathrm{sgn}$. Thus these techniques should apply to a calculation of cohomology of twisted Young modules $Y^\lambda \otimes \mathrm{sgn}$, and even to the so-called *signed* Young modules, which were defined by Donkin in [9].

If V is a connected vector space of finite type over any field k, then $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is isomorphic to the singular homology of a certain topological space studied in other contexts first arising in work of Nakaoka, Steenrod, Araki–Kudo, Dyer–Lashof and others [2,14,25,29]. That is the topological space $Q(X)$ described in the next section.

3.3. Recall that the reduced homology groups of the n-sphere S^n for $n > 0$

$$\bar{H}_i\big(S^n, k\big)$$

are all 0 for $i \neq n$ and is the field k in case $i = n$. In addition, a wedge of spheres

$$\bigvee_{n\in W} S^n$$

for an index set W has the property that there is an isomorphism

$$\bar{H}_i\Big(\bigvee_{n\in W} S^n, k\Big) \to \bigoplus_{n\in W} \bar{H}_i\big(S^n, k\big).$$

Thus given a graded, connected vector space V, there is a choice of a wedge of spheres

$$X = \bigvee_{n\in W} S^n$$

such that there is an isomorphism $\bar{\mathrm{H}}_\bullet(X, k) \to V$, where $\bar{\mathrm{H}}_\bullet(X, k)$ denotes the reduced homology of X with coefficients in k. In particular if V has basis $\{v_1(q), \ldots, v_{b_q}(q)\}$ in degree $q > 0$, then X may be chosen to be

$$X = \bigvee_{1\leqslant q<\infty} \Big(\bigvee_{b_q} S^q\Big).$$

The space $Q(X)$ is defined as

$$Q(X) = \bigcup_{0<m<\infty} \Omega^m \Sigma^m(X),$$

where $\Omega^m \Sigma^m(X)$ denotes the space of continuous, pointed functions from the m-sphere to the m-fold suspension of the space X. The space $Q(X)$ has been the subject of thorough investigation and admits many applications [2,6,14].

Next, restrict attention to the fields $\bar{\mathbb{F}}_p$ and $\mathbb{F}_p$. Notice that the universal coefficient theorem gives a natural isomorphism

$$\rho : \mathrm{H}_\bullet(Y, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \to \mathrm{H}_\bullet(Y, \bar{\mathbb{F}}_p)$$

for path-connected spaces of the homotopy type of a CW complex of finite type. Thus the homology of $Q(X)$ satisfies the property that there are isomorphisms

$$\rho : \mathrm{H}_\bullet\big(Q(X), \mathbb{F}_p\big) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \to \mathrm{H}_\bullet\big(Q(X), \bar{\mathbb{F}}_p\big).$$

3.4. A classical result due to Araki–Kudo and Dyer–Lashof is described first where homology is taken with coefficients in $\mathbb{F}_p$ [2,6,14]. Their results are then developed below both in Definition 3.4.4 and Theorem 3.4.6 to give the results required here with coefficients in $\bar{\mathbb{F}}_p$.

Theorem 3.4.1. *Let X denote a path-connected CW-complex with $V = \bar{\mathrm{H}}_\bullet(X, \mathbb{F}_p)$. There is a natural isomorphism*

$$H : \mathrm{H}_\bullet\big(Q(X), \mathbb{F}_p\big) \longrightarrow \bigoplus_{d \geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

Furthermore, $\mathrm{H}_\bullet(Q(X), \mathbb{F}_p)$ *is a known, explicit functor of* $\mathrm{H}_\bullet(X, \mathbb{F}_p)$ *described below.*

Therefore, the universal coefficient theorem together with Theorem 3.4.1 has the following consequence, where some additional structure is required to give the $GL(V)$-action.

Theorem 3.4.2. *Let X denote a path-connected CW-complex with $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$. There is a natural isomorphism*

$$\bar{H} : \mathrm{H}_\bullet\big(Q(X), \bar{\mathbb{F}}_p\big) \longrightarrow \bigoplus_{d \geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

Furthermore, the natural action of $GL(V)$ on $\bigoplus_{d \geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is described below.

Remark 3.4.3.

(1) A refined version of Theorem 3.4.1 is stated below as Theorem 7.2.1.
(2) The algebraic decomposition implied by Theorems 3.4.6, 3.4.2, and 3.4.1 correspond precisely to a geometric decomposition of the space $Q(X)$, at least after sufficient suspensions (as given in Theorem 7.2.1) [8,20,28].
(3) The main new ingredient here is the action of $GL(V)$; that action is described in Sections 4 and 5 in terms of operations known either as Dyer–Lashof operations or Araki–Kudo–Dyer–Lashof operations [2,14] with some properties listed next.

Recall the Araki–Kudo–Dyer–Lashof operations [2,6,14] given by homomorphisms

$$Q_i : \mathrm{H}_s\big(Q(X), \mathbb{F}_2\big) \to \mathrm{H}_{i+2s}\big(Q(X), \mathbb{F}_2\big),$$

and if p is odd,

$$Q_i : \mathrm{H}_s\big(Q(X), \mathbb{F}_p\big) \to \mathrm{H}_{i(p-1)+ps}\big(Q(X), \mathbb{F}_p\big)$$

with $s + i \equiv 0 \pmod 2$. The Q_i are linear maps over $\mathbb{F}_p$.

Definition 3.4.4. There are functions

$$\bar{Q}_i : \mathrm{H}_s\big(Q(X), \bar{\mathbb{F}}_2\big) \to \mathrm{H}_{i+2s}\big(Q(X), \bar{\mathbb{F}}_2\big),$$

and if p is odd,

$$\bar{Q}_i : \mathrm{H}_s\big(Q(X), \bar{\mathbb{F}}_p\big) \to \mathrm{H}_{i(p-1)+ps}\big(Q(X), \bar{\mathbb{F}}_p\big)$$

defined by the formula $\bar{Q}_i(x) = \rho(Q_i(x) \otimes 1)$.

Remark 3.4.5. The functions $\bar{Q}_i$ are additive, but not linear. The failure of linearity is the content of the next theorem which will be used to determine the $GL(V)$-action.

Theorem 3.4.6. *Let x, y denote elements in $\mathrm{H}_s(Q(X), \bar{\mathbb{F}}_p)$. The functions $\bar{Q}_i$ defined above satisfy the following properties.*

(a) $\bar{Q}_i(x + y) = \bar{Q}_i(x) + \bar{Q}_i(y)$,
(b) $\bar{Q}_i(\lambda x) = (\lambda)^p \bar{Q}_i(x)$ *for* $\lambda \in \bar{\mathbb{F}}_p$.

Let $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$. The action of $GL(V)$ on $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is derived from these formulas in Section 4 *in case $p = 2$ and Section* 5 *in case p is odd.*

The statements in this theorem follow at once from classic work [2,14] and the formula $\bar{Q}_i(x) = \rho(Q_i(x) \otimes 1)$ with the sole exception of the statement $\bar{Q}_i(\lambda x) = (\lambda)^p \bar{Q}_i(x)$ for $\lambda \in \bar{\mathbb{F}}_p$. A proof of this last statement is given in Lemma 6.1.2 below.

The explanation for why Theorem 3.4.6 suffices to specify the $GL(V)$-action on $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is the subject of the next subsection.

3.5. This section provides a sketch of the connection between Theorem 3.4.6 and the $GL(V)$-action on $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$.

Let $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$. The direct sum $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is naturally an algebra generated by elements in V and compositions of the operations $\bar{Q}_i(-)$. Thus to identify the $GL(V)$-action, it suffices to identify the induced action on elements

$$v \in V \tag{3.5.1}$$

together with the elements

$$\bar{Q}_I(v) \tag{3.5.2}$$

where $\bar{Q}_I(-)$ denotes compositions of the operations $\bar{Q}_i(-)$, and the action on

$$\text{products of the } \bar{Q}_I(v). \tag{3.5.3}$$

Thus the results in Theorem 3.4.6 suffice to give the requisite commutation formulas with elements in $GL(V)$ given by

$$\bar{Q}_i(x+y) = \bar{Q}_i(x) + \bar{Q}_i(y) \tag{3.5.4}$$

and

$$\bar{Q}_i(\lambda x) = (\lambda)^p \bar{Q}_i(x) \tag{3.5.5}$$

for $\lambda \in \bar{\mathbb{F}}_p$.

In Sections 4 and 5 below, $\bigoplus_{d \geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ is given by

(1) a polynomial ring with generators given by iterates of the operations $\bar{Q}_i$ applied to elements $x \in V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$ for $p = 2$, and
(2) a tensor product of a polynomial ring with an exterior algebra with generators given by iterates of the operations $\bar{Q}_i$ as well as Bocksteins applied to elements $x \in V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$ for $p > 2$.

Some additional information is given next concerning the connection between the operations above and elements in $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$. Assume that $v \in V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$, $w \in \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ with $u_1 \in \mathrm{H}_\bullet(\Sigma_{d_1}, V^{\otimes d_1})$, and $u_2 \in \mathrm{H}_\bullet(\Sigma_{d_2}, V^{\otimes d_2})$. Then

$$\bar{Q}_i(v) \in \mathrm{H}_\bullet\big(\Sigma_p, V^{\otimes p}\big), \tag{3.5.6}$$

$$\bar{Q}_i(w) \in \mathrm{H}_\bullet\big(\Sigma_{pd}, V^{\otimes pd}\big) \tag{3.5.7}$$

and

$$u_1 \cdot u_2 \in \mathrm{H}_\bullet\big(\Sigma_{(d_1+d_2)}, V^{\otimes (d_1+d_2)}\big). \tag{3.5.8}$$

Thus to give the $GL(V)$-action, it suffices to identify this action on the elements of V and to evaluate the extension of this action to composites of the $\bar{Q}_i$ together with their products as specified inductively by Theorem 3.4.6. This process is carried out in finer detail in the next two sections.

4. The prime 2

4.1. The purpose of this section is to describe the known homology groups $\mathrm{H}_\bullet(Q(X), \bar{\mathbb{F}}_2)$ and then to describe the $GL(V)$-action on $\mathrm{H}_\bullet(Q(X), \bar{\mathbb{F}}_2)$.

Let $\{v_\gamma\}$ denote a choice of basis for $V = \bar{\mathrm{H}}_\bullet(X, \mathbb{F}_2)$. Then there is an isomorphism of algebras

$$\mathrm{H}_\bullet\big(Q(X), \mathbb{F}_2\big) \to S\big[Q_I(v_\gamma)\big],$$

where the following hold:

(1) The algebra $S[Q_I(v_\gamma)]$ denotes the polynomial algebra with generators $Q_I(v_\gamma)$.
(2) The generators $Q_I(v_\gamma)$ are specified by $I = (i_1, i_2, \dots, i_t)$ with

$$0 < i_1 \leqslant i_2 \leqslant \dots \leqslant i_t < \infty$$

and

$$Q_I(v_\gamma) = Q_{i_i} Q_{i_2} \cdots Q_{i_t}(v_\gamma).$$

(3) The empty sequence $I = \emptyset$ is allowed and in this case $Q_\emptyset(v_\gamma)$ is defined to be v_γ.
(4) The weight of a product is defined by $w(X \cdot Y) = w(X) + w(Y)$.

Define the weight of a monomial $Q_I(v_\gamma)$ where $I = (i_1, i_2, \dots, i_t)$ to be

$$w\big(Q_I(v_\gamma)\big) = 2^t. \tag{4.1.1}$$

A basis for the symmetric algebra is given by choices of the products of monomials

$$\mathcal{B}(V) = \big\{Q_{I_1}(v_{\gamma_1}) \cdots Q_{I_k}(v_{\gamma_s})\big\}. \tag{4.1.2}$$

In addition, a basis for $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ are those monomials in $\mathcal{B}(V)$ with

$$w\big(Q_{I_1}(v_{\gamma_1})\big) + \dots + w\big(Q_{I_s}(v_{\gamma_s})\big) = d \tag{4.1.3}$$

as given in [6].

4.2. The required modification for coefficients in $\bar{\mathbb{F}}_2$ is stated next. These follow at once by tensoring with $\bar{\mathbb{F}}_2$, appealing to the universal coefficient theorem and quoting Theorem 3.4.6 concerning the $GL(V)$-action.

Let $\{v_\gamma\}$ denote a choice of basis for the reduced homology $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_2)$. Then there is an isomorphism of algebras

$$\mathrm{H}_\bullet\big(Q(X), \bar{\mathbb{F}}_2\big) \to S\big[\bar{Q}_I(v_\gamma)\big]$$

where the following hold:

(1) The algebra $S[\bar{Q}_I(v_\gamma)]$ denotes the polynomial algebra with generators $\bar{Q}_I(v_\gamma)$.
(2) The generators $\bar{Q}_I(v_\gamma)$ are specified by $I = (i_1, i_2, \dots, i_t)$ with

$$0 < i_1 \leqslant i_2 \leqslant \dots \leqslant i_t < \infty$$

and

$$\bar{Q}_I(v_\gamma) = \bar{Q}_{i_i} \bar{Q}_{i_2} \cdots \bar{Q}_{i_t}(v_\gamma).$$

(3) The formula

$$\bar{Q}_I(\alpha x) = \alpha^{2^t} \bar{Q}_I(x)$$

holds for $\alpha \in k$ and $I = (i_1, i_2, \dots, i_t)$ by Theorem 3.4.6.

(4) The empty sequence $I = \emptyset$ is allowed and in this case $\bar{Q}_\emptyset(v_\gamma)$ is defined to be v_γ.

Define the weight of a monomial $\bar{Q}_I(v_\gamma)$ to be $w(\bar{Q}_I(v_\gamma)) = 2^t$ where $I = (i_1, i_2, \dots, i_t)$. The weight of a product is defined by $w(X \cdot Y) = w(X) + w(Y)$.

A basis for the symmetric algebra $\mathrm{H}_\bullet(Q(X), \bar{\mathbb{F}}_2)$, isomorphic to $S[\bar{Q}_I(v_\gamma)]$, is given by a choice of products of monomials for a polynomial algebra

$$\mathcal{B}(V) = \{\bar{Q}_{I_1}(v_{\gamma_1}) \cdots \bar{Q}_{I_k}(v_{\gamma_k})\}.$$

This feature follows with coefficients in $\bar{\mathbb{F}}_2$ directly from the above remarks together with the universal coefficient theorem and the analogous results in [6] where coefficients are taken in $\mathbb{F}_2$.

Then a basis for $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d} \otimes_{\mathbb{F}_2} \bar{\mathbb{F}}_2)$ are those monomials in $\mathcal{B}(V)$ with $w(\bar{Q}_{I_1}(v_{\gamma_1})) + \cdots + w(\bar{Q}_{I_k}(v_{\gamma_k})) = d$. The action of $GL_n(k)$ on $\bigoplus_{d \geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ follows from the formula

$$\bar{Q}_I(\alpha \cdot x + \beta \cdot y) = \alpha^{2^t} \cdot \bar{Q}_I(x) + \beta^{2^t} \cdot \bar{Q}_I(y) \tag{4.2.1}$$

for scalars α and β with $I = (i_1, i_2, \dots, i_t)$ for $0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t$. These elements also have a degree which correspond to the natural degrees in $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ (cf. Section 8).

5. Odd primes

5.1. We now describe the known homology groups $\mathrm{H}_\bullet(Q(X), \mathbb{F}_p)$ and then describe the $GL(V)$-action on $\mathrm{H}_\bullet(Q(X), k)$ for p odd. The methods are similar to those in Section 4.

