

The Lie module and its complexity

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ABSTRACT

The complexity of a module is an important homological invariant that measures the polynomial rate of growth of its minimal projective resolution. For the symmetric group Σ_n , the Lie module $\text{Lie}(n)$ has attracted a great deal of interest in recent years. We prove here that the complexity of $\text{Lie}(n)$ in characteristic p is t , where p^t is the largest power of p dividing n , thus proving a conjecture of Erdmann, Lim and Tan. The proof uses work of Arone and Kankaanrinta which describes the homology $H_\bullet(\Sigma_n, \text{Lie}(n))$ and earlier work of Hemmer and Nakano on complexity for modules over Σ_n that involves restriction to Young subgroups.

1. Introduction

Let G be a finite group and k be an algebraically closed field of characteristic p . In 1977, Alperin [1] defined the *complexity* of M , denoted by $c_G(M)$, as the rate of growth of the minimal projective resolution of M . It is, in some sense, a measure of how far M is from being projective; in particular, M is projective if and only if $c_G(M) = 0$. Alperin's definition naturally led to the theory of support variety of modules in addition to the emphasis on homological and topological methods in representation theory. With such methods one can compute the complexity of modules without explicitly describing the minimal resolution. For an introduction to complexity and support varieties we refer the reader to [3].

Let Σ_n be the symmetric group on n -letters. The representation theory and combinatorics of the symmetric group via Young tableaux have been well-studied. In [6], Hemmer and Nakano provided a combinatorial formula for the complexity of Young modules over Σ_n via the removal of horizontal p -hooks. They also determined the complexity for completely splittable irreducible modules. One of their main tools was the development of a formula for $c_{\Sigma_n}(M)$ via branching to Young subgroups.

The goal of this paper is to compute the complexity of the $k\Sigma_n$ module $\text{Lie}(n)$, which we define next. For any commutative ring R and positive integer n , let $\text{Lie}_R(x_1, x_2, \dots, x_n)$ be the free Lie algebra over R generated by x_1, x_2, \dots, x_n and let $\text{Lie}_R(n)$ be the submodule spanned by all bracket monomials containing each x_i exactly once. Then $\text{Lie}_R(n)$ is a module for the symmetric group Σ_n acting by permuting the variables.

We will be interested in $\text{Lie}_k(n) := \text{Lie}(n)$. This module arises naturally in topology, for example as the top degree homology of the configuration space of n points in the plane tensored by the sign representation. In characteristic zero, there is a beautiful description of its complex character in terms of tableaux combinatorics, see [9, Chapter 8] for a thorough treatment. Furthermore, the representation $\text{Lie}(n)$ is a direct summand of $\mathbb{Q}\Sigma_n$. In characteristic p , very little is known about the module structure of $\text{Lie}(n)$ except in special cases, for example small n or when $p^2 \nmid n$. Over an arbitrary field k , $\text{Lie}(n)$ has dimension $(n-1)!$ and is free over $k\Sigma_{n-1}$.

Received 18 March 2015; published online 26 October 2015.

2010 *Mathematics Subject Classification* 20C30 (primary), 55S12, 55P47 (secondary).

Research of the second author was supported in part by NSF grant DMS-1068783. Research of the third author was supported in part by NSF grant DMS-1402271.

Erdmann, Lim and Tam [4] stated a conjecture for $c_{\Sigma_n}(\text{Lie}(n))$. Our strategy in proving this conjecture involves first employing the Hemmer–Nakano formula for the complexity of $k\Sigma_n$ -modules via branching to Young subgroups. In Section 2, we provide a homology version of this result. This lends itself well to applying the calculations due to Arone and Kankaanrinta [2] to give estimates on the rate of growth of the homology groups in the aforementioned complexity formula.

The authors acknowledge the hospitality of the Mathematics Institute of Extended Stay America where a majority of these results were obtained.

2. Complexity: interpretations via cohomology and homology

Throughout the paper G will denote a finite group, k an algebraically closed field and kG the group algebra for G . All kG -modules considered are finite-dimensional. If M is a kG -module, then M^* will denote the dual kG -module.

