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THE GROUP OF ENDOTRIVIAL MODULES FOR THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract We complete a classification of the groups of endotrivial modules for the modular group algebras of symmetric groups and alternating groups. We show that, for $n \geqslant p^2$, the torsion subgroup of the group of endotrivial modules for the symmetric groups is generated by the sign representation. The torsion subgroup is trivial for the alternating groups. The torsion-free part of the group is free abelian of rank 1 if $n \geqslant p^2 + p$ and has rank 2 if $p^2 \leqslant n < p^2 + p$. This completes the work begun earlier by Carlson, Mazza and Nakano.

Keywords: endotrivial module; endopermutation module; symmetric groups; alternating groups

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1. Introduction

Endotrivial modules were first defined for p-groups by Dade [**8**, **9**], though they had appeared earlier in a celebrated paper of Hall and Higman [**12**]. Early work saw them as the building blocks for the endopermutation modules which are the sources, in the sense of Green's theory of vertices and sources, of the irreducible modules of p-solvable groups. These modules occur in several other situations in modular representation theory. For p-groups a classification of the endotrivial modules was completed by Carlson and Thévenaz [**7**], building on the work of several others. Subsequently, the endopermutation modules were classified by Bouc [**3**].

In this paper, we consider endotrivial modules for symmetric and alternating groups. Our motivation comes from the fact that taking the tensor product with an endotrivial module is a self-equivalence (functor) on the stable module category, that is, the localized category of modules modulo projectives. Thus, the endotrivial modules define a distinguished subgroup of the Picard group of all self-equivalences of the stable module

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category. With that in mind, Carlson *et al*. determined the group of endotrivial modules for finite groups of Lie type [**5**] and made some progress on the symmetric and alternating groups. Specifically, the group of endotrivial modules for all symmetric and alternating groups in characteristic 2 and for all S_n and A_n for $n < p^2$ in the case in which p is odd was found in [**6**].

Our main result is that, in almost all cases for $n \geqslant p^2$, the group of endotrivial modules for the symmetric group S_n is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$, and for the alternating group A_n is isomorphic to $\mathbb{Z}$. The only exception is for $p^2 \leqslant n < p^2 + p$, where the torsion-free part of both groups is the sum of two copies of $\mathbb{Z}$ rather than only one. The class of the sign representation generates the copy of $\mathbb{Z}/2$ in the case of the symmetric groups. The class of the Heller shift $\Omega(k)$ of the trivial module k is a generator for the torsion-free part of both groups. In the case in which $p^2 \leqslant n < p^2 + p$, there is another generator for the torsion-free part of the group of endotrivial modules which remains somewhat elusive. We have some information on the structure of this generator, but it is not precise. A general discussion is given in the last section.

2. Notation and definitions

Let k be a field of characteristic p which is a splitting field for the symmetric group S_n and all of its subgroups. When defining subgroups of the symmetric group we assume the natural ordering on the letters unless otherwise indicated. For example, S_a is the collection of all permutations on $\{1, \dots, a\}$.

For two subgroups H and K of a finite group G, we let $[G/H]$ denote a complete set of representatives for the left H-cosets in G and we let $[H \setminus G/K]$ be a complete set of representatives for the H–K double cosets in G. For elements g, h of a group G and for a subgroup H of G, we write ${}^g h$ instead of ghg^{-1} and ${}^g H$ for gHg^{-1}.

We consider finitely generated left modules over group algebras. We denote by $\mathrm{mod}(kG)$ the category of finitely generated kG-modules and by $\mathrm{stmod}(kG)$ the corresponding stable module category. Given a group inclusion $H \hookrightarrow G$ we denote the induction and restriction functors between $\mathrm{mod}(kG)$ and $\mathrm{mod}(kH)$ by Ind_H^G and Res_H^G, respectively. If M and N are kG-modules, we write $\mathrm{Hom}_k(M, N)$ for the kG-module of all k-linear maps from M to N. If $N = M$, we write $\mathrm{End}_k M$ instead of $\mathrm{Hom}_k(M, M)$ and if $N = k$ is the trivial kG-module, we write $M^* = \mathrm{Hom}_k(M, k)$ for the k-linear dual of M. Let $M \otimes N$ be the tensor product of two modules M and N over the base field k with diagonal action of the group G. We write $M \mid N$ to mean that the module M is isomorphic to a direct summand of N.

For a kG-module M, let $\Omega^n(M)$ be the kernel of a projective cover $P \to M$ of M and let $\Omega^{-1}(M)$ be the cokernel of the injective hull $M \hookrightarrow Q$. Iterating, we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{n-1}(M))$. We remind the reader that kG is a self-injective ring, and hence injective modules are also projective.

Definition 2.1. A kG-module M is *endotrivial* provided that $\mathrm{End}_k M \cong k \oplus (\mathrm{proj})$ or, equivalently, $\mathrm{End}_k M \cong k$ in $\mathrm{stmod}(kG)$.

Recall that $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$ as kG-modules. Consequently, the tensor product of two endotrivial modules is endotrivial. This allows us to define the group of endotrivial modules whose elements are equivalence classes of endotrivial modules.