Recall the homology of $Q(X)$ over $\mathbb{F}_p$ for odd primes p. There are operations

$$Q_j : \mathrm{H}_n(Q(X), \mathbb{F}_p) \to \mathrm{H}_{j(p-1)+pn}(Q(X), \mathbb{F}_p)$$

for which

(1) $n + j \equiv 0 \pmod 2$ and
(2) $j > 0$.
(3) That is, each operation Q_j is defined on even dimensional classes in case $j > 0$ is even and on odd dimensional classes in case j is odd.

There is an additional operation given by the Bockstein

$$\beta : \mathrm{H}_n(Q(X), \mathbb{F}_p) \to \mathrm{H}_{n-1}(Q(X), \mathbb{F}_p)$$

for which β^0 denotes the identity map

$$\beta^0 : \mathrm{H}_n(Q(X), \mathbb{F}_p) \to \mathrm{H}_n(Q(X), \mathbb{F}_p).$$

Let $\{v_\alpha\}$ denote a choice of basis for the reduced homology groups $V = \bar{\mathrm{H}}_\bullet(X, \mathbb{F}_p)$. Then there is an isomorphism of algebras

$$\mathrm{H}_\bullet\big(Q(X), \mathbb{F}_p\big) \to S\big[Q_J(v_\alpha) \,\big|\, \deg\big(Q_J(v_\alpha)\big) \equiv 0 \ (\mathrm{mod}\ 2)\big] \otimes E\big[Q_J(v_\alpha) \,\big|\, \deg\big(Q_J(v_\alpha)\big) \equiv 1 (\mathrm{mod}\ 2)\big],$$

where

(4) $\deg(x)$ denotes the degree of an element x,
(5) $S[Q_J(v_\alpha)]$ denotes the polynomial algebra with generators $Q_J(v_\alpha)$ (which may contain Bocksteins as given in the next paragraph), and
(6) $E[Q_J(v_\alpha)]$ denotes the exterior algebra with generators $Q_J(v_\alpha)$ (which may contain Bocksteins as given in the next paragraph).

The elements $Q_J(v_\alpha)$ are described next where $J = (\epsilon_1, j_1, \epsilon_2, j_2, \ldots, \epsilon_t, j_t)$ with $0 < j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t < \infty$, $\epsilon_s = 0, 1$ with $1 \leqslant s \leqslant t$. Then

$$Q_J(v_\alpha) = \beta^{\epsilon_1} Q_{j_1} \beta^{\epsilon_2} Q_{j_2} \cdots \beta^{\epsilon_t} Q_{j_t}(v_\alpha),$$

whenever that operation is defined.

The monomials in $Q_J(v_\alpha)$ can be interpreted in terms of the homology of the symmetric groups with coefficients in the tensor powers of $\bar{\mathrm{H}}_\bullet(X, \mathbb{F}_p)$ as follows. Each monomial $Q_J(v_\alpha)$ has a weight given by $w(Q_J(v_\alpha)) = p^t$ for $J = (\epsilon_1, j_1, \epsilon_2, j_2, \ldots, \epsilon_t, j_t)$. Furthermore, the weight of a product is defined by $w(X \cdot Y) = w(X) + w(Y)$. In case of the empty sequence $J = \emptyset$, $Q_\emptyset(v_\alpha)$ is defined to be v_α.

Then the group $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ for $V = \bar{\mathrm{H}}_\bullet(X, \mathbb{F}_p)$ is the linear span of the product of monomials in

$$S\big[Q_J(v_\alpha) \,\big|\, \deg\big(Q_J(v_\alpha)\big) \equiv 0 \ (\mathrm{mod}\ 2)\big] \otimes E\big[Q_J(v_\alpha) \,\big|\, \deg\big(Q_J(v_\alpha)\big) \equiv 1 \ (\mathrm{mod}\ 2)\big]$$

of weight exactly d. As above, these elements also have a degree which correspond to the natural degrees in $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$.

5.2. The natural modifications for coefficients in $\bar{\mathbb{F}}_p$ are stated next. In this section, homology is taken with $\bar{\mathbb{F}}_p$-coefficients for odd primes p. There are operations

$$\bar{Q}_j : \mathrm{H}_n\big(Q(X), \bar{\mathbb{F}}_p\big) \to \mathrm{H}_{j(p-1)+pn}\big(Q(X), \bar{\mathbb{F}}_p\big)$$

for which

(1) $n + j \equiv 0 \ (\mathrm{mod}\ 2)$ and
(2) $j > 0$.
(3) That is, each operation $\overline{Q_j}$ is defined (i) on even dimensional classes in case j is even with $j > 0$ and (ii) on odd dimensional classes in case j is odd.

There is an additional operation given by the Bockstein

$$\beta : \mathrm{H}_n\big(Q(X), \bar{\mathbb{F}}_p\big) \to \mathrm{H}_{n-1}\big(Q(X), \bar{\mathbb{F}}_p\big)$$

for which β^0 denotes the identity map

$$\beta^0 : \mathrm{H}_n\big(Q(X), \bar{\mathbb{F}}_p\big) \to \mathrm{H}_n\big(Q(X), \bar{\mathbb{F}}_p\big).$$

Let $\{v_\alpha\}$ denote a choice of basis for the reduced homology groups $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$. Then there is an isomorphism of algebras

$$\begin{aligned}\mathrm{H}_\bullet\big(Q(X), \bar{\mathbb{F}}_p\big) \to S\big[\bar{Q}_J(v_\alpha) \,\big|\, \deg\big(\bar{Q}_J(v_\alpha)\big) \equiv 0 \ (\mathrm{mod}\ 2)\big] \\ \otimes E\big[\bar{Q}_J(v_\alpha) \,\big|\, \deg\big(\bar{Q}_J(v_\alpha)\big) \equiv 1 \ (\mathrm{mod}\ 2)\big],\end{aligned}$$

where

(4) $\deg(x)$ denotes the degree of an element x,
(5) $S[\bar{Q}_J(v_\alpha)]$ denotes the polynomial algebra with generators $\bar{Q}_J(v_\alpha)$ (which may contain Bocksteins), and
(6) $E[\bar{Q}_J(v_\alpha)]$ denotes the exterior algebra with generators $\bar{Q}_J(v_\alpha)$ (which may contain Bocksteins).

The elements $\bar{Q}_J(v_\alpha)$ are described next where $J = (\epsilon_1, j_1, \epsilon_2, j_2, \ldots, \epsilon_t, j_t)$ with $0 < j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t < \infty$, $\epsilon_s = 0, 1$ with $1 \leqslant s \leqslant t$. Then

$$\bar{Q}_J(v_\alpha) = \beta^{\epsilon_1} \bar{Q}_{j_1} \beta^{\epsilon_2} \bar{Q}_{j_2} \cdots \beta^{\epsilon_t} \bar{Q}_{j_t}(v_\alpha),$$

whenever that operation is defined.

The monomials in $\bar{Q}_J(v_\alpha)$ can be interpreted in terms of the homology of the symmetric groups with coefficients in the tensor powers of $\bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$ as follows. Each monomial $\bar{Q}_J(v_\alpha)$ has a weight given by $w(\bar{Q}_J(v_\alpha)) = p^t$ for $J = (\epsilon_1, j_1, \epsilon_2, j_2, \ldots, \epsilon_t, j_t)$. Furthermore, the weight of a product is defined by $w(X \cdot Y) = w(X) + w(Y)$. In case of the empty sequence $J = \emptyset$, $\bar{Q}_\emptyset(v_\alpha)$ is defined to be v_α.

Then the group $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ for $V = \bar{\mathrm{H}}_\bullet(X, \bar{\mathbb{F}}_p)$ is the linear span of the product of monomials in

$$S\big[\bar{Q}_J(v_\alpha) \,\big|\, d\big(\bar{Q}_J(v_\alpha)\big) \equiv 0 \ (\mathrm{mod}\ 2)\big] \otimes E\big[\bar{Q}_J(v_\alpha) \,\big|\, d\big(\bar{Q}_J(v_\alpha)\big) \equiv 1 \ (\mathrm{mod}\ 2)\big]$$

of weight exactly d.

Furthermore $\bar{Q}_i(\lambda x) = (\lambda)^p \bar{Q}_i(x)$ for $\lambda \in \bar{\mathbb{F}}_p$. Thus

$$\bar{Q}_J(\lambda v_\alpha) = \lambda^{p^t} \bar{Q}_J(v_\alpha)$$

for $J = (\epsilon_1, j_1, \epsilon_2, j_2, \ldots, \epsilon_t, j_t)$. This formula follows by iterating the formula

$$\bar{Q}_i(\lambda x) = \lambda^p \bar{Q}_i(x)$$

as stated in Theorem 3.4.6.

6. Homology operations for $Q(X)$ over $\bar{\mathbb{F}}_p$

6.1. In this section we work out the formula $\bar{Q}_i(\lambda x) = \lambda^p \bar{Q}_i(x)$ for $\lambda \in \bar{\mathbb{F}}_p$ as stated in Lemma 6.1.2 below. The proof of this lemma then finishes the proof of Theorem 3.4.6.

The operations Q_i, originally due to Araki and Kudo for $p = 2$ with odd primary versions due to Dyer and Lashof, were defined over the field with p elements [2,14]. These operations in the special case for the homology of the symmetric groups were implicit in Nakaoka's computations [25]. In addition, these operations admit extensions to homology taken with coefficients in $\bar{\mathbb{F}}_p$ as described above via the universal coefficient theorem. Properties of these operations over $\bar{\mathbb{F}}_p$ are obtained from the structure of $\mathrm{H}_\bullet(\Sigma_p, V^{\otimes p})$. These homology groups are easy to work out using the p-Sylow subgroup of Σ_p.

As preparation, it is convenient to first recall properties of $V^{\otimes p}$ where V is a vector space over a field k of characteristic p with further choices of either $k = \mathbb{F}_p$, or $k = \bar{\mathbb{F}}_p$ made explicit below. The cyclic group of order p generated by the p-cycle $\sigma_p = (1, 2, \dots, p)$ acts naturally on $V^{\otimes p}$. Thus $V^{\otimes p}$ is naturally a $k[\mathbb{Z}/p\mathbb{Z}]$-module.

Choose a totally ordered basis for V, say e_α for $\alpha \in S$. A basis for $V^{\otimes p}$ is described next. Consider the multi-index $A = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \in S$, with

$$e_A = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \in V^{\otimes p}. \tag{6.1.1}$$

Given the set S, consider the p-fold product $S^{\times p}$. Observe that $\mathbb{Z}/p\mathbb{Z}$ acts on the set $S^{\times p}$ via the p-cycle $\sigma_p = (1, 2, \dots, p)$. Let

$$\Delta^{\times p}(S) = \bigl\{(\alpha_1, \dots, \alpha_p) \in S^{\times p} \bigm| \alpha_i = \alpha_j \text{ for all } i, j\bigr\}. \tag{6.1.2}$$

Thus $\Delta^{\times p}(S)$ denotes the diagonal subset of $S^{\times p}$.

Let $\Gamma^{\times p}(S)$ denote the complement of $\Delta^{\times p}(S)$ in $S^{\times p}$. Thus

$$\Gamma^{\times p}(S) = \bigl\{(\alpha_1, \dots, \alpha_p) \in S^{\times p} \bigm| \alpha_i \neq \alpha_j \text{ for some } i < j\bigr\}. \tag{6.1.3}$$

The action of $\mathbb{Z}/p\mathbb{Z}$ on the set $S^{\times p}$ restricts to actions on both $\Delta^{\times p}(S)$, and $\Gamma^{\times p}(S)$. Thus the set $S^{\times p}$ is a disjoint union of sets with

$$S^{\times p} = \Delta^{\times p}(S) \sqcup \Gamma^{\times p}(S) \tag{6.1.4}$$

as a $\mathbb{Z}/p\mathbb{Z}$-set.

Define the set $T(S)$ by choosing one element in each $\mathbb{Z}/p\mathbb{Z}$-orbit in $\Gamma^{\times p}(S)$. Observe that in case S is finite of cardinality N, then

(1) $S^{\times p}$ has cardinality N^p,
(2) $\Delta^{\times p}(S)$ has cardinality N,
(3) $\Gamma^{\times p}(S)$ has cardinality $N^p - N$ which is divisible by p, and
(4) the cardinality of $T(S)$ is $(1/p)(N^p - N)$.

A specific choice of basis for $V^{\otimes p}$ consists of two types of elements as follows. Define

$$e_A = e_\alpha^{\otimes p} = \overbrace{e_\alpha \otimes \cdots \otimes e_\alpha}^{p \text{ times}} \tag{6.1.5}$$

with $A = (\alpha, \dots, \alpha) \in \Delta^{\times p}(S)$ for all $\alpha \in S$. Define

$$e_B = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \tag{6.1.6}$$

with $B = (\alpha_1, \dots, \alpha_p) \in T(S)$ where at least two of the α_j differ for all $B \in T(S)$.

Consider the $k[\mathbb{Z}/p\mathbb{Z}]$-modules given by

(1) the cyclic $k[\mathbb{Z}/p\mathbb{Z}]$-module spanned by $e_A = \overbrace{e_\alpha \otimes \cdots \otimes e_\alpha}^{p \text{ times}}$ denoted $\langle e_A \rangle$ for $A \in \Delta^{\times p}(S)$, and
(2) the cyclic $k[\mathbb{Z}/p\mathbb{Z}]$-module spanned by a *choice* of the elements $e_B = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p}$ denoted $\langle e_B \rangle$ with $B = (\alpha_1, \dots, \alpha_p) \in T(S)$.

The choices of basis elements above then give a direct sum decomposition of $V^{\otimes p}$. Let

$$V^{\otimes p}(\text{fix}) \tag{6.1.7}$$

denote the linear span in $V^{\otimes p}$ of the $\langle e_A \rangle$ for all $A \in \Delta^{\times p}(S)$. Let

$$V^{\otimes p}(\text{free}) \tag{6.1.8}$$

denote the linear span in $V^{\otimes p}$ of the $\langle e_B \rangle$ for all $B = (\alpha_1, \dots, \alpha_p) \in T(S)$.

Versions of the next lemma arose in work of Smith [27], and Steenrod [29] in which $V^{\otimes p}(\text{free})_{\mathbb{Z}/p\mathbb{Z}}$ denotes the natural module of co-invariants.

Lemma 6.1.1. *Let k be any field of characteristic p. The natural inclusion*

$$V^{\otimes p}(\text{fix}) \oplus V^{\otimes p}(\text{free}) \to V^{\otimes p}$$

is an isomorphism of $k[\mathbb{Z}/p\mathbb{Z}]$-modules. Thus there are induced isomorphisms

$$\mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}(\text{fix}) \oplus V^{\otimes p}(\text{free})\big) \to \mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}\big)$$

and

$$\mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}(\text{fix})\big) \oplus \mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}(\text{free})\big) \to \mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}\big).$$

Furthermore,

(1) $V^{\otimes p}(\text{fix})$ *is a trivial $k[\mathbb{Z}/p\mathbb{Z}]$-module, and*
(2) $V^{\otimes p}(\text{free})$ *is a free $k[\mathbb{Z}/p\mathbb{Z}]$-module.*

Thus there are isomorphisms

$$\big(\mathrm{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, k) \otimes_k V^{\otimes p}(\text{fix})\big) \oplus V^{\otimes p}(\text{free})_{\mathbb{Z}/p\mathbb{Z}} \to \mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}\big).$$

This information is used to prove the next lemma.