Let $V_\bullet = \{V_t : t \in \mathbb{N}\}$ be a sequence of finite-dimensional vector spaces. The *rate of growth* of V_\bullet , denoted by $\gamma(V_\bullet)$, is the smallest positive integer c such that $\dim V_t \leq K \cdot t^{c-1}$ for some constant K and for all t . For example, $\gamma(V_\bullet) = 1$ if and only if the dimensions of V_t are uniformly bounded.

We define suspension $\Sigma^1 V_\bullet$ to be $(\Sigma^1 V_\bullet)_{t+1} = V_t$. In particular, the degree of $\Sigma^1 V_t$ is $t + 1$. This corresponds algebraically to the behavior of the topological suspension of a space on the level of homology. One can now easily show that $\gamma(V_\bullet) = \gamma(\Sigma^1 V_\bullet)$. Let $c = \gamma(V_\bullet)$ and $e = \gamma(\Sigma^1 V_\bullet)$. Observe that $\dim V_t \leq K \cdot t^{c-1}$ so $\dim(\Sigma^1 V_\bullet)_t = \dim V_{t-1} \leq K \cdot (t-1)^{c-1}$, thus $e \leq c$.

We have $\dim(\Sigma^1 V_\bullet)_t \leq Q \cdot t^{e-1}$ for some positive constant Q . Then

$$\dim V_t \leq Q \cdot (t+1)^{e-1} = Q \cdot t^{e-1} + Q \cdot p(t),$$

where $p(t)$ is a polynomial of degree $e - 2$ with strictly positive coefficients. Therefore, there exists $Q' > 0$ such that $p(t) \leq Q' \cdot t^{e-1}$, and $\dim V_t \leq (Q + Q') \cdot t^{e-1}$. Consequently, $c \leq e$, and $c = e$. We have proved the following lemma.

LEMMA 2.1. *Let $V_\bullet = \{V_t : t \in \mathbb{N}\}$ be a sequence of finite-dimensional vector spaces. Then for any $i \geq 0$,*

$$\gamma(V_\bullet) = \gamma(\Sigma^i V_\bullet).$$

Let M be a kG -module and let $P_\bullet \rightarrow M$ be a minimal projective resolution of M . The complexity of M , denoted by $c_G(M)$, is $\gamma(P_\bullet)$. Since kG is a self-injective algebra, $c_G(M) = 0$ if and only if M is a projective kG -module.

We begin by following [5, Section 2.4]. Suppose that $P_\bullet \rightarrow M$ is a minimal projective resolution and S is a simple kG module. Then the differentials in the complexes defining Ext and Tor vanish and one gets:

$$\text{Tor}_n^{kG}(M, S) = P_n \otimes_{kG} S \tag{2.1}$$

and

$$\text{Ext}_{kG}^n(M, S) = \text{Hom}_{kG}(P_n, S). \tag{2.2}$$

Note that in (2.1) one considers P_n as a right kG -module via the usual $p \cdot g := g^{-1}p$. The dimension of $\text{Hom}_{kG}(P_n, S)$ is the number of summands of P_n isomorphic to the projective cover of S . Using the usual adjoint associativity between Hom and tensor product, it follows

that

$$\begin{aligned} \dim_k(P_n \otimes_{kG} S) &= \dim_k \operatorname{Hom}_k(P_n \otimes_{kG} S, k) \\ &= \dim_k \operatorname{Hom}_{kG}(P_n, \operatorname{Hom}_k(S, k)) \\ &= \dim_k \operatorname{Hom}_{kG}(P_n, S^*). \end{aligned}$$

Therefore, we obtain the following relationship:

$$\dim_k \operatorname{Ext}_{kG}^n(M, S) = \dim_k \operatorname{Tor}_n^{kG}(M, S^*), \tag{2.3}$$

where S is a simple kG -module and M is an arbitrary kG -module.