Definition 2.2. Two endotrivial kG-modules are *equivalent* if they are isomorphic in $\mathrm{stmod}(kG)$. That is, $[M] = [N]$ if $M \oplus P \cong N \oplus Q$ for projective modules P and Q. The group of endotrivial kG-modules is the set $T(G)$ of equivalence classes $[M]$ of endotrivial kG-modules M, with the operation given by the rule $[M] + [N] = [M \otimes N]$.

Clearly, $T(G)$ is abelian, and we have that $0 = [k]$ and $-[M] = [M^*]$. Furthermore, if p does not divide the order of G, then every module is projective. In this case, the definition of an endotrivial module does not have much meaning, as every object in the stable category is equivalent to the zero object, and also every module is an endotrivial module, by a strict interpretation of the definition. In that case, we set $T(G) = \{0\}$.

3. Properties of endotrivial modules

In this section we recall some basic properties of the group $T(G)$ that will be of use to us.

Theorem 3.1. *Let G be a finite group. The group $T(G)$ is finitely generated. Thus, the torsion subgroup $TT(G)$ of $T(G)$ is finite and there is a torsion-free subgroup $TF(G)$ of $T(G)$ of finite rank, such that $T(G) \cong TT(G) \oplus TF(G)$.*

(a) *The modules $\Omega^n(k)$ are endotrivial and their classes form a cyclic direct summand of $T(G)$ [**6**, Theorem 2.3 (a)].*

(b) *Let n denote the number of conjugacy classes of maximal elementary abelian p-subgroups of p-rank 2 in G [**6**, Theorem 2.3 (b)]. Then the rank of $TF(G)$ is n if G has p-rank at most 2 and is $n + 1$ if the p-rank of G is greater than 2.*

(c) *Let P be a Sylow p-subgroup of G [**6**, Theorem 2.3 (d)].*

  (i) *The torsion subgroup $TT(P)$ is trivial except in the case in which P is cyclic, quaternion or semi-dihedral.*

  (ii) *If $TT(P)$ is trivial, then $TT(G)$ is generated by the classes $[M]$ of indecomposable endotrivial kG-modules M such that $\mathrm{Res}^G_P M \cong k \oplus (\mathrm{proj})$, for a projective kP-module* (proj).

Note that, in general, a module with vertex P and trivial source is not endotrivial. Nevertheless, for a subgroup H which contains a Sylow p-subgroup, the groups $T(G)$ and $T(H)$ are related to each other by the following.

Proposition 3.2 (Carlson *et al.* [5, Proposition 2.6]). *Let H be a subgroup of G that contains a Sylow p-subgroup P of G, and let M be an indecomposable endotrivial kG-module. The following hold.*

(a) *If $N_G(P) \leqslant H$, then the restriction map $\mathrm{Res}^G_H : T(G) \to T(H)$ is injective. The kH-module $\mathrm{Res}^G_H M$ is endotrivial and has a unique indecomposable non-projective direct summand. This summand has vertex P and is isomorphic to the kH-Green correspondent of M.*

(b) *Suppose that H is a normal subgroup of G. Then $\mathrm{Res}^G_H M$ is endotrivial and indecomposable. Thus, if P is non-cyclic and is a normal Sylow p-subgroup of G, then $TT(G)$ is isomorphic to the group of one-dimensional kG-modules, i.e. $TT(G) \cong G/G'P$.*

More generally, for any indecomposable endotrivial kG-module M and for any subgroup H of G, we have that $\mathrm{Res}^G_H M \cong M_0 \oplus (\mathrm{proj})$, where M_0 is an indecomposable endotrivial kH-module and (proj) is a projective kH-module. In particular, for any endotrivial kG-module M, there is a unique indecomposable endotrivial direct summand M_0 of M such that $M \cong M_0$ in $\mathrm{stmod}(kG)$. Notice also that $\mathrm{Dim}\, M \equiv \pm 1 \bmod |P|$ if p is odd, whereas $\mathrm{Dim}\, M \equiv \pm 1 \bmod |P|/2$ if $p = 2$.

4. Subgroup structure

In this section we collect some information concerning the p-local structure of the symmetric and alternating groups for p an odd prime. We write G for the symmetric group S_n of degree n, for an integer n greater than or equal to p^2, and we write A for the alternating subgroup of the same degree n as G. Hence, the Sylow p-subgroups of G are not abelian and are all contained in A.

For H and W two finite groups, with W a transitive subgroup of some symmetric group S_n, the *wreath product* of H and W is the group $G = H \wr W$, isomorphic to a semidirect product

$$(H^{(1)} \times \cdots \times H^{(n)}) \rtimes W \quad \text{with } H^{(i)} \cong H \text{ for all } i$$

and with W acting on the set $\{H^{(i)} \mid 1 \leqslant i \leqslant n\}$ by permutation of the superscripts $\{1, \ldots, n\}$. The normal subgroup $H^{(1)} \times \cdots \times H^{(n)}$ of G is called the *base subgroup.* More generally, we define inductively *iterated wreath products* $H^{\wr i} = (H^{\wr(i-1)}) \wr H$, for all $i \geqslant 2$, and for all transitive subgroups H of some symmetric group.