Lemma 6.1.2. *Assume that $k = \bar{\mathbb{F}}_p$ with $x \in V = \mathrm{H}_\bullet(X, \bar{\mathbb{F}}_p)$. The formula $\bar{Q}_i(\lambda x) = \lambda^p \bar{Q}_i(x)$ for $\lambda \in \bar{\mathbb{F}}_p$ is satisfied.*

Proof. Assume that all homology groups are taken with coefficients in $k = \bar{\mathbb{F}}_p$. Observe that the natural map

$$\mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}\big) \to \mathrm{H}_\bullet\big(\Sigma_p, V^{\otimes p}\big)$$

is an epimorphism as $\mathbb{Z}/p\mathbb{Z}$ is the p-Sylow subgroup of Σ_p and $\bar{\mathbb{F}}_p$ has characteristic p. Furthermore,

$$\mathrm{H}_\bullet\big(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}\big)$$

is described classically (by Smith and by Steenrod) as in Lemma 6.1.1 obtained from isomorphisms of $\mathbb{F}[\mathbb{Z}/p\mathbb{Z}]$-modules

$$V^{\otimes p} \to V^{\otimes p}(\mathrm{fix}) \oplus V^{\otimes p}(\mathrm{free}).$$

Let $B_\bullet$ denote the classical minimal free resolution of k regarded as a trivial $k[\mathbb{Z}/p\mathbb{Z}]$-module with basis for $B_i = k[\mathbb{Z}/p\mathbb{Z}]$ given by g_i in degree i. Next, consider the homology of the chain complex

$$B_\bullet \otimes_{k[\mathbb{Z}/p\mathbb{Z}]} \langle e_A \rangle$$

for $A = (\alpha, \ldots, \alpha) \in \Delta^{\times p}(S)$. Thus

$$e_A = \overbrace{e_\alpha \otimes \cdots \otimes e_\alpha}^{p \text{ times}} = e_\alpha{}^{\otimes p}$$

as described in formula (6.1.5).

Furthermore, the class of $g_i \otimes (\lambda \cdot e_\alpha)^{\otimes p}$ is equal to the class of $\lambda^p g_i \otimes (e_\alpha)^{\otimes p}$ in the homology of $B_\bullet \otimes_{k[\mathbb{Z}/p\mathbb{Z}]} \langle e_A \rangle$. Since the element $\bar{Q}_i(\lambda x)$ is identified with the image of $g_i \otimes (\lambda \cdot e_\alpha)^{\otimes p}$ up to a scalar multiple depending on i and the degree of x in the homology of $Q(X)$, the lemma follows. □

Some remarks about graded vector spaces V and the sign representation are given in the next section.

Remark 6.1.3. The cyclic $k[\mathbb{Z}/p\mathbb{Z}]$-modules spanned by $e_A = \overbrace{e_\alpha \otimes \cdots \otimes e_\alpha}^{p \text{ times}}$ denoted $\langle e_A \rangle$ are all trivial $k[\mathbb{Z}/p\mathbb{Z}]$-modules where each such module is concentrated in degree $p \cdot \deg(e_\alpha)$, where $\deg(e_\alpha)$ denotes the degree of the element e_α.

The modules $\langle e_A \rangle$ are not always trivial $k\Sigma_p$-modules as the action depends on the degree of e_A. In particular, there are isomorphisms of $k\Sigma_p$-modules given by

$$e_A = \begin{cases} k & \text{if } \deg(e_\alpha) \text{ is even, and} \\ \mathrm{sgn} & \text{if } \deg(e_\alpha) \text{ is odd.} \end{cases}$$

The structure of the operations $Q_i(-)$ in case $k = \mathbb{F}_p$, and $\bar{Q}_i(-)$ in case $k = \bar{\mathbb{F}}_p$ 'record' this structure, and thus give information about coefficients in the sign representation. This trick has been used extensively in working out the homology of certain mapping class groups with coefficients in sgn.

7. Topological analogues

7.1. The purpose of this section is to explain the relationship in more detail between the homology of $Q(X)$ and the groups $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ where homology is taken with coefficients in any field k. First, recall that if V is a connected graded vector space over k, then a choice of space X was given in Section 3 which has reduced homology given by V: one choice of X is a wedge of spheres with reduced homology given by V. This choice of X as a wedge of spheres has a second feature which is addressed next.

Namely, the reduced homology groups of a wedge of spheres is a free abelian group. Consider the case for which V is obtained by tensoring a free module M over the integers $\mathbb{Z}$ with the field k, namely

$$V = M \otimes_{\mathbb{Z}} k.$$

In this case X may be chosen to have the analogous property given by $M = \bar{H}_*(X, \mathbb{Z})$ and

$$V = \bar{H}_*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} k.$$

The point of view here is that, with mild conditions concerning the graded vector space V, there is a topological space X with the property that the homology of the space $Q(X)$, a functor of X, has homology given by

$$\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

This feature then provides a transparent way to determine the $GL(V)$-action on $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$.

To describe this isomorphism in more detail, it is useful from the topological point of view to have base-points. Thus this section will be restricted to path-connected, pointed CW-complexes $(X, *)$ where $*$ is the base-point of X. Maps are required to preserve base-points denoted $*$. Furthermore, a wedge of pointed spaces has a natural base-point. Thus the example of a wedge of spheres given in Section 3 suffices.

Next let $E\Sigma_d$ denote a contractible Hausdorff space which has a free (right) action of Σ_d. Let X^d denote the d-fold product and $X^{(d)}$ the d-fold smash product given by $X^{(d)} = X^d/S(X^d)$ where $S(X^d)$ denotes the subspace of X^d given by

$$S\big(X^d\big) = \big\{(x_1, \ldots, x_d) \;\big|\; x_i = * \text{ for some } 1 \leqslant i \leqslant d\big\}.$$

In addition, let $*$ also denote the class of the base-point in $X^{(d)}$.

Consider the functor from "pointed spaces" to "pointed spaces" which sends the space X to the space given by

$$\big(E\Sigma_d \times_{\Sigma_d} X^{(d)}\big)/\big(E\Sigma_d \times_{\Sigma_d} \{*\}\big).$$

This last construction is denoted $D_d(X)$ in what follows.

Some properties of the space $D_d(X)$ are listed next where it is assumed that X is a path-connected CW-complex with $V = \bar{\mathrm{H}}_\bullet(X,k)$, the reduced homology groups of X over the field k.

(1) The construction $D_d(X)$ gives a functor $D_d(-)$ with values $D_d(X)$ for any pointed space X. Thus $D_d(-)$ is a functor from pointed spaces to pointed spaces.
(2) The reduced (singular) homology of $X^{(d)}$ with any field coefficients k is isomorphic to $V^{\otimes d}$ as a (left) Σ_d-module.
(3) The reduced homology of $D_d(X)$ is isomorphic to $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$.
(4) Consider the functor from pointed spaces to pointed spaces which sends the space X to the space given by $Q(X)$. Then $Q(X)$ satisfies the property that there is an isomorphism of algebras

$$\mathrm{H}_\bullet\big(Q(X),\mathbb{F}\big) \to \bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

7.2. A theorem originally due to Kahn [20], subsequently proven by Snaith [28] with extensions to related configuration spaces as well as variations in [8] can be stated as follows.

Theorem 7.2.1. *Let X denote a path-connected CW-complex. Then there is a natural stable homotopy equivalence*

$$H : Q(X) \longrightarrow \bigvee_{d\geqslant 0} D_d(X).$$

Thus there are isomorphisms in homology, natural for pointed spaces $(X,*)$,

$$E_*\big(Q(X)\big) \to E_*\Big(\bigvee_{d\geqslant 0} D_d(X)\Big)$$

for any homology theory $E_*(-)$. In particular, if $V = \bar{\mathrm{H}}_\bullet(X,k)$ is regarded as a graded vector space over a field k, there are isomorphisms

$$\mathrm{H}_\bullet\big(Q(X),\mathbb{F}\big) \to \bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, V^{\otimes d}\big).$$

7.3. Some remarks about where these results can be found are listed next. An efficient computation of the homology of $Q(X)$ is given in the proof of Theorem 4.2 on pages 40–47 of [6]. The stable decomposition in Theorem 7.2.1 is proven in [20,28]. A short two page proof is in the appendix of [7].

That the algebraic weights for the homology of $Q(X)$ agree with those arising from the stable decompositions follows from the way in which the operations are defined. One sketch is in [6], pages 237–243.

Finally, a remark about notation: there are two different notations for homology operations in use. The 'lower notation' $Q_i(-)$ is used above while 'upper notation' $Q^s(-)$ is used in [6]. The translation is as follows:

(1) $p = 2$: $Q_{s-q}(x) = Q^s(x)$ if $s > q = \text{degree}(x)$.
(2) $p > 2$: $Q_{(2s-q)}(x) = cQ^s(x)$ for a nonzero scalar c if $2s > q = \text{degree}(x)$, a choice of notation slightly different than that in [6], foot of page 7.

8. A degree shift

8.1. In the work above in which the $GL(V)$-action on $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ was analyzed, it was specifically assumed that V is the reduced homology of a path-connected topological space X, namely

$$V = \bar{\mathrm{H}}_\bullet(X, k).$$

However, the reduced homology of any path-connected space is concentrated in degrees greater than 0.

Thus to address the structure of $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ where W is a graded vector space concentrated in degree 0, some technical modifications are required. This modification is achieved through a formal degree shift which is addressed in this section. One way to achieve this modification is as follows.

Definition 8.1.1. Let W be a vector space over k concentrated in degree 0. Given a fixed natural number $n \in \mathbb{N}$ define a graded vector space over k denoted

$$(n, W)$$

which is

(1) concentrated in degree n, namely

$$(n, W) = \{0\}$$

in degrees not equal to n, and
(2) (n, W) in degree n is isomorphic to W.

Thus there is a morphism of vector spaces (which does not respect gradation) given by

$$\sigma_n : W \to (n, W),$$

where

$$\sigma_n(x) = (n, x).$$

Elementary features of $\sigma_n : W \to (n, W)$ are stated next.

Lemma 8.1.2. *The morphism of vector spaces (which does not respect gradation) given by*

$$\sigma_n : W \to (n, W)$$

is an isomorphism of underlying vector spaces. Furthermore, if the rank of W is at least one, the induced map given by the d-fold tensor product, $d \geqslant 2$, of σ_n,

$$(\sigma_n)^{\otimes d} : W^{\otimes d} \to (n, W)^{\otimes d}$$

is an isomorphism of underlying modules over $k\Sigma_d$ provided n is even.

Remark 8.1.3. If n is odd, and $d \geqslant 2$, then the map of vector spaces

$$(\sigma_n)^{\otimes d} : W^{\otimes d} \to (n, W)^{\otimes d}$$

is not an isomorphism of underlying modules over $k\Sigma_d$. In particular if n is odd and W is of rank one, then

(1) $W^{\otimes 2}$ is a vector space of rank one which is the trivial representation of Σ_2, and
(2) $(n, W)^{\otimes 2}$ is still a vector space of rank one, but is the sign representation of Σ_2.

Furthermore, if n is even, the map

$$(\sigma_n)^{\otimes d} : W^{\otimes d} \to (n, W)^{\otimes d}$$

is both an isomorphism of vector spaces and of modules over $k\Sigma_d$. The most economical way to achieve this is by setting $n = 2$ as is done below.

Definition 8.1.1 provides a natural degree shift for the isomorphism given in Sections 4 and 5 above. This will ensure the degrees match with the natural degrees in $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$. Namely, start with a graded vector space W concentrated in degree 0.

Then

$$(\sigma_{2n})^{\otimes d} : W^{\otimes d} \to (2n, W)^{\otimes d}$$

induces an isomorphism of underlying Σ_d-modules (which does not preserve degrees). Thus there is an induced natural shift map on the level of homology which gives isomorphisms

$$\Theta_d(2n) : \mathrm{H}_s\big(\Sigma_d, W^{\otimes d}\big) \to \mathrm{H}_{s+2nd}\big(\Sigma_d, (2n, W)^{\otimes d}\big)$$

as well as isomorphisms

$$\bigoplus_{d\geqslant 0} \Theta_d(2n) : \bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, W^{\otimes d}\big) \to \bigoplus_{d\geqslant 0} \mathrm{H}_{\bullet+2nd}\big(\Sigma_d, (2n, W)^{\otimes d}\big).$$

This all is described by the following theorem, which uses Lemma 6.1.2 with $k = \bar{\mathbb{F}}_p$.

Theorem 8.1.4.

Let W be a graded vector space concentrated in degree 0 with basis $\{w_\gamma \mid \gamma \in S\}$ over $\bar{\mathbb{F}}_p$. Then

$$\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, W^{\otimes d}\big)$$

is a graded,commutative algebra given as follows.

(a) *If* $p = 2$, *then* $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ *is isomorphic to the polynomial algebra* $S[\bar{Q}_I(w_\gamma)]$ *with generators* $\bar{Q}_I(w_\gamma)$, $I = (i_1, i_2, \dots, i_t)$, $0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t < \infty$ *where the degree of* $\bar{Q}_I(w_\gamma)$ *is*

$$i_1 + 2i_2 + 4i_3 + \cdots + 2^{i_t-1} i_t.$$

(b) *If* $p > 2$, *then* $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ *is isomorphic to the symmetric algebra*

$$S\big[\bar{Q}_J(w_\gamma) \bigm| \deg\big(\bar{Q}_J(w_\gamma)\big) \equiv 0 \pmod 2\big] \otimes E\big[\bar{Q}_J(w_\gamma) \bigm| \deg\big(\bar{Q}_J(w_\gamma)\big) \equiv 1 \pmod 2\big]$$

for which $J = (\epsilon_1, j_1, \epsilon_2, j_2, \dots, \epsilon_t, j_t)$ *with* $0 < j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t < \infty$, $\epsilon_k = 0, 1$ *with* $1 \leqslant k \leqslant t$,

(i) $S[\bar{Q}_J(w_\gamma)]$ *denotes the polynomial algebra with generators* $\bar{Q}_J(w_\gamma)$ *(which may contain Bocksteins but must start with* $\bar{Q}_{j_t}(w_\gamma)$ *and is not allowed to start with* $\beta(w_\gamma)$*)*,

(ii) $E[\bar{Q}_J(w_\gamma)]$ *denotes the exterior algebra with generators* $\bar{Q}_J(w_\gamma)$ *(which may contain Bocksteins but must start with* $\bar{Q}_{j_t}(w_\gamma)$ *and is not allowed to start with* $\beta(w_\gamma)$*), and*

(iii) *the degree of* $\bar{Q}_J(w_\gamma)$ *for* $J = (\epsilon_1, j_1, \epsilon_2, j_2, \dots, \epsilon_t, j_t)$ *is equal to*

$$\big(-\epsilon_1 + j_1(p-1)\big) + p\big(-\epsilon_2 + j_2(p-1)\big) + \cdots + p^{t-1}\big(-\epsilon_t + j_t(p-1)\big).$$

The action of $GL(W)$ *on*

$$\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet\big(\Sigma_d, W^{\otimes d}\big)$$

is given in terms of the formulas in Section 4 *in case* $p = 2$ *and Section* 5 *in case* p *is odd for which*

$$\bar{Q}_i(\lambda x) = \lambda^p \bar{Q}_i(x)$$

with $\lambda \in \bar{\mathbb{F}}_p$.

8.2. Let $k = \bar{\mathbb{F}}_p$. The description of the action of $GL(W)$ from Sections 4 and 5, shows that the action on monomials of the form $\bar{Q}_i(w)$ or $\bar{Q}_{\epsilon_1,i_1}(w)$ is through a Frobenius twist. That is, this action gives $GL(W)$-modules which are isomorphic to $W^{(1)}$. A monomial $\bar{Q}_{i_1,i_2}(w)$ would correspond to the $GL(W)$-module $W^{(2)}$. Monomials $\bar{Q}_i(w)\bar{Q}_i(w)$ would give a $S^2(W^{(1)})$. In odd characteristic, the square of a monomial in the exterior algebra part would give zero, so we obtain modules of the form $\Lambda^c(W^{(c)})$ but no symmetric powers of such modules. Thus Theorem 8.1.4 implies the following.