Now consider the case of modules over the symmetric group Σ_n . Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vDash n$ be a composition of n . The associated Young subgroup is $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots \times \Sigma_{\lambda_l}$. We apply the results of [6], which let one interpret the complexity of a $k\Sigma_n$ module in terms of its homology or cohomology on restriction to Young subgroups.

THEOREM 2.2. *Let M be a $k\Sigma_n$ -module. The following are equivalent:*

- (a) $c_{\Sigma_n}(M)$;
- (b) $\max_{\lambda \vDash n} \{\gamma(\mathbf{H}^\bullet(\Sigma_\lambda, M))\}$;
- (c) $\max_{\lambda \vDash n} \{\gamma(\mathbf{H}_\bullet(\Sigma_\lambda, M))\}$.

Proof. The statement that (a) is equivalent to (b) is [6, Theorem 4.3.1]. To prove that (b) is equivalent to (c), first observe that for $n \geq 0$,

$$\begin{aligned} \dim \mathbf{H}^n(\Sigma_\lambda, M) &= \dim \operatorname{Ext}_{k\Sigma_\lambda}^n(k, M) \\ &= \dim \operatorname{Ext}_{k\Sigma_\lambda}^n(M^*, k) \\ &= \dim \operatorname{Tor}_n^{k\Sigma_\lambda}(M^*, k) \\ &= \dim \mathbf{H}_n(\Sigma_\lambda, M^*). \end{aligned}$$

In order to complete the proof, use the fact that $c_{\Sigma_\lambda}(M) = c_{\Sigma_\lambda}(M^*)$ (cf. [3, Proposition 5.7.3]). □

3. The computation of $c_{\Sigma_n}(\operatorname{Lie}(n))$

We first reduce the computation of the complexity of the Lie module $\operatorname{Lie}(n)$ to the computation of $\gamma(\mathbf{H}_\bullet(\Sigma_{p^r}, \operatorname{Lie}(p^r)))$ for all $r \leq t$, where p^t is the largest power of p dividing n . This will be accomplished by using the work in [2] in conjunction with Theorem 2.2(c).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vDash n$ and set $\hat{\lambda} = \gcd(\lambda_1, \lambda_2, \dots, \lambda_l)$. Moreover, let $j(\lambda)$ be the largest integer such that $p^{j(\lambda)}$ divides $\hat{\lambda}$. According to [2, p. 4],

$$\mathbf{H}_\bullet(\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots \times \Sigma_{\lambda_l}, \operatorname{Lie}(n)) \cong \bigoplus_{r=0}^{j(\lambda)} \mathbf{H}_\bullet(\Sigma_{p^r}, \operatorname{Lie}(p^r))^{\oplus C_r}, \tag{3.1}$$

where $C_r \geq 1$ depends only on p, λ and n . In particular, the value of the various C_r does not change the rate of growth. We can conclude that

$$\gamma(\mathbf{H}_\bullet(\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots \times \Sigma_{\lambda_l}, \operatorname{Lie}(n))) = \max_{0 \leq r \leq j(\lambda)} \gamma(\mathbf{H}_\bullet(\Sigma_{p^r}, \operatorname{Lie}(p^r))). \tag{3.2}$$

Finally, one can apply Theorem 2.2 with (3.2) to deduce the following result.

THEOREM 3.1. *Let t be the largest positive integer such that $p^t \mid n$. Then*

$$c_{\Sigma_n}(\operatorname{Lie}(n)) = \max_{0 \leq r \leq t} \gamma(\mathbf{H}_\bullet(\Sigma_{p^r}, \operatorname{Lie}(p^r))).$$

Let

$$M_r = H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r)) = \text{Tor}_\bullet^{k\Sigma_{p^r}}(k, \text{Lie}(p^r)).$$

Arone and Kankaanrinta gives a basis for the $(r+1)$ st suspension $\Sigma^{1+r}M_r$ in terms of ‘completely inadmissible Dyer–Lashof words of length r ’. Their results are summarized in Theorem 3.2.