Some detail of the structure of the Sylow p-subgroup of G and its normalizer can be found in [**1**]. Let N be the normalizer of a Sylow p-subgroup P of G. The normalizer N_A of P in A has index 2 in N. For $i \geqslant 0$, write $N_i = N_{S_{p^i}}(P_i)$ for the normalizer of a Sylow p-subgroup $P_i \cong {C_p}^{\wr i}$ in S_{p^i}. Then, $N_i \cong P \rtimes (C_{p-1})^i$. In the general case, we write

$$n = \sum_{0 \leqslant i \leqslant s} a_i p^i \quad \text{with } s \geqslant 2 \text{ and } a_s \neq 0$$

for the p-adic expansion of n. There are isomorphisms

$$P \cong \prod_{1 \leqslant i \leqslant s} (P_i^{a_i}) \quad \text{and} \quad N \cong \prod_{0 \leqslant i \leqslant s} (N_i \wr S_{a_i}).$$

For a group H, let us denote by H' its derived subgroup. (We thank Jørn Olsson for providing the proof of the next lemma.)

Lemma 4.1. *Assume that $n = p^t$ for some integer $t \geqslant 1$ and set $N = N_t$ and $P = P_t$. Then, $N' = P$ and ${N_A}' = P$.*

Proof. Since the factor group N/P is abelian, we have that P contains N'.

Conversely, we need to show that N/N' has order prime to p. We proceed by induction on t for $t \geqslant 1$. If $t = 1$, then $N_1 \cong P_1 \rtimes C_{p-1}$ and we can easily verify that the p-cycle generating P_1 is a commutator in N_1. So $P \subseteq N'$. Assume now that $t > 1$ and set $Q = Q_t$ for the base subgroup of P_t. That is $Q_t \cong P_{t-1}^p$. By [**14**, Lemma 4.2], we have that $Q_t \lhd N_t$, which implies that $N_t = N_{H_t}(P_{t-1} \wr P_1)$, where H_t is the subgroup of G containing P_t and which is isomorphic to $S_{p^{t-1}} \wr S_p$. Now, by [**14**, Proposition 1.5], the factor group N_t/Q_t' is isomorphic to $N_{S_{p^{t-1}}}(P_{t-1})/P_{t-1}' \times N_1$. Since $Q_t' \leqslant N_t'$, we obtain, by induction, that $N_t/N_t' \cong (C_{p-1})^{t-1} \times C_{p-1}$.

The statement for the alternating group is immediate. □

We end with an important observation that will be useful in the next section.

Proposition 4.2. *Consider the above notation and assume that $n = p^s$ for $s \geqslant 1$. Let H be a subgroup of G isomorphic to S_{n-1}. There exist elements $\sigma_1, \sigma_2, \dots, \sigma_s \in H$, each of order $p-1$, such that*

$$N = \langle P, \sigma_1, \sigma_2, \dots, \sigma_s \rangle \cong P_s \rtimes (C_{p-1})^s.$$

Furthermore, if $\sigma_{i,j} = \sigma_i \sigma_j$ for all $1 \leqslant i, j \leqslant s$, then $N_A = \langle P, \sigma_{i,j},\ 1 \leqslant P, \sigma_{i,j};\ 1 \leqslant i, j \leqslant s \rangle$, and the in a given subgroup of A isomorphic to A_{n-1}.

Proof. We proceed by induction on s. Clearly, the statement holds in the case when $s = 1$, where σ_1 is just a $(p-1)$-cycle. Assume that $s \geqslant 1$ and that the statement holds for $n = p^{s-1}$. By [**1**, (1.3)],

$$\frac{N_{S_{p^s}}(P_s)}{P_s} = \frac{N_{S_{p^{s-1}.p}}(P_{s-1} \wr P_1)}{P_{s-1} \wr P_1} = \frac{N_{S_{p^{s-1}}}(P_{s-1})}{P_{s-1}} \times \frac{N_{S_p}(P_1)}{P_1}.$$

With the above inductive statement we can construct specific instances of the normalizer of the Sylow subgroup. That is, the normalizer of P_1 is generated by the cycles

$$\sigma_1 = (1, 2, \dots, p) \quad \text{and} \quad \mu_{1,1} = (\sigma_1 \mapsto \sigma_1^\ell),$$

where ℓ generates the group of units in $\mathbb{F}_p$. For example, if $p = 5$, we can take $\ell = 2$ and hence $\mu_{1,1} = (1, 2, 4, 3)$, which fixes the letter 5. In general, $\mu_{1,1}$ can be taken to be the $(p-1)$-cycle, $(1, \ell, \ell^2, \dots, \ell^{p-1})$, where ℓ^i should be read as the residue of ℓ^i modulo p, or as the element in $\mathbb{F}_p$. The cycle fixes the letter p (which is zero in $\mathbb{F}_p$).