Corollary 8.2.1. *Let* $k = \bar{\mathbb{F}}_p$, *and let* W *denote a vector space over* k *of* rank n *concentrated in degree zero, so* $GL(W) = GL_n(k)$. *In characteristic two, the* $GL_n(k)$*-module* $\mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ *is a direct sum of modules of the form:*

$$S^{a_1}(W) \otimes S^{a_2}\big(W^{(c_2)}\big) \otimes \cdots \otimes S^{a_s}\big(W^{(c_s)}\big) \tag{8.2.1}$$

where each $a_i \geqslant 0$, $c_i > 0$ *and* $d = a_1 + \sum_{j=2}^{s} a_j 2^{c_j}$.

In odd characteristic the $GL_n(k)$*-module* $\mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ *is a direct sum of modules of the form*

$$S^{a_1}(W) \otimes S^{a_2}\big(W^{(c_2)}\big) \otimes \cdots \otimes S^{a_s}\big(W^{(c_s)}\big) \otimes \Lambda^{d_2}\big(W^{(d_2)}\big) \otimes \cdots \otimes \Lambda^{d_t}\big(W^{(d_t)}\big), \quad (8.2.2)$$

where each $a_i \geqslant 0$, *each* $c_i, d_i > 0$ *and where* $d = a_1 + \sum_{j=2}^{s} a_j p^{c_j} + \sum_{j=2}^{t} d_j p^{d_j}$.

Remark 8.2.2. Corollary 8.2.1 cannot be used alone to determine $\mathrm{H}_i(\Sigma_d, W^{\otimes d})$ as a $GL(W)$-module, additional details on degrees of monomials in Theorem 8.1.4 are needed. To compute $\mathrm{H}_i(\Sigma_d, W^{\otimes d})$, one needs to determine all monomials inside $\bigoplus_{d\geqslant 0} \mathrm{H}_\bullet(\Sigma_d, W^{\otimes d})$ which lie in degree i and have weight d. To each such monomial, one obtains a corresponding $GL_n(k)$-module of the form described in Corollary 8.2.1. In several of the applications below it is enough to know that each summand in (8.2.1) and (8.2.2) is of the form $S^a(W) \otimes M^{(1)}$.

Remark 8.2.3. We remark that as a $GL_n(k)$-module, $[V^{(b)}]^{\otimes d} \cong [V^{\otimes d}]^{(b)}$ because if $F^b : GL_n(k) \to GL_n(k)$ is the bth iteration of the Frobenius map then the action of G on $[V^{(b)}]^{\otimes d}$ is given by

$$g.(v_1 \otimes \cdots \otimes v_d) = (g.v_1 \otimes \cdots \otimes g.v_d) = \big(F^b(g)v_1 \otimes \cdots \otimes F^b(g)v_d\big) = F^b(g)(v_1 \otimes \cdots \otimes v_d),$$

which coincides with the action of $GL_n(k)$ on $[V^{\otimes d}]^{(b)}$. This argument also shows that one has natural $GL_n(k)$-module isomorphisms between $S^d(V^{(b)}) \cong S^d(V)^{(b)}$ and $\Lambda^d(V^{(b)}) \cong \Lambda^d(V)^{(b)}$.

Remark 8.2.4. The results in this section are stated for $\bar{\mathbb{F}}_p$, but by base change arguments will hold for any algebraically closed field of prime characteristic. More specifically, the homology of Σ_d with coefficients in $V^{\otimes d}$ can be computed over any field of characteristic p by using the Universal Coefficient Theorem. Henceforth, we will assume that k is an arbitrary algebraically closed field of characteristic p.

9. Cohomology between Young modules

9.1. The remainder of the paper consists of applications of Theorem 8.1.4 and Corollary 8.2.1 to the representation theory of the symmetric group. Since any $n \geqslant d$ gives the same results, we will assume henceforth that $n = d$. Further, we henceforth always let V denote the natural $GL_d(k)$ module, as we will be applying Theorem 8.1.4 and Corollary 8.2.1 but will have no need of the more general results where V is not concentrated in degree zero. Theorem 2.3.1(b) will allow us to compute $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ from our description of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$. However much more information can be obtained from this calculation.

In this section, we observe that with additional knowledge about decomposition numbers for the Schur algebra $S(d,d)$ one can compute $\mathrm{Ext}^\bullet_{\Sigma_d}(Y^\lambda, Y^\mu)$ for arbitrary partitions $\lambda, \mu \vdash d$. The results in this section do not use the topological information from the previous sections.

9.2. The decomposition numbers for $S(d,d)$ are precisely the multiplicities $[V(\lambda) : L(\mu)]$ for $\lambda, \mu \vdash d$ where $V(\lambda)$ is the Weyl module of highest weight λ. We will first explain the connection between the decomposition numbers for $S(d,d)$ and homomorphisms between Young modules.

Proposition 9.2.1. *Knowing the decomposition matrix for the Schur algebra $S(d,d)$ is equivalent to knowing* $\dim \operatorname{Hom}_{\Sigma_d}(Y^\lambda, Y^\mu)$ *for all* $\lambda, \mu \vdash d$.

Proof. Recall from [13, 7.1] that $\mathcal{G}_{\mathrm{Hom}}(Y^\mu) = P(\mu)$, where $P(\mu)$ is the indecomposable projective cover of $L(\lambda)$ in the category of $S(d,d)$-modules. This implies that

$$\operatorname{Hom}_{\Sigma_d}\big(Y^\lambda, Y^\mu\big) \cong \operatorname{Hom}_{S(d,d)}\big(P(\lambda), P(\mu)\big)$$

for all $\lambda, \mu \vdash d$. The dimensions of the latter Hom-spaces give the Cartan matrix for $S(d,d)$ (i.e., multiplicities of composition factors in projective indecomposable modules). It is well known that the decomposition matrix times its transpose gives the Cartan matrix for $S(d,d)$. Therefore, from the decomposition matrix we can get the dimensions of the homomorphisms between Young modules.

It remains to prove that from the Cartan matrix for $S(d,d)$ one can deduce the decomposition numbers. This will follow by using reverse induction on the ordering on the weights, which in this setting can be taken as just the lexicographic order on partitions of d. Note for a maximal weight λ, $P(\lambda) = V(\lambda)$ and

$$\big[V(\lambda) : L(\mu)\big] = \big[P(\lambda) : L(\mu)\big]$$

is known for all $\mu \vdash d$. In general, $P(\lambda)$ has a filtration by Weyl modules with one copy of $V(\lambda)$ on top, and successive subquotients of the form $V(\mu)$ for $\mu > \lambda$. We let $[P(\lambda) : V(\mu)]$ denote the multiplicity of $V(\mu)$ in any such filtration, which is well-defined by [19, 4.19].

Now assume that $[V(\sigma) : L(\mu)]$ is known for all $\sigma > \lambda$, $\mu \vdash d$. We want to be able to deduce $[V(\lambda) : L(\mu)]$ for $\mu \vdash d$. Observe that

$$\begin{aligned}
\big[P(\lambda) : L(\mu)\big] &= \sum_{\sigma \vdash d} \big[P(\lambda) : V(\sigma)\big]\big[V(\sigma) : L(\mu)\big] \\
&= \big[P(\lambda) : V(\lambda)\big]\big[V(\lambda) : L(\mu)\big] + \sum_{\sigma > \lambda} \big[P(\lambda) : V(\sigma)\big]\big[V(\sigma) : L(\mu)\big] \\
&= \big[V(\lambda) : L(\mu)\big] + \sum_{\sigma > \lambda} \big[V(\sigma) : L(\lambda)\big]\big[V(\sigma) : L(\mu)\big].
\end{aligned}$$

The last equality follows by using the reciprocity law [23, 4.5]: $[P(\lambda) : V(\sigma)] = [V(\sigma) : L(\lambda)]$, and $[P(\lambda) : V(\lambda)] = [V(\lambda) : L(\lambda)] = 1$. It follows that from the equation above that $[V(\lambda) : L(\mu)]$ can be computed using the induction hypothesis and the Cartan matrix. □

9.3. According to Proposition 9.2.1, knowing the dimensions of Hom-spaces between Young modules for Σ_d is equivalent to knowing the decomposition numbers for $S(d,d)$. We will now demonstrate a striking result: it is enough to know $\operatorname{Hom}_{\Sigma_t}(Y^\rho, Y^\tau)$ for all $\rho, \tau \vdash t \leqslant d$ (or equivalently the decomposition numbers for all $S(t,t)$, $t \leqslant d$) in order to compute $\operatorname{Ext}^\bullet_{\Sigma_d}(Y^\lambda, Y^\mu)$ for all $\lambda, \mu \vdash d$. This relies heavily on our computation of $\mathrm{H}_n(\Sigma_d, V^{\otimes d})$ as a GL_d-module.

It is an easy consequence of Mackey's theorem that the tensor product of two Young modules is a direct sum of Young modules. The direct sum decomposition can be determined from the decomposition matrix of $S(d,d)$:

Proposition 9.3.1. *Suppose the decomposition matrix for the Schur algebra $S(d,d)$ is known. Then for $\lambda, \mu \vdash d$, one can compute the decomposition of $Y^\lambda \otimes Y^\mu$ into Young modules.*

Proof. The Young modules have a filtration by Specht modules with multiplicities determined by decomposition numbers for $S(d,d)$. In particular Y^λ has a Specht filtration with submodule S^λ and other successive subquotients of the form S^μ with $\mu > \lambda$, where $[Y^\lambda : S^\mu] = [V(\mu) : L(\lambda)]$, see [23, 4.6.4]. Thus the decomposition numbers of $S(d,d)$ together with the ordinary character table of the symmetric group Σ_d (which is easily computed), allow one to compute the ordinary character of Y^λ:

$$\operatorname{ch} Y^\lambda = \sum_{\mu \geqslant \lambda} \big[V(\mu) : L(\lambda)\big] \chi^\mu, \tag{9.3.1}$$

where χ^μ denotes the ordinary (irreducible) character of S^μ. Multiplying the two characters together gives us:

$$\operatorname{ch}\big(Y^\lambda \otimes Y^\mu\big) = \sum m_\tau \chi^\tau. \tag{9.3.2}$$

Finally, the triangular nature of the decomposition matrix of $S(d,d)$ allows one to recover, from the character of $Y^\lambda \otimes Y^\mu$, the direct sum decomposition of $Y^\lambda \otimes Y^\mu$ into Young modules, using (9.3.1) and (9.3.2). □

Define numbers $g_\sigma^{\lambda,\mu}$ by

$$Y^\lambda \otimes Y^\mu = \bigoplus_{\sigma \vdash d} g_\sigma^{\lambda,\mu} Y^\sigma. \tag{9.3.3}$$

Then we have the following:

$$\begin{aligned} \dim \operatorname{Ext}^n_{\Sigma_d}\big(Y^\lambda, Y^\mu\big) &= \dim \operatorname{Ext}^n_{\Sigma_d}\big(k, Y^\lambda \otimes Y^\mu\big) \text{ since } Y^\lambda \text{ is self-dual,} \\ &= \sum_{\sigma \vdash d} g_\sigma^{\lambda,\mu} \dim \operatorname{Ext}^n_{\Sigma_d}\big(k, Y^\sigma\big) \\ &= \sum_{\sigma \vdash d} g_\sigma^{\lambda,\mu} \big[\mathrm{H}_n\big(\Sigma_d, V^{\otimes d}\big) : L(\sigma)\big] \text{ by Theorem 2.3.1(b).} \end{aligned} \tag{9.3.4}$$

We can collect the previous results into a theorem stating that knowing only the dimension of spaces of homomorphisms between Young modules, one can compute $\operatorname{Ext}^i_{\Sigma_d}(Y^\lambda, Y^\mu)$ for all $i \geqslant 0$. We wish to use Theorem 2.3.1(b) and Corollary 8.2.1 to compute $\dim \operatorname{Ext}^i_{\Sigma_d}(Y^\lambda, Y^\mu)$. The composition factors of $S^a(V^{(b)})$ are known from work of Doty [10]. The module $\Lambda^a(V)$ is irreducible, isomorphic to $L(1^a) = L((1,1,\dots,1))$. Thus $\Lambda^a(V^{(b)}) \cong L((p^b, p^b, \dots, p^b))$. Thus the only remaining difficulty is computing the composition multiplicities in the tensor products of irreducibles. Since these can be determined from decomposition numbers, we obtain the following:

Theorem 9.3.2. *Suppose* $\dim \operatorname{Hom}_{\Sigma_t}(Y^\rho, Y^\tau)$ *is known for all $t \leqslant d$ and for all $\rho, \tau \vdash t$. Then there is an algorithm to compute* $\dim \operatorname{Ext}^i_{\Sigma_d}(Y^\lambda, Y^\mu)$ *for all $i \geqslant 0$ and all $\lambda, \mu \vdash d$.*

Proof. Suppose all the dimensions of the Hom spaces are known. Then by Proposition 9.2.1, all the decomposition matrices for the Schur algebras $S(t,t)$ can be computed for $t \leqslant d$. The Littlewood–Richardson rule lets one compute the multiplicities in a Weyl filtration of $V(\lambda) \otimes V(\mu)$. The unitriangular nature of the decomposition matrix of $S(d,d)$ then lets one compute the composition factor multiplicities in tensor products of irreducible modules. This is the step that requires the decomposition numbers not just for $S(d,d)$, but for all $S(t,t)$ with $t \leqslant d$, since we will be computing tensor products of $L(\mu)$'s with $\mu \vdash t \leqslant d$. These multiplicities are all that is needed to determine the multiplicities in $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$, since the multiplicities in $S^a(V^{(b)})$ and $\Lambda^a(V^{(b)})$ are known, so we can use Corollary 8.2.1 to compute $\mathrm{Ext}^i_{\Sigma_d}(k, Y^\lambda)$. Now Proposition 9.3.1 says one can obtain all the $\mathrm{Ext}^i_{\Sigma_d}(Y^\lambda, Y^\mu)$. □

In the next section we will illustrate this algorithm in an example, see Table 10.4.1 for the result.

10. A complete example: $d = 6$, $p = 2$

10.1. Before proving some general results, we apply the description of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ and Theorem 2.3.1 to determine $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ for all $\lambda \vdash 6$ in characteristic two. In the next section we describe a method for determining these groups for $\lambda \vdash d$ where d is arbitrary and $p = 2$. In general the method requires one to compute some tensor products of simple modules for $GL_t(k)$ where t is considerably smaller than d. For example to compute Young module cohomology for $d = 16$ and $p = 2$ in all degrees the most difficult computation is that of $L(2,2,1) \otimes L(1^3)$ for $GL_8(k)$, which is easily handled.

We then demonstrate how, using the known decomposition matrices for $S(6,6)$ and the character table of Σ_6, one can use the results of Section 9 to determine $\mathrm{Ext}^\bullet_{\Sigma_6}(Y^\lambda, Y^\mu)$ for arbitrary $\lambda, \mu \vdash 6$.

10.2. The first step is to determine the structure of the $GL_6(k)$-module $\mathrm{H}_\bullet(\Sigma_6, V^{\otimes 6})$. This V will be concentrated in degree zero, so we will be using the description given in Theorem 8.1.4(a) as a polynomial algebra with certain generators. We can first determine all possible "shapes" for monomials which contribute to $d = 6$ in the description of $\bigoplus_{d \geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ in Theorem 8.1.4. Then for each monomial shape it is easy to work out the corresponding $GL_6(k)$-module structure from the formulas given in Section 4 and the fact that we have a polynomial algebra. For example, monomials of the form $\bar{Q}_2(v)\bar{Q}_3(v) \cdot v \cdot v$ give a summand of $\mathrm{H}_5(\Sigma_6, V^{\otimes d})$ which is isomorphic to $V^{(1)} \otimes V^{(1)} \otimes S^2(V)$ as a $GL_6(k)$-module. Monomials of the form $\bar{Q}_{1,3}(v) \cdot v \cdot v$ give a $GL_6(k)$-module isomorphic to $V^{(2)} \otimes S^2(V)$ as a direct summand of $\mathrm{H}_7(\Sigma_6, V^{\otimes d})$. For $d = 6$ and $p = 2$ the possible monomials are listed in Table 10.2.1, along with the corresponding $GL_6(k)$-module structure. In all such formulas, one should assume the "v" 's are arbitrary vectors in V, not assumed to be equal.