Then $N_{S_{p^2}}(P_2)$ is generated by

$$\sigma_1, \sigma_2 = \prod_{i=1}^{p} (i, i+p, \dots, i+(p-1)p), \quad \mu_{2,1} = \prod_{i=0}^{p-1} \sigma_2^i \mu_{1,1} \sigma_2^{-i} \quad \text{and} \quad \mu_{2,2} = (\sigma_2 \mapsto \sigma_2^\ell).$$

Here σ_1 and $\mu_{1,1}$ are the cycles given exactly as above, but they are now considered to be elements of S_{p^2}. The elements σ_1 and σ_2 generate P_2. The element $\mu_{2,1}$ is the product of the conjugates of $\mu_{1,1}$ by powers of σ_2. These conjugates are disjoint cycles and hence $\mu_{2,1}$ commutes with σ_2 and normalizes the normal subgroup of P_2 generated by σ_1 and its conjugates by powers of σ_2. The elements σ_2 and $\mu_{2,2}$ generate a subgroup isomorphic to $N_{S_p}(P_1)$.

The general case is similar. The group P_s is generated by $\sigma_1, \ldots, \sigma_s$, where σ_s is the product of p^{s-1} disjoint p-cycles:

$$\sigma_s = \prod_{i=1}^{p^{s-1}} (i, i + p^{s-1}, \ldots, i + (p-1)p^{s-1}).$$

Then the normalizer $N_{S_{p^s}}(P_s)$ is generated by P_s, by

$$\mu_{s,j} = \prod_{i=0}^{p-1} \sigma_s^i \mu_{s-1,j} \sigma_s^{-i} \quad \text{for } j = 1, \ldots, s-1 \text{ and by } \mu_{s,s} = (\sigma_s \mapsto \sigma_s^\ell).$$

The element $\mu_{s,s}$ is a product of p^{s-1} $(p-1)$-cycles, conjugation by each one of which takes the corresponding p-cyclic factor of σ_s to its ℓth power and fixes the other factors. The elements $\sigma_1, \ldots, \sigma_s$ in the statement of the proposition can be taken to be the elements $\mu_{s,1}, \ldots, \mu_{s,s}$ in the above construction. It is a straightforward exercise to show that these elements commute with one another. It is also clear that every one of these elements stabilizes the letter $n = p^s$, and hence they are contained in S_{p^s-1} as asserted. The normalizer of any other Sylow p-subgroup is conjugate to this one, and hence the conjugate elements stabilize some letter.

The last statement of the proposition is straightforward from this analysis and from [**1**, Equation (2.1)]. □

5. The torsion subgroup

Let H be a subgroup of $G = S_n$ isomorphic to S_{n-1}. Let N be the normalizer of a Sylow p-subgroup P of G. We let A, N_A, $H_A = H \cap A$ be as in the previous section.

By Proposition 3.2, the indecomposable torsion endotrivial kG-modules are among the Green correspondents of the one-dimensional kN-modules. So suppose that χ is a one-dimensional kN-module which has an endotrivial kG-Green correspondent M. In this section we show that χ, and thus M, is necessarily either k or ε, where ε denotes the one-dimensional sign representation.

Throughout this section let χ and M be as above. Assume that χ_A is a one-dimensional kN_A-module with kA-Green correspondent M_A which we assume to be endotrivial.

We recall the results from [**5**, Theorems A and B].

Proposition 5.1. *$TT(S_n) = \langle [\varepsilon] \rangle$ and $TT(A_n) = \{[k]\}$ for all $3p \leqslant n < p^2$.*

Our objective is to prove the following theorem.

Theorem 5.2. *Let $p \neq 2$ and assume that $n \geqslant p^2$. Then $TT(S_n) = \langle[\varepsilon]\rangle$ and $TT(A_n) = \{[k]\}$.*

We proceed by an inductive argument which varies depending on the p-adic expansion of n. The most difficult cases are those in which the expansion has only one term. These situations are treated first. The base case of the induction is given by Proposition 5.1.

Suppose that $n = p^s$. The next result is needed for this case.

Lemma 5.3.

(a) *There are exactly $(p-1)^s$ one-dimensional kN-modules, corresponding to a choice of a $(p-1)$st root of unity for each σ_i.*

(b) *The restrictions to $H \cap N$ of the one-dimensional kN-modules remain pairwise non-isomorphic.*

(c) *There are exactly $(p-1)^s/2$ one-dimensional kN_A-modules, all of which are pairwise non-isomorphic upon restriction to $H_A \cap N_A$.*

Proof. The first statement is straightforward from Proposition 3.2 and Lemma 4.1. The second claim follows from the choice of the elements σ_i that determine the action of N on any one-dimensional kN-module. Namely, by Proposition 4.2, $\sigma_i \in H \cap N$ for all i. Part (c) follows from parts (a) and (b). □

The case when $p = 3$ and $n = 9$ is treated separately, because Proposition 5.1 does not apply and S_8 has torsion endotrivial modules which are not one dimensional. In this case, N has four one-dimensional modules. Explicitly, we checked using the algebra software Magma [**2**] that the Green correspondents for these modules are k, ε, M and $M \otimes \varepsilon$, where M has dimension 118, which is not congruent to $\pm 1 \bmod 81$. Hence, M is not endotrivial. A similar statement holds for M_A.

Thus, we have proved the smallest case ($p^s = 9$) of the following.

Proposition 5.4. *Let $s \geqslant 2$. If $p^s > 9$, then assume also that $TT(S_{p^s-1}) = \langle[\varepsilon]\rangle$, and $TT(A_{p^s-1}) = \{[k]\}$. We have that $TT(S_{p^s}) = \langle[\varepsilon]\rangle$ and $TT(A_{p^s}) = \{[k]\}$.*

Proof. Assume that $p^s > 9$. We know that $M \mid \mathrm{Ind}_N^G \chi$. Tensoring by ε if necessary, we can assume without loss of generality that $\mathrm{Res}_H^G M \cong k \oplus (\mathrm{proj})$, and so there is a non-trivial map in $\mathrm{Hom}_{kH}(k, \mathrm{Res}_H^G M)$. Likewise, $\mathrm{Hom}_{kH_A}(k, \mathrm{Res}_{H_A}^A M_A)$ is non-zero. Using the Mackey Formula and the Eckmann–Shapiro Lemma, we get that

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{kH}(k, \mathrm{Res}_H^G \mathrm{Ind}_N^G \chi) \\ &\cong \mathrm{Hom}_{kH}\left(k, \bigoplus_{x \in [H\backslash G/N]} \mathrm{Ind}^H_{{}^xN \cap H} \mathrm{Res}^{{}^xN}_{{}^xN \cap H} {}^x\chi\right) \\ &\cong \prod_{x \in [H\backslash G/N]} \mathrm{Hom}_{k({}^xN \cap H)}(k, \mathrm{Res}^{{}^xN}_{{}^xN \cap H} {}^x\chi). \end{aligned} \tag{5.1}$$

By Lemma 5.3 (b), the latter is non-zero only if $\chi = k$, in which case $M \cong k$. Thus, $TT(G) = \langle[\varepsilon]\rangle$, as desired. Similarly, $\mathrm{Hom}_{kH_A}(k, \mathrm{Res}_{H_A}^A \mathrm{Ind}_{N_A}^A \chi_A)$ is non-zero if and only if $\chi_A = k$. □

Suppose that $n = 2p^s$ and $s \geqslant 2$. The normalizer of the Sylow p-subgroup P has the form $N \cong N_s \wr S_2$, where N_s is the normalizer of the Sylow p-subgroup of S_{p^s}. Let J be the subgroup of G containing N and which is isomorphic to a wreath product $S_{p^s} \wr S_2$. We proceed as in [**6**, §§ 6 and 8]. Let $S = S_{p^s} \times S_{p^s}$ be the Young subgroup of G for the partition (p^s, p^s). Note that $J = SN$ and S is a normal subgroup of index 2 in J. Write $N_S = N \cap S$. Likewise, let us set $A = A_{2p^s}$ and $J_A = J \cap A$.

Proposition 5.5. *Assume that $TT(S_{p^s}) = \langle[\varepsilon]\rangle$, $TT(A_{p^s}) = \{[k]\}$. Assume also that $TT(S_{2p^s-2}) = \langle[\varepsilon]\rangle$. Then we have that $TT(S_{2p^s}) = \langle[\varepsilon]\rangle$ and $TT(A_{2p^s}) = \{[k]\}$.*

Proof. Let χ be a one-dimensional kN-module with an endotrivial kJ-Green correspondent L. By the Green correspondence and the Mackey Formula, we have that

$$\mathrm{Res}^J_S L \mid \mathrm{Res}^J_S \mathrm{Ind}^J_N \chi \cong \mathrm{Ind}^S_{N_S} \chi_{N_S}$$

since $J = SN$, and where $\chi_{N_S} = \mathrm{Res}^N_{N_S} \chi$. Therefore, $\mathrm{Res}^J_S L$ is a direct summand of $\mathrm{Ind}^S_{N_S} \chi_{N_S}$.

Now, the conditions that S is normal in J and that L is an indecomposable endotrivial module imply that $\mathrm{Res}^J_S L$ is an indecomposable endotrivial module, by Proposition 3.2. Thus, $L_S = \mathrm{Res}^J_S L$ is the kS-Green correspondent of χ_{N_S}. Note that the Green correspondence is well defined in this case, since $N_S = N_S(P)$.

Let K be a subgroup of S containing N_S and which is isomorphic to a direct product $S_{p^s} \times N_s$, where N_s is the normalizer of the Sylow p-subgroup of S_{p^s}. By our assumption, L_S is an indecomposable endotrivial module. So $\mathrm{Res}^S_K L_S \cong U \oplus (\mathrm{proj})$ for some indecomposable endotrivial kK-module U which also satisfies the condition that $\mathrm{Res}^K_{N_S} U \cong \chi_S \oplus (\mathrm{proj})$. Now, because K has a non-trivial normal p-subgroup, $\mathrm{Res}^K_{N_S} U$ has no non-zero projective summand. That is, U must be a direct summand of χ_S induced to K, and the restriction back to N_S consists entirely of modules whose vertices contain that normal subgroup. Therefore, $\mathrm{Res}^K_{N_S} U \cong \chi_S$ and U has dimension 1. A similar argument, using the fact that a set of coset representatives of K in S can be taken to normalize a p-subgroup of K, shows that $\mathrm{Res}^S_K L_S \cong U$, and so L_S and L also have dimension 1. Likewise, the indecomposable torsion endotrivial kJ_A-modules have dimension 1.