The modules which occur in Table 10.2.1 (and indeed for any d in characteristic two) are all tensor products of Frobenius twists of symmetric powers of the natural module. In [10], Doty determined the entire submodule structure of $S^m(V)$. It is always multiplicity free, and there is a simple rule for determining which $L(\lambda)$ appear. In the Grothendieck group we have:

$$[S^1(V)] = [L(1)],$$
$$[S^2(V)] = [L(2)] + [L(1^2)],$$

Table 10.2.1
$GL_6(k)$-module summands of $\mathrm{H}_\bullet(\Sigma_6, V^{\otimes 6})$, $\bullet > 0$.

Monomial shape		$GL_6(k)$ structure	Degree $\bullet$
$\bar{Q}_i(v)\cdot v^4$	$1 \leqslant i$	$V^{(1)} \otimes S^4(V)$	i
$\bar{Q}_{i,j}(v)\cdot v^2$	$1 \leqslant i \leqslant j$	$V^{(2)} \otimes S^2(V)$	$i+2j$
$\bar{Q}_i(v)\cdot \bar{Q}_j(v)\cdot v^2$	$1 \leqslant i < j$	$V^{(1)} \otimes V^{(1)} \otimes S^2(V)$	$i+j$
$\bar{Q}_i(v)\cdot \bar{Q}_i(v)\cdot v^2$	$1 \leqslant i$	$S^2(V^{(1)}) \otimes S^2(V)$	$2i$
$\bar{Q}_{i,j}(v)\cdot \bar{Q}_k(v)$	$1 \leqslant i \leqslant j, 1 \leqslant k$	$V^{(2)} \otimes V^{(1)}$	$i+2j+k$
$\bar{Q}_i(v)\cdot \bar{Q}_j(v)\cdot \bar{Q}_k(v)$	$1 \leqslant i < j < k$	$V^{(1)} \otimes V^{(1)} \otimes V^{(1)}$	$i+j+k$
$\bar{Q}_i(v)\cdot \bar{Q}_i(v)\cdot \bar{Q}_j(v)$	$1 \leqslant i, j, i \neq j$	$S^2(V^{(1)}) \otimes V^{(1)}$	$2i+j$
$\bar{Q}_i(v)\cdot \bar{Q}_i(v)\cdot \bar{Q}_i(v)$	$1 \leqslant i$	$S^3(V^{(1)})$	$3i$

$$\begin{aligned}&[S^3(V)] = [L(3)] + [L(1^3)],\\ &[S^4(V)] = [L(4)] + [L(3,1)] + [L(2^2)] + [L(1^4)].\end{aligned} \tag{10.2.1}$$

To calculate the tensor products in Table 10.2.1, one can often make use of the Steinberg Tensor Product Theorem (STPT), but we need some information on tensor products of simple modules. The next lemma follows from [22], where the entire submodule structure of the Weyl module $V(2^a, 1^b)$ is determined in any characteristic.

Lemma 10.2.1. *Let $p = 2$ and let $a \leqslant n$. Then, in the Grothendieck group we have:*

$$V(2, 1^{a-2}) = \begin{cases} [L(2, 1^{a-2})] & \text{if } a \text{ is odd,} \\ [L(2, 1^{a-2})] + [L(1^a)] & \text{if } a \text{ is even.} \end{cases} \tag{10.2.2}$$

$$V(2, 2, 1^{a-4}) = \begin{cases} [L(2, 2, 1^{a-4})] + [L(2, 1^{a-2})] & \text{if } a \equiv 0 \mod 4, \\ [L(2, 2, 1^{a-4})] + [L(1^a)] & \text{if } a \equiv 1 \mod 4, \\ [L(2, 2, 1^{a-4})] + [L(2, 1^{a-2})] + [L(1^a)] & \text{if } a \equiv 2 \mod 4, \\ [L(2, 2, 1^{a-4})] & \text{if } a \equiv 3 \mod 4. \end{cases} \tag{10.2.3}$$

The Littlewood–Richardson rule can be used to compute the multiplicities in a Weyl filtration of $V(1^a) \otimes V(1^b) \cong L(1^a) \otimes L(1^b)$, and all terms have the form $V(2^x, 1^y)$. Since the composition factor multiplicities in these "two-column" Weyl modules are all known, one can determine the composition factor multiplicities in any $L(1^a) \otimes L(1^b)$. We will only need those in the following lemma, which follows easily from the Littlewood–Richardson rule and Lemma 10.2.1.

Lemma 10.2.2. *Suppose $p = 2$ and let $n \geqslant a + 1$ in the first case and $n \geqslant a + 2$ in the second. Then in the Grothendieck group we have:*

$$[L(1) \otimes L(1^a)] = \begin{cases} [L(2, 1^{a-1})] + [L(1^{a+1})] & \text{if } a \equiv 0 \mod 2, \\ [L(2, 1^{a-1})] + 2[L(1^{a+1})] & \text{if } a \equiv 1 \mod 2, \end{cases}$$

Table 10.2.2
Simple module multiplicities in direct summands of $\mathrm{H}_\bullet(\Sigma_6, V^{\otimes 6})$.

	$L(6)$	$L(51)$	$L(42)$	$L(33)$	$L(31^3)$	$L(2^3)$
$V^{(1)} \otimes S^4(V)$	1	1	1	2	1	1
$V^{(2)} \otimes S^2(V)$	1	1	0	0	0	0
$V^{(1)} \otimes V^{(1)} \otimes S^2(V)$	1	1	2	2	0	2
$S^2(V^{(1)}) \otimes S^2(V)$	1	1	1	1	0	1
$V^{(2)} \otimes V^{(1)}$	1	0	0	0	0	0
$V^{(1)} \otimes V^{(1)} \otimes V^{(1)}$	1	0	2	0	0	2
$S^2(V^{(1)}) \otimes V^{(1)}$	1	0	1	0	0	1
$S^3(V^{(1)})$	1	0	0	0	0	1

$$[L(1^2) \otimes L(1^a)] = \begin{cases} [L(2^2, 1^{a-2})] + 2[L(2, 1^a)] + 3[L(1^{a+2})] & \textit{if } a \equiv 0 \mod 4, \\ [L(2^2, 1^{a-2})] + [L(2, 1^a)] + [L(1^{a+2})] & \textit{if } a \equiv 1 \mod 4, \\ [L(2^2, 1^{a-2})] + 2[L(2, 1^a)] + 2[L(1^{a+2})] & \textit{if } a \equiv 2 \mod 4, \\ [L(2^2, 1^{a-2})] + [L(2, 1^a)] + 2[L(1^{a+2})] & \textit{if } a \equiv 3 \mod 4. \end{cases}$$

For this section we only use the following special case of Lemma 10.2.2:

$$\begin{aligned} [L(1) \otimes L(1)] &= [L(2)] + 2[L(1^2)], \\ [L(1^2) \otimes L(1)] &= [L(2, 1)] + [L(1^3)]. \end{aligned} \tag{10.2.4}$$

Remark 10.2.3. Since we are assuming $n = d$, the tensor products $L(\lambda) \otimes L(\mu)$ which arise in (8.2.1) and (8.2.2) only occur when $\lambda \vdash a$, $\mu \vdash b$ with $a + b \leqslant d$. Thus *all statements about such tensor products of polynomial $GL_d(k)$-modules will implicitly assume that d is large*. For example (10.2.4) would be false if $n = 2$, in this case there is no module $L(1^3)$. However for us writing $L(1^2) \otimes L(1)$ implies n is at least three. Similarly Lemma 10.2.2 would be false without the assumptions on n and a.

Using (10.2.4) and the STPT, we can calculate the tensor products in Table 10.2.1 and determine the composition multiplicities in each summand. These are given in Table 10.2.2.

10.3. The final step is to determine how many copies of each direct summand appear in each degree. Here is an example:

Lemma 10.3.1. *The $GL_6(k)$ module $V^{(2)} \otimes S^2(V)$ appears $\lfloor \frac{t-1}{2} \rfloor - \lfloor \frac{t-1}{3} \rfloor$ times in $\mathrm{H}_t(\Sigma_6, V^{\otimes 6})$, where $\lfloor\ \rfloor$ is the greatest integer function.*

Proof. From line two in Table 10.2.2, the multiplicity in degree t will be the number of ways to write $t = i + 2j$ with $1 \leqslant i \leqslant j$, equivalently to write $t - 1 = i + 2j$ with $0 \leqslant i < j$. Expressing j as $i + c$ for $c > 0$, we immediately see this is the number of ways to write $t - 1$ as the sum of a positive even integer and a nonnegative multiple of 3. This is just sequence number A008615 in [26], and the formula given is one of several possible. For example when $t = 9$ the monomials of shapes $\bar{Q}_{1,4}(v) \cdot v^2$ and $\bar{Q}_{3,3}(v) \cdot v^2$ contribute the two such summands. □

The other summands can be handled similarly and, since d is fairly small, have reasonable closed form multiplicity formulas. For instance $V^{(2)} \otimes V^{(1)}$ will occur in degree t with multiplicity the number of ways to write t as $i + 2j + k$ with $i \leqslant j$. One fairly easily determines this is $\lfloor \frac{(t-1)^2}{12} \rfloor$, the number of partitions of $t-1$ with exactly 3 parts.

We now have all the information to determine the nontrivial Young module cohomology, the remaining calculations we leave to the reader. For notational convenience in the formulas below set

$$a = a(j) = \frac{(j+1)(j+2)}{6} \tag{10.3.1}$$

and let $\lceil\,\rceil$ be the ceiling function.

Theorem 10.3.2. *Let $p = 2$. The dimensions of the nonzero cohomology groups $\mathrm{H}^j(\Sigma_6, Y^\lambda)$ are given by the following table:*

Table 10.3.1
Cohomology of Young modules for Σ_6, $p = 2$.

λ	$\dim \mathrm{H}^j(\Sigma_6, Y^\lambda)$	λ	$\dim \mathrm{H}^j(\Sigma_6, Y^\lambda)$
(6)	$\lceil a \rceil$	$(5,1)$	$1 + \lfloor \frac{2j}{3} \rfloor$
$(4,2)$	$\lfloor a \rfloor$	$(3,3)$	$j+1$
$(3,1^3)$	1	(2^3)	$\lceil a \rceil$

Corollary 10.3.3. *In characteristic two, $\mathrm{H}^j(\Sigma_6, Y^{(4,1^2)}) = 0$ for all $j \geqslant 0$.*

Observe that $Y^{(4,1^2)}$ is the only nonprojective Young module in the principal block with this property. Thus its support variety contributes to the so called *representation theoretic nucleus*. This corollary could be obtained "by hand" but it is already quite complicated. The module $Y^{(4,1^2)}$ is 48-dimensional. It is uniserial of length 9 and its projective cover is a nonsplit extension of $Y^{(4,1^2)}$ by itself. Thus its projective resolution is periodic. In Theorem 11.1.1 we will give a necessary and sufficient condition on λ for vanishing of $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$.

We also know that $Y^{(6)} \cong k$ and remark the formula for $\mathrm{H}^\bullet(\Sigma_6, Y^{(6)}) \cong \mathrm{H}^\bullet(\Sigma_6, k)$ agrees with previous results, for instance [1, IV.5] where the entire ring structure of $\mathrm{H}^\bullet(\Sigma_6, k)$ is determined.

10.4. For $d = 6$ and $p = 2$ we implemented the procedure described in Section 9. The tensor product decompositions together with Theorem 10.3.2 allow one to determine all the $\mathrm{Ext}^i_{\Sigma_6}(Y^\lambda, Y^\mu)$. (Notice the results of Theorem 10.3.2 are needed in the computation (9.3.4).) All nonvanishing cohomology between all Young modules for Σ_6 is given in Table 10.4.1.

Table 10.4.1
Dimensions of $\mathrm{Ext}^j_{\Sigma_6}(Y^\lambda, Y^\mu)$, $p = 2$.

λ/μ	6	51	42	41^2	3^2	31^3	2^3
6	$\lceil a \rceil$						
51	$1 + \lfloor \frac{2j}{3} \rfloor$	$2 + 2\lfloor \frac{2j}{3} \rfloor$					
42	$\lfloor a \rfloor$	$j+1$	$1 + \lfloor a \rfloor + \lceil a \rceil$				
41^2	0	1	2	3			
3^2	$j+1$	$2j+2$	$2j+2$	2	$4j+4$		
31^3	1	2	2	2	4	4	
2^3	$\lceil a \rceil$	$j+1$	$1 + 2\lfloor a \rfloor$	2	$2+2j$	2	$1 + 2\lceil a \rceil$

Notice that $\lfloor a \rfloor$ and $\lceil a \rceil$ differ only when j is a multiple of 3, reflecting the contribution of the $S^3(V^{(1)})$ terms in these degrees.

10.5. Recall that the *complexity* of a module M can be defined as the minimal c such that there exists a constant $K > 0$ with:

$$\dim \operatorname{Ext}^j_{\Sigma_d}(M, M) \leqslant K j^{c-1}.$$

Notice from (10.3.1) that a is quadratic in j. Thus the diagonal entries in Table 10.4.1 prove that $Y^{(6)}$, $Y^{(4,2)}$ and $Y^{(2,2,2)}$ have complexity 3, while $Y^{(5,1)}$ and $Y^{(3.3)}$ have complexity 2. Finally, $Y^{(4,1,1)}$ and $Y^{(3,1,1,1)}$ have complexity 1 while the remaining Young modules are all projective. These results agree with [18], where the complexity of any Young module in any characteristic was determined.

10.6. Although the computations are too long to include in detail, we will attempt to convince the reader that even for $d \leqslant 16$, one can apply this method to determine $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for any i and any $\lambda \vdash d$. For $d = 16$ one can quickly write down the equivalent of Table 10.2.1, it is just much larger. Each corresponding $GL_{16}(k)$-module will be a tensor product of modules of the form $S^a(V^{(b)})$. Since the constituents of these modules are all known, the only obstacle is calculating the tensor products of the irreducibles which occur. The decomposition matrices for the Schur algebra $S(n, d)$ in characteristic 2 are well-known for d up to at least 10, for example, see the appendix of [24]. Most of the cases reduce using the STPT to very small computations. For the few difficult cases one can use the Littlewood–Richardson rule to compute the tensor products of Weyl modules, and then use the decomposition matrices to handle the simple modules. One of the "larger" cases which occurs is calculating the composition factor multiplicities of $L(2^2, 1) \otimes L(1^3)$ as a $GL_8(k)$-module. This computation arises from the $L(4^2, 2) \otimes L(2^3)$ inside the summand

$$V^{(1)} \otimes V^{(1)} \otimes V^{(1)} \otimes V^{(1)} \otimes V^{(1)} \otimes S^6(V)$$

corresponding to monomials of the form $\bar{Q}_i(v) \cdot \bar{Q}_j(v) \cdot \bar{Q}_k(v) \cdot \bar{Q}_l(v) \cdot \bar{Q}_m(v) \cdot v^6$. This computation is easily handled using the known decomposition matrices for the Schur algebras $S(5, 5)$ and $S(8, 8)$ in characteristic two, together with the Littlewood–Richardson rule.