We first handle the case of the symmetric groups. There are exactly four one-dimensional kJ-modules, which form a Klein four-group. That is, $TT(J)$ is generated by the sign representation and $\chi = k \wr \varepsilon$, which is the one-dimensional module on which S acts by the trivial representation and elements not in S act by multiplication by -1. Relative projectivity shows that the kG-Green correspondent M of χ is a Young module. Namely, M is isomorphic to a direct summand of the permutation module $\mathrm{Ind}^G_S k = M^{(p^s,p^s)}$. It is well known that the indecomposable summands of a permutation module M^λ are Young modules labelled by partitions greater than or equal to λ in the dominance order. In addition, the Young module Y^λ occurs exactly once. Now, S is the only proper Young subgroup of G of index prime to p, and therefore $Y^{(p^s,p^s)}$ is the only indecomposable direct summand of $M^{(p^s,p^s)}$ with vertex P. By the Krull–Schmidt Theorem, we conclude that M is isomorphic to $Y^{(p^s,p^s)}$.

Therefore, the question is reduced to determining whether $Y^{(p^s,p^s)}$ is an endotrivial module. From [**13**, Theorem 5.1] we prove that this is not the case. Explicitly, if L is a subgroup of G isomorphic to S_{2p^s-1}, then $\mathrm{Res}^G_L Y^{(p^s,p^s)}$ has a direct summand V of the form

$$V = \bigoplus_{0\leqslant i\leqslant s} Y^{(p^s+p^i-1,p^s-p^i)}.$$

Recall that $s \geqslant 2$ and that a Young module Y^λ is projective if and only if λ is p-restricted; that is, the difference of any two consecutive parts of λ is less than p (cf. [**11**, Theorem 2]). In particular, V has at least two direct summands which are not projective, and so $\mathrm{Res}^G_L Y^{(p^s,p^s)}$ is not endotrivial. *A fortiori*, neither is $Y^{(p^s,p^s)}$. This shows that $TT(G) = \langle[\varepsilon]\rangle \cong \mathbb{Z}/2$, whenever $G = S_{2p^s}$, with $s \geqslant 2$.

We now turn to the alternating groups. As in [**6**], the above argument does not apply to the non-trivial one-dimensional kJ_A-modules. These form a Klein four-group, generated by the restriction of a two-dimensional kJ-module. Namely, let V be the subgroup of J of index 8 that is isomorphic to a direct product $A_{p^s} \times A_{p^s}$. Then V is normal in J. Because the factor group J/V is dihedral of order 8, there is a simple kJ-module U of dimension 2. The same argument in [**6**, § 8] says that $\mathrm{Res}^J_{J_A} U$ splits as the direct sum of two one-dimensional conjugate modules $\chi \oplus {}^x\chi$, and that $\mathrm{Res}^J_S U \cong \lambda \oplus (\varepsilon\lambda)$, where $\lambda \cong k_{S^s_p} \otimes \varepsilon_{p^s}$ affords a signed permutation module, usually denoted as $M^{(p^s|p^s)}$, and with Young vertex S, in the sense of Grabmeier (cf. [**10**, § 1.3]). Moreover, $M^{(p^s|p^s)}$ contains the kG-Green correspondent Y of U (in fact, $Y = M^{(p^s|p^s)}$, as the latter is indecomposable). We have that $\mathrm{Res}^G_A Y = Y_A \oplus {}^xY_A$, where Y_A is isomorphic to the kA-Green correspondent of χ. Now, if Y_A is endotrivial, then so is $\mathrm{Res}^A_B Y_A$, for any subgroup B of A. Hence, take $B \cong S_{2p^s-2}$ and assume that $\mathrm{Res}^A_B Y_A$ is endotrivial. By hypothesis, we have that $\mathrm{Res}^A_B Y_A \cong \mu \oplus (\mathrm{proj})$, where $\mu \cong \varepsilon$, or $\mu \cong k$. In particular, it follows that $0 \neq \mathrm{Hom}_B(\mu, \mathrm{Res}^A_B Y_A)$. By the Mackey Formula and the Eckmann–Shapiro Lemma and using the fact that $A = J_A B$, we obtain

$$\begin{aligned} 0 &\neq \mathrm{Hom}_B(\mu, \mathrm{Res}^A_B \mathrm{Ind}^A_{J_A} \chi) \\ &\cong \mathrm{Hom}_B(\mu, \mathrm{Ind}^B_{J_A\cap B} \mathrm{Res}^{J_A}_{J_A\cap B} \chi) \\ &\cong \mathrm{Hom}_{J_A\cap B}(k, \mathrm{Res}^{J_A}_{J_A\cap B} \chi), \end{aligned}$$

since $\mathrm{Res}^B_{J_A\cap B} \mu = k$. However, a direct computation shows that $\mathrm{Res}^{J_A}_{J_A\cap B} \chi$ is not trivial. So we have a contradiction to the assumption that Y_A is endotrivial. □

Suppose that $n = ap^s$ *for* $3 \leqslant a < p$ *and* $s \geqslant 2$. Again, we assume without loss of generality and by induction that $\mathrm{Res}^G_H M \cong k \oplus (\mathrm{proj})$ and that $\mathrm{Res}^A_{H_A} M_A \cong k \oplus (\mathrm{proj})$, where $H \cong S_{ap^s-1}$ as before. Moreover, $\mathrm{Res}^G_N M \cong \chi \oplus (\mathrm{proj})$ and $\mathrm{Res}^A_{N_A} M_A \cong \chi_A \oplus (\mathrm{proj})$, by the Green correspondence. In this case, $N \cong N_s \wr S_a$, which has $2(p-1)^s$ modules of dimension 1, as in the previous case. However, these modules are distinguished by their restrictions to the two subgroups N_s and S_{a-1}. Both of these subgroups are contained in H, and hence the restrictions of M to both of these subgroups must be a trivial module plus a projective module. We conclude that $\chi = k$ is the trivial kN-module and its Green correspondent M is the trivial kG-module.

Suppose the p-adic expansion of n has two or more terms. We write

$$n = a_0 + a_1 p + \cdots + a_s p^s \quad \text{for } s \geqslant 2,\ a_s \neq 0 \text{ and } n \neq a_s p^s.$$

Recall that

$$N \cong \prod_{i \geqslant 0} (N_i \wr S_{a_i}).$$

We know that the kN-Green correspondent of M has dimension 1. Without loss of generality, we assume by induction that

$$\operatorname{Res}^G_{S_{a_s p^s}} M \cong k \oplus (\text{proj}) \quad \text{and} \quad \operatorname{Res}^A_{A'} M_A \cong k \oplus (\text{proj}),$$

where $A' = S_{a_s p^s} \cap A$. Now, each of the subgroups $N_i \wr S_{a_i}$ for $i < s$ is conjugate to a subgroup of $S_{a_s p^s}$, and the corresponding claim holds for the corresponding subgroups of N_A. Thus, $\operatorname{Res}^G_N M \cong k \oplus (\text{proj})$ and $\operatorname{Res}^A_{N_A} M_A \cong k \oplus (\text{proj})$. Hence, M and M_A are the Green correspondents of k and are trivial modules.

With all of this we can complete the proof of the main theorem.

Proof of Theorem 5.2. We perform induction on s, where s is the largest degree in the p-adic expansion of n, beginning with $s = 2$. For $n = p^2$, we invoke Propositions 5.1 and 5.4. We use the proof of the case in which the p-adic expansion of n has more than one term to prove the theorem for $n < 2p^s$. Then we apply Proposition 5.5. Now we use the proof for the case in which $n = ap^s$ for $a \geqslant 3$ and the proof for the case of more than one term in the p-adic expansion of n. Applied in the proper order, these results prove the theorem for all n such that $n < p^{s+1}$. We can now use Proposition 5.4 to prove the theorem for $n = p^{s+1}$. The latter is the induction step. □

6. Torsion-free complements

Recall our assumption that $p > 2$. It is an easy calculation to see that, for $n \geqslant p^2 + p$, the groups $G = A_n$ and $G = S_n$ have no maximal elementary abelian p-subgroups of p-rank 2. As a consequence, by Theorem 3.1, the torsion-free part $TF(G)$ of the group $T(G)$ is isomorphic to $\mathbb{Z}$ and is generated by $[\Omega(k)]$. This is the major part of our investigation of $TF(G)$.

In the case when $p^2 \leqslant n < p^2 + p$, a Sylow p-subgroup P of G has the form $C_p \wr C_p$, and P has two conjugacy classes of maximal elementary abelian subgroups. These are the base subgroup $E_1 \cong C_p^p$, which is normal in P, and $E_2 = \langle x, y \rangle \cong C_p^2$, where $\langle x \rangle = E_1 \cap E_2$ is the centre of P and y is a non-central p-element not in E_1. We can take y to be a generator of the second C_p in the wreath product expression for P.

We have proved the first part of the following.

Theorem 6.1. *Let G denote either the symmetric group S_n or the alternating A_n.*

(i) *If $n \geqslant p^2 + p$, then $TF(G) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}$.*

(ii) *Suppose that $p^2 \leqslant n < p^2 + p$. Then $TF(G) \cong \mathbb{Z}^2$. The class $[\Omega(k)]$ generates one direct summand of $TF(G)$. The other summand is generated by the class $[M]$ of* *an indecomposable endotrivial kG-module M, having the property that*

$$\mathrm{Res}^G_{E_1} M \cong k \oplus (\mathrm{proj}) \quad \textit{and} \quad \mathrm{Res}^G_{E_2} M \cong \Omega^{2pr}(k) \oplus (\mathrm{proj})$$

for some integer r with $1 \leqslant r \leqslant p-1$.

Proof. Theorem 3.1 shows that the rank of $TF(G)$ in case (ii) is 2, once we have verified that E_1 and E_2 are maximal elementary abelian p-subgroups of G. That is, the only question is whether or not E_2 is conjugate to a subgroup of E_1 in S_n. This is not possible because of the cycle structure of the elements of E_2. Specifically, every non-identity element of E_2 is the product of p p-cycles. On the other hand, there is a subgroup F of index p in E_1 which has no element that is a product of p p-cycles. Explicitly, consider $H = E_1 \cap S_{p^2-p}$. Then, any subgroup of order p^2 in E_1 has a non-trivial intersection with H and hence contains a non-identity element which does not have the cycle structure of the non-identity elements of E_2.