The power of the method is that a small number of tensor product computations for much smaller values of d allows one to compute $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ in arbitrary degree. Of course results corresponding to Lemma 10.3.1 and closed form formulas like those in Table 10.3.1 will not be obtained. One would need to solve combinatorial problems like "Find a closed form formula for the number of ways to express an integer m in the form $3i + 2j + r + s + t$ where i, j, r, s, t are distinct."

11. Cohomology vanishing theorems

11.1. In this section we prove some general results about Young module cohomology. Sullivan [30] calculated the composition factors of $S^d(V)$, in particular determining that it is multiplicity free. Doty [10] then calculated the entire submodule structure of $S^d(V)$ and gave a nice way to describe the composition factors. Corollary 8.2.1 allows us to determine precisely which Young modules have no cohomology, in arbitrary characteristic; the answer being closely related to the results of Doty and Sullivan. Recall that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is p-restricted if $\lambda_i - \lambda_{i+1} < p$ for all i. Any $\lambda \vdash d$ can be written uniquely as $\lambda_{(0)} + p\mu$ where $\lambda_{(0)}$ is p-restricted and μ is a partition. The modules $\{Y^\lambda \mid \lambda \text{ is} p\text{-restricted}\}$ form a complete set of indecomposable projective (hence injective) $k\Sigma_d$-modules.

Theorem 11.1.1. *Let $\lambda \vdash d$. Write $\lambda = \lambda_{(0)} + p\mu$ with $\lambda_{(0)} \vdash a$ p-restricted and $p = \operatorname{char} k$ arbitrary. Then*

$$\mathrm{H}^\bullet\big(\Sigma_d, Y^\lambda\big) \neq 0$$

if and only if $[S^a(V) : L(\lambda_{(0)})] \neq 0$ or $a = 0$.

Proof. If $\mu = \emptyset$ then Y^λ is injective so $\mathrm{H}^i(\Sigma_d, Y^\lambda) = 0$ for $i > 0$. The $i = 0$ case is Proposition 12.1.1.

So now assume $\mu \neq \emptyset$ and suppose Y^λ has nonvanishing cohomology in some positive degree. By Theorem 2.3.1, $L(\lambda)$ is a constituent of some summand of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$. According to Corollary 8.2.1, these summands are twists of tensor products of symmetric and exterior powers of the natural module, where at most one summand, a symmetric tensor, is not twisted. Thus $L(\lambda)$ is a constituent of $S^u(V) \otimes M^{(1)}$ for some $GL_d(k)$-module M. The result about λ follows from the STPT.

Now suppose $\lambda = \lambda_{(0)} + p\mu$ as above, with either $a = 0$ or $[S^a(V) : L(\lambda_{(0)})] \neq 0$ and $\mu \vdash t$. Since $L(\mu)$ is a constituent of $V^{\otimes t}$ then $L(\lambda)$ is a constituent of

$$S^a(V) \otimes V^{(1)} \otimes \cdots \otimes V^{(1)} \cong S^a(V) \otimes \big(V^{\otimes t}\big)^{(1)}.$$

This module corresponds to monomials of the form

$$\bar{Q}_{i_1}(v) \cdot \bar{Q}_{i_2}(v) \cdots \bar{Q}_{i_t}(v) \cdot v^a \quad 1 \leqslant i_1 < i_2 < \cdots < i_t.$$

contributing to degree $(p-1)\sum i_u$. Choosing $i_s = s$ we see that

$$\mathrm{H}^{(p-1)t(t+1)/2}\big(\Sigma_d, Y^\lambda\big) \neq 0. \qquad \square$$

Theorem 11.1.1 guarantees that many nonprojective Young modules have vanishing cohomology. For example Σ_{16} in characteristic two has 118 nonprojective Young modules in the principal block. Theorem 11.1.1 proves that exactly 47 of them have vanishing cohomology!

11.2. For a finite group G the *p-rank* is the maximal rank of an elementary abelian p subgroup. It is also the maximal complexity of a G-module in characteristic p and is the complexity of the trivial module. Let $b = \lfloor \frac{d}{p} \rfloor$, which is the p-rank of Σ_d. It follows from work of Benson [3] that there are no modules M with complexity b such that $\mathrm{H}^\bullet(\Sigma_d, M) = 0$. Furthermore, in characteristic two there can be none of complexity $b-1$ either. The next corollary says that among Young modules with vanishing cohomology, all other complexities do occur.

Corollary 11.2.1. *Let p be arbitrary. For every $1 \leqslant c \leqslant b-2$, there is a Young module Y^λ in the principal block of $k\Sigma_d$ which has complexity c and such that* $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda) = 0$. *There is also such a Young module of complexity $b-1$ precisely when $p > 2$.*

Proof. Recall from [18] that for $\lambda = \lambda_{(0)} + p\mu$ where $\lambda_{(0)}$ is p-restricted, the complexity of Y^λ is c where $\mu \vdash c$. We will describe how to choose λ to satisfy the theorem. Consider first the case of characteristic two. There are no partitions of 2 or 3 which are 2-restricted, which lie in the principal block and are not of the form (1^a). Thus the smallest choice of $\lambda_{(0)}$ is $(2, 1, 1)$, so there are no Young modules of complexity $b - 1$ or b with vanishing cohomology.

In general, for d even, choose $\lambda_{(0)} = (2, 1^{2(b-c)-2})$ with μ arbitrary. For d odd choose $\lambda_{(0)} = (2, 2, 1^{2(b-c)-3})$ and μ arbitrary. In both cases Y^λ lies in the principal block and has the desired complexity c and vanishing cohomology.

Now suppose p is odd and let $d = bp + s$ with $0 \leqslant s < p$. Choose $\lambda_{(0)} = (s, 1^{ep})$ with $e \geqslant 1$. Then Y^λ is in the principal block of Σ_d. Since $[S^{ep+s}(V) : L(\lambda_{(0)})] = 0$, the module Y^λ has vanishing cohomology. Choosing $e = 1, 2, \ldots, b$ gives modules of complexity $b-1, b-2, \ldots, 0$ respectively which have vanishing cohomology. □

12. Cohomology in low degrees

12.1. In this section we compute $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for $i = 1, 2$ and arbitrary d and λ, in characteristic two. For arbitrary characteristic, $\mathrm{H}^i(\Sigma_d, Y^\lambda) = 0$ for $1 \leqslant i < 2p - 3$ by [21, Cor. 6.3]. In particular for any λ in odd characteristic, we have $\mathrm{H}^1(\Sigma_d, Y^\lambda) = \mathrm{H}^2(\Sigma_d, Y^\lambda) = 0$. One could likely work out the first few nonvanishing degrees $i = 2p - 3, 2p - 2, 2p - 1$ in the same way we do below for $p = 2$, which we leave to the reader. The case $i = 0$ is already known:

Proposition 12.1.1. *(See [17, 6.5].)*

$$\dim \mathrm{Hom}_{\Sigma_d}\big(k, Y^\lambda\big) = \big[S^d(V) : L(\lambda)\big].$$

Doty determined these multiplicities in all characteristics. For $p = 2$ they are given in Proposition 12.2.1.

12.2. We first observe that Doty's result takes on a particularly nice form in characteristic two:

Proposition 12.2.1. *Let $\lambda \vdash s$ have a 2-adic expansion*

$$\lambda = \sum_{i=0}^{m} 2^i \lambda_{(i)},$$

where each $\lambda_{(i)}$ is 2-restricted. Then $L(\lambda)$ is a constituent of $S^s(V)$ if and only if each $\lambda_{(i)}$ is of the form (1^{a_i}) for $a_i \geqslant 0$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and let $\lambda_i = \sum_u c_{iu} 2^u$ where the $c_{iu} \in \{0, 1\}$. Doty's theorem says that $L(\lambda)$ is a constituent of $S^s(V)$ if and only if λ is maximal among partitions of s with its *carry pattern* (see [10] for definition). This is easily seen to be equivalent to the condition that whenever $c_{iu} = 1$ then $c_{i1} = c_{i2} = \cdots = c_{iu} = 1$. Informally, if one does the addition $\lambda_1 + \cdots + \lambda_t = s$ in binary, all the ones in each column are as far towards the top of the column as possible. Given such a partition, it is clear that the 2-adic expansion can be read off from this addition, and it is of the form desired. For example, consider (15, 5, 5, 1). In binary the addition $15 + 5 + 5 + 1 = 26$ takes the form:

$$\begin{array}{r} {\scriptstyle 1\ \ 2\ \ 1\ \ 2} \\ 1\ 1\ 1\ 1 \\ 0\ 1\ 0\ 1 \\ 0\ 1\ 0\ 1 \\ +\ 0\ 0\ 0\ 1 \\ \hline 1\ 1\ 0\ 1\ 0 \end{array}$$

and this is the largest partition of 26 with carry pattern 1, 2, 1, 2. Thus $[S^{26}(V) : L(15, 5, 5, 1)] \neq 0$. The 2-adic expansion corresponds to the columns:

$$(15, 5, 5, 1) = \left(1^4\right) + 2 \cdot (1) + 4 \cdot \left(1^3\right) + 8 \cdot (1).$$

The partition (13, 7, 5, 1) has the same carry pattern but has 2-adic expansion

$$(13, 7, 5, 1) = (3, 3, 1, 1) + 2 \cdot (1) + 4 \cdot (2, 1, 1)$$

and thus $[S^{26}(V) : L(13, 7, 4, 1)] = 0$. □

12.3. From Theorem 8.1.4 we see the only monomials contributing to degree one are of the form $Q_1(v) \cdot v^{d-2}$. Thus the following is immediate.

Proposition 12.3.1. *Assume $p = 2$. The dimension of $\mathrm{H}^1(\Sigma_d, Y^\lambda)$ is $[S^{d-2}(V) \otimes V^{(1)} : L(\lambda)]$.*

Since the composition factors of $S^{d-2}(V)$ are given by Proposition 12.2.1, we need to calculate the tensor product of each with $V^{(1)} \cong L(2)$. We start with a theorem, valid in arbitrary characteristic, that lets us ignore the 2-restricted part of λ in characteristic two.

Theorem 12.3.2. *Let p be arbitrary and let $\lambda = \lambda_{(0)} + p\mu \vdash d$ where $\lambda_{(0)} \vdash a$ is p-restricted. Assume $[S^a(V) : L(\lambda_{(0)})] \neq 0$. Then*

$$\mathrm{H}^i\left(\Sigma_d, Y^\lambda\right) \cong \mathrm{H}^i\left(\Sigma_{d-a}, Y^{p\mu}\right)$$

as k-vector spaces.

Proof. From Corollary 8.2.1 we see that each $GL_d(k)$-module direct summand of $\mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ can be written in the form $S^a(V) \otimes M^{(1)}$ for some M. The result follows from the STPT. □

12.4. Recall that if $[S^a(V) : L(\lambda_{(0)})] = 0$ then $\mathrm{H}^\bullet(\Sigma_d, Y^\lambda)$ is identically 0 by Theorem 11.1.1. Thus Theorem 12.3.2 reduces the problem of computing $\mathrm{H}^1(\Sigma_d, Y^\lambda)$ in characteristic two to the case where $\lambda = 2\mu$. Next we reduce to the case where λ is not of the form 4μ. From Lemma 10.2.2 we have:

$$\begin{aligned} \left[L(2) \otimes L\left(2^{2a}\right)\right] &= \left[L\left(4, 2^{2a-1}\right)\right] + \left[L\left(2^{2a+1}\right)\right], \\ \left[L(2) \otimes L\left(2^{2a+1}\right)\right] &= \left[L\left(4, 2^{2a}\right)\right] + 2\left[L\left(2^{2a+2}\right)\right]. \end{aligned} \tag{12.4.1}$$

This allows us to obtain the following stability result.

Theorem 12.4.1. *Suppose $\lambda \vdash d$ and $p = 2$. Then:*

$$\mathrm{H}^1(\Sigma_{2d}, Y^{2\lambda}) \cong \mathrm{H}^1(\Sigma_{4d}, Y^{4\lambda})$$

as k-vector spaces.

Proof. We will give a bijection between appearances of $L(4\lambda)$ in $L(2) \otimes S^{4d-2}(V)$ and appearances of $L(2\lambda)$ in $L(2) \otimes S^{2d-2}(V)$. Since $V^{(1)} \cong L(2)$, this together with Proposition 12.3.1 will establish the result.

Suppose $L(4\lambda)$ is a constituent of $L(2) \otimes L(\mu)$ for $L(\mu)$ a constituent of $S^{4d-2}(V)$. By Proposition 12.2.1, $\mu = (2^{a_1}) + (4^{a_2}) + (8^{a_3}) + \cdots$. So by the STPT and (12.4.1), $L(2) \otimes L(\mu)$ does not have any constituents of the form $L(4\lambda)$ unless $a_1 = 1$. In this case

$$\begin{aligned} L(2) \otimes L(\mu) &\cong L(2) \otimes L(2) \otimes L(4^{a_2}) \otimes \cdots \\ &= (L(4) + 2L(2,2)) \otimes L(4^{a_2}) \otimes \cdots. \end{aligned} \tag{12.4.2}$$

Now any $L(4\lambda)$ composition factors must come from the $L(4) \otimes L(4^{a_2}) \otimes \cdots$ in (12.4.2). Thus we have:

$$\begin{aligned} [L(2) \otimes L(\mu) : L(4\lambda)] &= [L(4) \otimes L((4^{a_2} + 8^{a_3} + \cdots)) : L(4\lambda)] \\ &= [(L(2) \otimes L((2^{a_2} + 4^{a_3} + \cdots)))^{(1)} : L(2\lambda)^{(1)}] \\ &= [L(2) \otimes L((2^{a_2} + 4^{a_3} + \cdots)) : L(2\lambda)]. \end{aligned} \tag{12.4.3}$$

Thus each $L(4\lambda)$ in $L(2) \otimes L(\mu)$ corresponds to an $L(2\lambda)$ in $L(2) \otimes L((2^{a_2} + 4^{a_3} + \cdots))$ inside $L(2) \otimes S^{2d-2}(V)$ and similarly in reverse, so the multiplicities are the same and the result follows. □

The previous two results are now enough to completely determine $\mathrm{H}^1(\Sigma_d, Y^\lambda)$:

Theorem 12.4.2. *Let $p = 2$ and let $\lambda \vdash d$ be in the principal block with λ not 2-restricted (otherwise Y^λ is projective). Write the 2-adic expansion of λ as*

$$\lambda = \lambda_{(0)} + 2^s \lambda_{(s)} + 2^{s+1} \lambda_{(s+1)} + \cdots + 2^r \lambda_{(r)},$$

where $s \geqslant 1$, $\lambda_{(s)} \neq \emptyset$ and each $\lambda_{(i)}$ is 2-restricted. Then:

(a) *If $\{\lambda_{(i)}\colon i \neq s\}$ are all of the form (1^{a_i}) and $\lambda_{(s)} = (2, 1^a)$ or (1^b) with b odd then*

$$\dim \mathrm{H}^1(\Sigma_d, Y^\lambda) = 1.$$

(b) *If $\{\lambda_{(i)}\colon i \neq s\}$ are all of the form (1^{a_i}) and if $\lambda_{(s)} = (1^b)$ with b even then*

$$\dim \mathrm{H}^1(\Sigma_d, Y^\lambda) = 2.$$

(c) *Otherwise* $\mathrm{H}^1(\Sigma_d, Y^\lambda) = 0$.

Proof. Since any constituent of $S^{d-2}(V)$ is of the form $L(\tau)$ for $\tau = (1^a) + 2\mu$, it is immediate that $\lambda_{(0)}$ must be of the form (1^a) for the cohomology to be nonzero. Now Theorem 12.3.2 says we can assume without loss that $\lambda_{(0)} = \emptyset$. With this assumption we can apply Theorem 12.4.1 to assume, again without loss of generality, that $s = 1$, i.e. λ is of the form 2μ but not of the form 4μ.

We know $\mathrm{H}^1(\Sigma_d, Y^\mu)$ is nonzero if and only if $L(\mu)$ occurs in $L(2) \otimes S^{d-2}(V)$. Referring to (12.4.2), and our assumption that λ is not of the form 4μ, we need to know when $L(\lambda)$ occurs inside

$$L(2) \otimes L(2^a) \otimes L(4^{a_2}) \otimes \cdots .$$

Now the STPT and (12.4.1) give the result. □

Example 12.4.3. In characteristic two, $\mathrm{H}^1(\Sigma_{47}, Y^{(17,13,13,4)}) \cong k$.

This follows since $(17, 13, 13, 4) = (1^3) + 2^2(2, 1, 1) + 2^3(1^3)$ so $s = 2$ and we are in case (a) of the theorem. Similarly since $(57, 41, 9, 8) = (1^3) + 2^3(1^4) + 2^4(1) + 2^5(1^2)$ we conclude that

Example 12.4.4. In characteristic two, $\dim \mathrm{H}^1(\Sigma_{115}, Y^{(57,41,9,8)}) = 2$.

12.5. Next we compute $\mathrm{H}^2(\Sigma_d, Y^\lambda)$ in characteristic two.

Proposition 12.5.1. *Let $p = 2$. The dimension of $\mathrm{H}^2(\Sigma_d, Y^\lambda)$ is the multiplicity of $L(\lambda)$ in*

$$L(2) \otimes S^{d-2}(V) \oplus S^2(V^{(1)}) \otimes S^{d-4}(V).$$

Proof. The only monomials which contribute to degree two are those of the form $Q_2(v) \cdot v^{d-2}$ and $Q_1(v) \cdot Q_1(v) \cdot v^{d-4}$. □

The multiplicity of $L(\lambda)$ in $L(2) \otimes S^{d-2}(V)$ is $\dim \mathrm{H}^1(\Sigma_d, Y^\lambda)$, which we have already determined. Theorem 12.3.2 makes it sufficient to determine $\mathrm{H}^2(\Sigma_d, Y^{2\lambda})$.

Next we prove another stability result:

Theorem 12.5.2. *Let $p = 2$ and suppose $\lambda \vdash d$. Then*

$$\mathrm{H}^2(\Sigma_{4d}, Y^{4\lambda}) \cong \mathrm{H}^2(\Sigma_{8d}, Y^{8\lambda}).$$

Proof. By Proposition 12.5.1, it is enough to show the two equalities:

$$\big[L(2) \otimes S^{8d-2}(V) : L(8\lambda)\big] = \big[L(2) \otimes S^{4d-2}(V) : L(4\lambda)\big], \tag{12.5.1}$$

$$\big[S^2(V^{(1)}) \otimes S^{8d-2}(V) : L(8\lambda)\big] = \big[S^2(V^{(1)}) \otimes S^{4d-2}(V) : L(4\lambda)\big]. \tag{12.5.2}$$

Eq. (12.5.1) was obtained in the proof of Theorem 12.4.1 so we show Eq. (12.5.2) holds.

We know $[S^2(V^{(1)})] = [L(4)] + [L(2^2)]$. To show the multiplicity of $L(4\lambda)$ coming from the $L(4)$ term is equal to that of $L(8\lambda)$, the proof proceeds exactly as the proof of Theorem 12.4.1, everything is just twisted once.

Now suppose $L(2^2) \otimes L(\mu)$ has a constituent $L(4\lambda)$, where $\mu = 2^{a_1} + 4^{a_2} + \cdots$. By Lemma 10.2.2, this only happen when $a_1 = 2$, in which case $[L(2^2) \otimes L(2^2)] = [L(4^2)] + 2[L(4, 2^2)] + 2[L(2^4)]$. Thus the $L(4\lambda)$ which occur are exactly the constituents of $L(4^2) \otimes L(4^{a_2}) \otimes L(8^{a_3}) \cdots$.

Next suppose $L(2^2) \otimes L(\mu)$ has a constituent $L(8\lambda)$, where $\mu = (2^{a_1}) + (4^{a_2}) + (8^{a_3}) + \cdots$. Arguing as above we see that we must have $a_1 = a_2 = 2$. By Lemma 10.2.2, $L(2^2) \otimes L(2^2) \otimes L(4^2)$ has $L(8^2)$ as its only term divisible by 8. Thus the $L(8\lambda)$ which occur are the constituents of $L(8^2) \otimes L(8^{a_3}) \otimes \cdots$, and the result is proved as in (12.4.3). □

Proposition 12.5.1 gives three modules to analyze, namely $L(2) \otimes S^{d-2}(V)$, $L(4) \otimes S^{d-4}(V)$ and $L(2^2) \otimes S^{d-4}(V)$. For a given 2λ, we must find the number of times $L(2\lambda)$ occurs in each. For $L(2) \otimes S^{d-2}(V)$, we know that $L(2\lambda)$ occurs the dimension of $\mathrm{H}^1(\Sigma_d, Y^{2\lambda})$ times, which was already determined. Next we must determine the constituents of $L(4) \otimes S^{d-4}(V)$ and $L(2^2) \otimes S^{d-4}(V)$. The former is straightforward:

Lemma 12.5.3. *Suppose $L(2\lambda)$ is a constituent of $L(4) \otimes S^{d-4}(V)$. Then 2λ is of the form $(2^{a_1}) + 4\mu$ for some $a_1 \geqslant 0$ and $\mu \neq \emptyset$. The multiplicity of $L(2\lambda)$ is the dimension of $\mathrm{H}^1(\Sigma_{2c}, Y^{2\mu})$, given by Theorem* 12.4.2*, where $c = d/2 - a_1$.*

Proof. By Proposition 12.2.1, we are considering constituents of tensor products of the form $L(4) \otimes L(2^{a_1}) \otimes L(4^{a_2}) \otimes \cdots$. Rearranging, we get

$$L\left(2^{a_1}\right) \otimes \left(L(2) \otimes L\left(2^{a_2}\right) \otimes L\left(4^{a_2}\right) \otimes \cdots\right)^{(1)}. \tag{12.5.3}$$

The constituent $L(2\mu)$ appears in $L(2) \otimes L(2^{a_2}) \otimes L(4^{a_2}) \cdots$ exactly the dimension of $\mathrm{H}^1(\Sigma_{2c}, Y^{2\mu})$ times, by Proposition 12.3.1, so the result follows. □

Finally we turn to the constituents of $L(2^2) \otimes S^{d-4}(V)$ of the form $L(2\lambda)$. Thus we must consider

$$L\left(2^2\right) \otimes L\left(2^{a_1}\right) \otimes L\left(4^{a_2}\right) \otimes \cdots.$$

Suppose $a_1 \neq 2$. We know from Lemma 10.2.2 that these constituents will be all of the form $L((\mu) + (4^{a_2}) + (8^{a_3}) + \cdots)$ where μ is of the form $(4, 2^a)$, $(4^2, 2^a)$ or (2^a), and the multiplicities depending on the congruence class of a mod 4. When $a_1 = 2$ we also have an $L(4, 4)$ term in the $L(2, 2) \otimes L(2, 2)$, so we also get higher twists of the constituents above.

We now have all the information necessary to completely determine $\mathrm{H}^2(\Sigma_d, Y^\lambda)$.

Theorem 12.5.4. *Let $p = 2$ and let*

$$\lambda = \lambda_{(0)} + 2\lambda_{(1)} + 2^s\lambda_{(s)} + \cdots + 2^r\lambda_{(r)}$$

be the 2-adic expansion of λ, where $\lambda_{(s)}$ is nonempty unless $\lambda = \lambda_{(0)} + 2\lambda_{(1)}$.

(a) *If* $\mathrm{H}^2(\Sigma_d, Y^\lambda) \neq 0$ *then* $\lambda_{(t)}$ *is of the form* (1^{a_t}), $a_t \geqslant 0$ *for all* $t \neq 1, s$.
(b) *Suppose the* $\lambda_{(t)}$*'s are as in* (a)*. Then the choices of* $\lambda_{(1)}$ *and* $\lambda_{(s)}$ *which give nonzero* $\mathrm{H}^2(\Sigma_d, Y^\lambda)$ *are given in Table* 12.5.1*, together with the dimension:*

Table 12.5.1
Nonzero $\dim \mathrm{H}^2(\Sigma_d, Y^\lambda)$ sorted by $\lambda_{(1)}$ and $\lambda_{(s)}$, $a, b, c \geqslant 1$.

$\lambda_{(s)} \setminus \lambda_{(1)}$	1	1^2	1^{4a-1}	1^{4a}	1^{4a+1}	1^{4a+2}	21^{2c-1}	21^{2c}	221^a
1^{2b}	3	5	4	6	5	7	2	3	1
1^{2b-1}	2	4	3	5	4	6	2	3	1
21^b	1	1	1	1	1	1	0	0	0
$\emptyset$	1	3	2	4	3	5	2	3	1

$\lambda_{(1)} \setminus \lambda_{(s)}$	$\emptyset$	1	1^2	1^{4a-1}	1^{4a}	1^{4a+1}	1^{4a+2}	1^{4a+3}	21^{2c-1}	21^{2c}	221^c
$\emptyset$	0	2	5	3	6	4	7	3	3	4	1

Proof. Suppose $\mathrm{H}^2(\Sigma_d, Y^\lambda) \neq 0$. Then $\lambda_{(0)}$ must be of the desired form by Theorem 11.1.1. In this case we can assume $\lambda_{(0)} = \emptyset$ by Theorem 12.3.2. So suppose $\lambda = 2\mu$. Proposition 12.5.1 and the discussion before Lemma 12.5.3 tell us that $L(\lambda)$ is a constituent of at least one of $L(2) \otimes S^{d-2}(V)$, $L(4) \otimes S^{d-4}(V)$ or $L(2^2) \otimes S^{d-4}(V)$. It is a constituent of $L(2) \otimes S^{d-2}(V)$ precisely when $H^1(\Sigma_d, Y^\lambda) \neq 0$, and the λ that occur are given by Theorem 12.4.2.

The second case is handled in Lemma 12.5.3, we see that λ must have the form $(2^{a_1}) + 4\mu$ where $H^1(\Sigma_s, Y^{2\mu}) \neq 0$.

The constituents in the final case are determined just after the proof of Lemma 12.5.3. The three cases together give part (a) of the theorem.

Part (b) is just a matter of applying what we have already figured out. For example, we explain the “7” in the top row of Table 12.5.1. From Proposition 12.5.1, we need to figure out the multiplicity of $L(2\lambda)$ in

$$L(2) \otimes S^{d-2}(V) \oplus S^2\big(V^{(1)}\big) \otimes S^{d-4}(V),$$

where 2λ is of the form $((2^{4a+2}) + (4^{2b}) + (8^{a_3}) + (16^{a_4}) + \cdots)$, i.e.

$$L(2\lambda) \cong L\big(2^{4a+2}\big) \otimes L\big(4^{2b}\big) \otimes L\big(8^{a_3}\big) \otimes \cdots.$$

Since $4a + 2$ is even, $L(2\lambda)$ occurs twice in $L(2) \otimes S^{d-2}(V)$. Now since $2b$ is even, Lemma 12.5.3 tells us that $L(2\lambda)$ appears twice in $L(4) \otimes S^{d-4}(V)$.

Finally consider $L(2^2) \otimes S^{d-4}(V)$. $L(2\lambda)$ will only appear in the $L(2^2) \otimes L((2^{4a}) + (4^{a_2}) + \cdots)$ term, and the multiplicity will be 3 by (12.4.1). Thus we have a total multiplicity of seven. The other cases are similar. □

12.6. Before concluding this section we make an observation about the versions of Propositions 12.3.1 and 12.5.1 which would appear in calculating higher degree cohomology groups. We will apply it the next section.

Proposition 12.6.1. *Fix $i > 0$ and let p be arbitrary. There is a finite list of partitions $\{\mu_s \vdash j_s \mid 1 \leqslant s \leqslant t_i\}$, not necessarily distinct, such that for $\lambda \vdash d$,*

$$\dim \mathrm{H}^i\left(\Sigma_d, Y^\lambda\right) = \sum_{s=1}^{t_i} \left[L(p\mu_s) \otimes S^{d-2j_s}(V) : L(\lambda)\right]$$

and where each $j_s \leqslant i$.

For example if $p = 2$ and $i = 1$ the list is $\{(1)\}$ (cf. Proposition 12.3.1) while for $i = 2$ the list is: $\{(1), (2), (1, 1)\}$ (cf. Proposition 12.5.1).

Proof. From Theorem 8.1.4, we see that there are only finitely many shapes for the "Q" part of monomials in $\bigoplus_{d \geqslant 0} \mathrm{H}_\bullet(\Sigma_d, V^{\otimes d})$ which have degree i. Thus the $GL_d(k)$-summands that contribute to degree i are each (by Corollary 8.2.1) of the form $S^{d-a}(V) \otimes M^{(1)}$, where M is a possibly complicated tensor product of symmetric and exterior powers of V, and where there are only finitely many choices for M. The list then consists of the constituents of the various M which arise. □

13. Generic cohomology for Young modules

13.1. From the preceding sections one could imagine even more elaborate formulas for $\mathrm{H}^3(\Sigma_d, Y^\lambda)$ and higher degrees. As i grows, the number of possible monomial shapes continues to grow, as does the number of tensor products one must compute. Rather than continuing in this direction, we instead observe that the stability behavior for low degree cohomology exists in all degrees and all characteristics. We will also prove that, for a given i, only a finite number of tensor product calculations are required to produce formulas for $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ valid for d and λ arbitrary.

In the process of computing low degree cohomology, we have obtained (cf. Propositions 12.2.1, 12.1.1, and Theorems 12.4.1, 12.5.2) some "stability" results which we collect below:

Proposition 13.1.1. *Let $\lambda \vdash d$ and $p = 2$.*

(a) $\mathrm{H}^0(\Sigma_d, Y^\lambda) \cong \mathrm{H}^0(\Sigma_{2d}, Y^{2\lambda})$.
(b) $\mathrm{H}^1(\Sigma_{2d}, Y^{2\lambda}) \cong \mathrm{H}^1(\Sigma_{4d}, Y^{4\lambda})$.
(c) $\mathrm{H}^2(\Sigma_{4d}, Y^{4\lambda}) \cong \mathrm{H}^2(\Sigma_{8d}, Y^{8\lambda})$.

We will show that this stability behavior extends to H^i for all $i \geqslant 0$ and in any characteristic. We begin with a generalization of Proposition 12.2.1:

Proposition 13.1.2. *Let $\lambda \vdash s$ have a p-adic expansion*

$$\lambda = \sum_{i=0}^{m} p^i \lambda_{(i)},$$

where each $\lambda_{(i)}$ is p-restricted. Then $L(\lambda)$ is a constituent of $S^s(V)$ if and only if each $\lambda_{(i)}$ is of the form $((p-1)^{a_i}, b_i)$ for $a_i \geqslant 0$ and $0 \leqslant b_i < p - 1$.

Proof. The argument is the same as for Proposition 12.2.1, except each column in the addition must have all $(p-1)$'s pushed to the top, a single entry between zero and $p-2$, and then all zeros. □

To prove the general stability result, we need to understand the highly twisted simple modules which occur in $L(\lambda)\otimes S^t(V)$, i.e. modules of the form $L(p^c\mu)$ for large c. The next lemma shows that, for c large enough, μ is completely determined.