In case (ii) we know that the class of $\Omega(k)$ generates a summand of $TF(G)$. The sum of the restriction maps

$$TF(G) \to TF(E_1) \oplus TF(E_2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

is an injection. Let $[M]$ be the other generating class. By replacing M with a suitable Heller translate, we may assume that the restriction of the class of M to E_1 is zero in $TF(E_1)$. Hence, we can assume that the restriction of M to E_1 is the direct sum of k plus a projective module. The restriction to E_2 is isomorphic to the direct sum of $\Omega^t(k)$ plus a projective module, with $t \neq 0$. By taking a dual if necessary we can assume that $t > 0$.

The restriction map $\mathrm{Res}^G_{E_2}$ factors through the restriction map Res^G_P to the Sylow p-subgroup P of G that contains E_1 and E_2. Consequently, t is bounded below by the same value that is obtained for $TF(P)$. This value is $2p$ by [**7**]. It also follows that $t = 2pr$ is a multiple of $2p$. On the other hand, t is bounded above by the minimal degree of a cohomology element of $H^*(G,k)$ whose restriction to the centre Z of P is not nilpotent (see the proof of [**5**, Theorem 3.1]). Now, by the analysis of [**4**, Proposition 5.1], there is an element γ of degree $2p(p-1)$ in the integral cohomology whose restriction to Z is not zero. Because Z is cyclic and γ has even degree, this element is not nilpotent and thus γ restricts to a non-zero element in the mod-p cohomology. Therefore, $2p \leqslant t \leqslant 2p(p-1)$ and t is divisible by $2p$, as asserted. □

We conclude with some partial information on the missing generator. From [**5**, **7**], we can find explicit generators for $T(N_G(P))$. In the case of the symmetric group S_n, with $p^2 \leqslant n < p^2 + p$, this latter result can be improved in the following way. Consider P, E_1 and $E_2 = \langle x, y \rangle$ as above. Set $H = N_G(E_1)$. Then, H contains $N_G(P)$ and has the form $H \cong (N_1 \wr S_p) \times S_a$, with $a = n - p^2$, in the notation of § 4. Notice that the inflation of an endotrivial module from $N_1 \wr S_p$ to H is endotrivial, and that any endotrivial kH module can be obtained up to equivalence in this way. Thus, our task is reduced to finding generators for $T(H)$ in the case in which $n = p^2$, which we assume henceforth. Now, [**7**, Theorem 3.1] gives us that

$$T(P) = \langle [\Omega(k)], [M] \rangle \quad \text{with } M = \Omega^{-2}(\Omega^2_{P/\langle y \rangle}(k)).$$

Here, $\Omega^2_{P/\langle y \rangle}(k)$ is the unique indecomposable direct summand of the tensor product $\Omega_{P/\langle y \rangle}(k) \otimes \Omega_{P/\langle y \rangle}(k)$, where $\Omega_{P/\langle y \rangle}(k)$ is the kernel of the map $k[P/\langle y \rangle] \to k$ which sends a coset $u\langle y \rangle \in P/\langle y \rangle$ to 1. We observe that $\Omega_{P/\langle y \rangle}(k)$ extends to H. Indeed, let C be a complement of E_1 in the base subgroup of H. Thus, $C \cong C^p_{p-1}$. In fact, $H = E_1 \rtimes CS_p$, with $\langle y \rangle \leqslant H$. Consider the permutation module $L = k[H/CS_p]$. Then

$$\operatorname{Res}^H_P L \cong \operatorname{Res}^H_P \operatorname{Ind}^H_{CS_p} k \cong \bigoplus_{x \in [P \backslash H/CS_p]} \operatorname{Ind}^P_{{}^x(CS_p) \cap P} k \cong k[P/\langle y \rangle]$$

by the Mackey Formula and since $H = PCS_p$. Thus, the relative syzygy $\Omega_{H/CS_p}(k)$, that is, the kernel of the map $L \to k$, restricts to P to an indecomposable module, isomorphic to $\Omega_{P/\langle y \rangle}(k)$. Consequently, $\Omega^2_{P/\langle y \rangle}(k)$ extends to $\Omega^2_{H/CS_p}(k)$, proving simultaneously that the latter is endotrivial. The same holds for the translate $M = \Omega^{-2}(\Omega^2_{H/CS_p}(k))$. Obviously, the restriction of M to $A \cap H$ is an indecomposable endotrivial module. This proves the following.

Proposition 6.2. *Consider the symmetric group $G = S_n$ and the alternating group $A = A_n$, with $p^2 \leqslant n < p^2 + p$. In the same notation as above, with $H = N_G(E_1)$ and $H_A = N_A(E_1)$, we have that*

$$T(H) = TT(H) \oplus \langle [\Omega(k)], [M] \rangle \quad \text{and} \quad T(H_A) = TT(H_A) \oplus \langle [\Omega(k)], [\operatorname{Res}^H_{H_A} M] \rangle,$$

where $TT(H)$ and $TT(H_A)$ are generated by the one-dimensional modules, and where the module M satisfies

$$\operatorname{Res}^H_{E_1} M \cong k \oplus (\text{proj}) \quad \text{and} \quad \operatorname{Res}^H_{E_2} M \cong \Omega^{2p}(k) \oplus (\text{proj}).$$

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