Lemma 13.1.3. *Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ with $\lambda_1\leqslant p^c$. Suppose that $\rho=\sum_{i=0}^{c-1}p^i\rho_{(i)}\vdash a$ is such that $[S^a(V):L(\rho)]\neq 0$. Consider $M=L(\lambda)\otimes L(\rho)$. Then $[M:L(p^c\mu)]\neq 0$ implies that $\mu=(1^w)$ for some $w\geqslant 1$.*

Proof. This follows from considering the highest weight τ which occurs in M. Proposition 13.1.2 tells us each $\rho_{(i)}$ has first part at most $p-1$. Thus τ has first part

$$\begin{aligned}\tau_1&\leqslant \lambda_1+p-1+p(p-1)+\cdots+p^{c-1}(p-1)\\&=\lambda_1+p^c-1\\&\leqslant p^c+p^c-1\\&<2p^c.\end{aligned}$$

Thus M has no dominant weight of the form $p^c\mu$ unless $\mu_1=1$, and the result follows. □

We need another lemma about twisted constituents in tensor products:

Lemma 13.1.4. *Let $\rho=((p-1)^a,b)$ with $0\leqslant b<p-1$. Then $L(1^w)\otimes L(\rho)$ contains no constituents of the form $L(p\mu)$ unless $b=0$ and $a=w$, in which case the only such constituent is a single copy of $L((p)^a)=L((p,p,\ldots,p))$.*

Proof. This is clear from considering the weight spaces in $L(1^w)\otimes L(\rho)$, as there is not even a nonzero weight space of the form $p\mu$ unless $a=w$, in which case $(p)^a$ is the highest weight, and the only dominant weight of that form. Thus there is a single copy of $L((p)^a)$. □

The preceding lemmas let us prove that the calculations in the proof of Theorem 12.5.2 generalize. Specifically we have the following generalization of (12.5.2).

Proposition 13.1.5. *Suppose $\lambda=p\sigma\vdash a$ with $\lambda_1\leqslant p^c$, and $\tau\vdash d$. Then:*

$$\bigl[L(\lambda)\otimes S^{p^cd-a}(V):L\bigl(p^c\tau\bigr)\bigr]=\bigl[L(\lambda)\otimes S^{p^{c+1}d-a}(V):L\bigl(p^{c+1}\tau\bigr)\bigr]. \tag{13.1.1}$$

Proof. We explain how occurrences of $L(p^{c+1}\tau)$ in the right side of (13.1.1) and $L(p^c\tau)$ in the left hand side correspond bijectively. Since we are assuming $\lambda=p\sigma$, twisted constituents of $L(\lambda)\otimes S^{p^{c+1}d-a}(V)$ come from modules of the form:

$$L(\lambda)\otimes L(\mu_{(1)})^{(1)}\otimes L(\mu_{(2)})^{(2)}\otimes\cdots\otimes L(\mu_{(c)})^{(c)}\otimes L(\mu_{(c+1)})^{(c+1)}\otimes\cdots, \tag{13.1.2}$$

where $\mu_{(i)} = ((p-1)^{a_i}, b_i)$ as in Proposition 13.1.2.

Lemma 13.1.3 implies that

$$L(\lambda) \otimes L(\mu_{(1)})^{(1)} \otimes L(\mu_{(2)})^{(2)} \otimes \cdots \otimes L(\mu_{(c-1)})^{(c-1)}$$

has no constituents of the form $p^c\rho$ unless $\rho = 1^w$. Then Lemma 13.1.4 implies that constituents in (13.1.2) of the form $L(p^{c+1}\tau)$ occur only when $\mu_{(c)} = ((p-1)^w)$.

Thus we end up counting occurrences of $L(p^{c+1}\tau)$ in

$$L\big((p^{c+1})^w\big) \otimes L(\mu_{(c+1)})^{(c+1)} \otimes L(\mu_{(c+2)})^{(c+2)} \otimes \cdots . \tag{13.1.3}$$

If we repeat the analysis above for counting occurrences of $L(p^c\tau)$ in the left hand side of (13.1.1), we end up counting occurrences of $L(p^c\tau)$ in

$$L\big((p^c)^w\big) \otimes L(\mu_{(c)})^{(c)} \otimes L(\mu_{(c+1)})^{(c+1)} \otimes \cdots,$$

which is the same as (13.1.3) by the STPT. □

13.2. We can now state our main stability theorem.

Theorem 13.2.1. *Fix $i > 0$ and let p be arbitrary. Then there exists $s(i) > 0$ such that for any d and $\lambda \vdash d$ we have*

$$\mathrm{H}^i\big(\Sigma_{p^a d}, Y^{p^a\lambda}\big) \cong \mathrm{H}^i\big(\Sigma_{p^{a+1}d}, Y^{p^{a+1}\lambda}\big),$$

whenever $a \geqslant s(i)$.

Proof. This follows from Propositions 12.6.1 and 13.1.5. □

Remark 13.2.2. Suppose $p = 2$ and let $2^{a-1} < i \leqslant 2^a$. When calculating H^i we must determine the multiplicities in modules of the form $M^{(1)} \otimes S^{d-a}(V)$, where the $M^{(1)}$ is determined by the shapes of the various monomials contributing to degree i. One easily sees that if $[M : L(\mu)] \neq 0$, then μ has degree at most $2i$. In particular $\mu_1 \leqslant 2^{a+1}$, and so by Proposition 13.1.5, we can choose $s(i) = i + 1$ in Theorem 13.2.1.

Remark 13.2.3. The choice of $s(i)$ in Remark 13.2.2, can be seen to be best possible in all degrees, in the sense that the stability would not hold for $2^i\lambda$. For example we observe that in characteristic two the stability for H^3 does not work for $Y^{4\lambda}$ and $Y^{8\lambda}$ as the following example demonstrates.

Example 13.2.4. Let $p = 2$. Then

$$\dim \mathrm{H}^3\big(\Sigma_{12}, Y^{(4^3)}\big) = 7,$$
$$\dim \mathrm{H}^3\big(\Sigma_{24}, Y^{(8^3)}\big) = 8.$$

The difference comes from the $L(6) \otimes L(2) \otimes L(8^2)$ term in $L(6) \otimes S^{18}(V)$. It contributes an $L(8^3)$ for which there is no corresponding contribution of an $L(4^3)$ inside $L(6) \otimes S^6(V)$.

Theorem 13.2.1 is highly reminiscent of the generic cohomology results of [5]. In the generic cohomology setting where G is a reductive group and M a finite-dimensional G-module, one has a series of injective maps [19, II.10.14]

$$\mathrm{H}^i(G, M) \to \mathrm{H}^i\big(G, M^{[1]}\big) \to \mathrm{H}^i\big(G, M^{[2]}\big) \to \cdots.$$

In [5] it was shown that this sequence stabilizes to the *generic cohomology* of M. In our setting then, one might expect injections $\mathrm{H}^i(\Sigma_d, Y^\lambda) \to \mathrm{H}^i(\Sigma_{pd}, Y^{p\lambda})$ which stabilize. The following example shows there are not necessarily injections and the dependence on the degree is quite crucial.

Example 13.2.5. In characteristic two it follow from Theorem 12.4.2 that:

$$\mathrm{H}^1\big(\Sigma_8, Y^{(5,3)}\big) \cong k, \qquad \mathrm{H}^1\big(\Sigma_{16}, Y^{(10,6)}\big) = 0.$$

13.3. We only had to perform a small number of tensor product calculations to get a formula for $\mathrm{H}^2(\Sigma_d, Y^\lambda)$ that was valid for any d or λ. The answer was given in terms of the 2-adic expansion of λ and only finitely many possibilities occurred (cf. Table 12.5.1). Suppose one wanted a formula for $\mathrm{H}^i(\Sigma_d, Y^\lambda)$. One would have to compute the tensor products which appear in Proposition 12.6.1, and try to determine the multiplicity of $L(\lambda)$. However Lemma 13.1.3 tells us, essentially, that the appearance of $L(\lambda)$ will be determined by the beginning of the p-adic expansion of λ. That is, if we compute

$$L(\tau) \otimes L(\mu_{(1)})^{(1)} \otimes L(\mu_{(2)})^{(2)} \otimes \cdots \otimes L\big((\mu_{(c)})^{(c)}\big)$$

for c large enough, we get a finite list of simple modules which are not of the form $L(p^c\mu)$, and then some copies of $L(p^c(1^w))$. Consequently we can state the following:

Theorem 13.3.1. *Fix $i > 0$ and let p be arbitrary. Then there is a $t > 0$ such that computing the multiplicities $[L(p\mu_s) \otimes S^t(V) : L(\lambda)]$ for each μ_s appearing in Proposition* 12.6.1 *is enough to determine $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for any d and any $\lambda \vdash d$. Furthermore, for fixed i the dimension of $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ is bounded uniformly, independent of d and λ.*

Theorem 13.3.1 says that for each fixed i there is some table like Table 12.5.1, just larger and depending on more initial terms in the p-adic expansion of λ.

14. Cohomology of permutation modules

14.1. We have seen that $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ is the multiplicity of the simple module $L(\lambda)$ in an explicitly given $GL_d(k)$-module. In this section we will see that the cohomology $\mathrm{H}^i(\Sigma_d, M^\lambda)$ of the permutation module M^λ is determined by the same $GL_d(k)$-module, however this time by the λ-weight space of the module.

Substitute $M = S^\lambda(V)$ and $N = k$ into the spectral sequence (2.1.1). Since $S^\lambda(V)$ is injective as an $S(d,d)$-module, the spectral sequence collapses and we get:

Proposition 14.1.1. *Let p be arbitrary. Then*

$$\dim \mathrm{H}^i\big(\Sigma_d, M^\lambda\big) = \dim \mathrm{Hom}_{GL_d(k)}\big(\mathrm{H}_i\big(\Sigma_d, V^{\otimes d}\big), S^\lambda(V)\big).$$

By [11, 2.1–2.3] the dimension of $\mathrm{Hom}_{GL_d(k)}(U, S^\lambda(V))$ is just the dimension of the λ-weight space U_λ. Thus we have the following result, where we restate the corresponding Young module result Theorem 2.3.1(b) for comparison. Cohomology for Y^λ is controlled by the composition multiplicities in a certain $GL_d(k)$ module whereas for the permutation module it is controlled by weight spaces in the *same* module.

Theorem 14.1.2. *Let p be arbitrary.*

(a) $\dim \mathrm{H}^i(\Sigma_d, M^\lambda) = \dim \mathrm{H}_i(\Sigma_d, V^{\otimes d})_\lambda$.
(b) $\dim \mathrm{H}^i(\Sigma_d, Y^\lambda) = [\mathrm{H}_i(\Sigma_d, V^{\otimes d}) : L(\lambda)]$.

Remark 14.1.3. The weight space dimensions in Theorem 14.1.2(a) are in principal known, as we have described the module $\mathrm{H}_i(\Sigma_d, V^{\otimes d})$ as a tensor product of modules with known weight space decompositions. In principal one can calculate $\mathrm{H}^i(\Sigma_d, M^\lambda)$ from Nakaoka's work and repeated application of the Künneth theorem. However we find Theorem 14.1.2 a much more conceptual and pleasing interpretation.

References

[1] A. Adem, R.J. Milgram, Cohomology of Finite Groups, Grundlehren Math. Wiss., vol. 309, 2nd ed., Springer, 2004.
[2] S. Araki, T. Kudo, Topology of H_n-spaces operations and H-squaring operations, Mem. Fac. Sci. Kyushu Univ. Ser. A 10 (1956) 85–120.
[3] D.J. Benson, Cohomology of modules in the principal block of a finite group, New York J. Math. 1 (1995) 196–206.
[4] D.J. Benson, An algebraic model for the chains on $\Omega(BG_p^\wedge)$, Trans. Amer. Math. Soc. 361 (2009) 2225–2242.
[5] E. Cline, B. Parshall, L. Scott, W. van der Kallen, Rational and generic cohomology, Invent. Math. 39 (1977) 143–163.
[6] F.R. Cohen, The homology of C_{n+1}-spaces, in: The Homology of Iterated Loop Spaces, in: Lecture Notes in Math., vol. 533, Springer, Berlin, 1976, pp. 207–351.
[7] F.R. Cohen, The unstable decomposition of $\Omega^2\Sigma^2X$ and its applications, Math. Z. 182 (4) (1983) 553–568.
[8] F.R. Cohen, J.P. May, L.R. Taylor, Splitting of certain spaces CX, Math. Proc. Cambridge Philos. Soc. 84 (1978) 465–496.
[9] S. Donkin, Symmetric and exterior powers, linear source modules and representations of Schur superalgebras, Proc. Lond. Math. Soc. (3) 83 (2001) 647–680.
[10] S.R. Doty, The submodule structure of certain Weyl modules for groups of type A_n, J. Algebra 95 (1985) 373–383.
[11] S.R. Doty, G. Walker, Modular symmetric functions and irreducible modular representations of general linear groups, J. Pure Appl. Algebra 82 (1992) 1–26.
[12] S.R. Doty, K. Erdmann, D.K. Nakano, Extensions of modules over Schur algebras, symmetric groups, and Hecke algebras, Algebr. Represent. Theory 7 (2004) 67–100.
[13] J. Du, B. Parshall, L. Scott, Quantum Weyl reciprocity and tilting modules, Comm. Math. Phys. 195 (2) (1998) 321–352.
[14] E. Dyer, R.K. Lashof, Homology of iterated loop spaces, Amer. J. Math. 84 (1962) 35–88.
[15] M. Feshbach, The mod 2 cohomology of symmetric groups and rings of invariants, Topology 41 (2002) 57–84.
[16] J.A. Green, Polynomial Representations of GL_n, Lecture Notes in Math., vol. 830, Springer-Verlag, Berlin, 1980.
[17] D.J. Hemmer, Fixed-point functors for symmetric groups and Schur algebras, J. Algebra 280 (2004) 295–312.
[18] D.J. Hemmer, D.K. Nakano, Support varieties for modules over symmetric groups, J. Algebra 254 (2002) 422–440.
[19] J.C. Jantzen, Representations of Algebraic Groups, Math. Surveys Monogr., vol. 107, 2nd ed., Amer. Math. Soc., Providence, RI, 2003.

[20] D.S. Kahn, On the stable decomposition of $\Omega^{\infty} S^{\infty} A$, in: Geometric Applications of Homotopy Theory, Proc. Conf., vol. II, Evanston, IL, 1977, in: Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 206–214.
[21] A.S. Kleshchev, D.K. Nakano, On comparing the cohomology of general linear and symmetric groups, Pacific J. Math. 201 (2001) 339–355.
[22] A.S. Kleshchev, J. Sheth, On extensions of simple modules over symmetric and algebraic groups, J. Algebra 221 (1999) 705–722.
[23] S. Martin, Schur Algebras and Representation Theory, Cambridge Tracts in Math., vol. 112, Cambridge University Press, 1993.
[24] A. Mathas, Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group, Univ. Lecture Ser., vol. 15, American Mathematical Society, 1991.
[25] M. Nakaoka, Homology of the infinite symmetric group, Ann. of Math. (2) 73 (1961) 229–257.
[26] N.J.A. Sloane, The on-line encyclopedia of integer sequences, published electronically at http://www.research.att.com/~njas/sequences/, 2007.
[27] P.A. Smith, A theorem on fixed points for periodic transformations, Ann. of Math. (2) 35 (1934) 572–578.
[28] V.P. Snaith, A stable decomposition of $\Omega^n S^n X$, J. Lond. Math. Soc. (2) 7 (1974) 577–583.
[29] N.E. Steenrod, Cohomology operations derived from the symmetric group, Comment. Math. Helv. 31 (1957) 195–218.
[30] J.B. Sullivan, Some representation theory for the modular general linear groups, J. Algebra 45 (2) (1977) 516–535.