THEOREM 3.2 [2, Theorem 3.2]. *The following elements constitute a basis for $\Sigma^{1+r}M_r$: if $p > 2$*

$$\{\beta^{\epsilon_1}Q^{s_1} \cdots \beta^{\epsilon_r}Q^{s_r}u \mid s_r \geq 1, s_j > ps_{j+1} - \epsilon_{j+1} \forall 1 \leq j < r\},$$

if $p = 2$

$$\{Q^{s_1} \cdots Q^{s_r}u \mid s_r \geq 1, s_j > 2s_{j+1} \forall 1 \leq j < r\}.$$

Here u is of dimension 1, the $Q^{s_j}s$ are Dyer–Lashof operations and the β s are the homology Bocksteins. Thus Q^s increases dimension by s if $p = 2$ and by $2s(p-1)$ if $p > 2$, and β decreases dimension by 1.

As a special case, we remark that the basis element $Q^{s_1}Q^{s_2} \cdots Q^{s_r}u$ lies in degree $2(p-1)(s_1 + s_2 + \cdots + s_r)$ for p odd and $(s_1 + s_2 + \cdots + s_r)$ for $p = 2$.

REMARK 3.3. The operations in Theorem 3.2 are not required to satisfy the Adem relations. For example, $Q^4Q^1(u)$ vanishes by the Adem relations but is non-zero in the current setting of the homology.

The results above are now employed to compute $\gamma(H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r)))$. We would like to thank Nick Kuhn for acquainting us with the long history of the Lie module in the topology literature. For example, work on this Poincaré series of $H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))$ dates back to Mitchell and Priddy [8], and also can be found in more recent work of Kuhn [7].

THEOREM 3.4. *For all $r \geq 0$, $\gamma(H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))) = r$.*

Proof. As observed in Lemma 2.1, the rate of growth does not change by taking suspensions so it suffices to look at the rate of growth of $\Sigma^{1+r}M_r$, where a basis is given in Theorem 3.2.

We first prove that $\gamma(H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))) \leq r$. We do this by embedding the basis of $H_m(\Sigma_{p^r}, \text{Lie}(p^r))$, for each m , in a sequence of larger vector spaces that have rate of growth r .

For $p = 2$, using Theorem 3.2, we see that the number of basis elements is at most the number of monomials of degree m in the various Q s. This is bounded by the number of compositions of m of the form $\mu = (s_1, s_2, \dots, s_r) \vDash m$. The number of such compositions has rate of growth r , because it coincides with the Krull dimension of the polynomial ring in r variables.

For p odd the number of basis elements in degree m is bounded by the number of monomials of degree m in the ring:

$$\Lambda(e_1, \dots, e_r) \otimes k[x_1, x_2, \dots, x_r]. \quad (3.3)$$

The exterior algebra generators in (3.3) have degree -1 and correspond to the Bocksteins while the polynomial generators have degree $2(p-1)$ and correspond to the Q s. It is clear that taking the tensor product with the finite-dimensional algebra $\Lambda(e_1, \dots, e_r)$ does not change the rate of growth, nor does shifting the generators x_i from degree 1 to degree $2(p-1)$. So the rate of growth is again bounded above by r , the Krull dimension of the polynomial ring. Therefore, in the p odd case, one also has $\gamma(H_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))) \leq r$.

Next we show that $\gamma(\mathbf{H}_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))) \geq r$ by constructing a sufficiently large number of basis vectors in each dimension. Fix p and r and let $x \in \mathbb{N}$. Define

$$i = 2(p-1)(p^{r-1} + 2p^{r-2} + 3p^{r-3} + \dots + (r-1)p + r-1)x. \tag{3.4}$$

One can describe a subset of the basis for $\mathbf{H}_i(\Sigma_{p^r}, \text{Lie}(p^r))$ that has x^{r-1} elements. Since $i = Cx$, where C is a constant, depending only on p and r , this will prove that $\gamma(\mathbf{H}_\bullet(\Sigma_{p^r}, \text{Lie}(p^r))) \geq r$ as desired.

We proceed by constructing x^{r-1} distinct basis elements that all contribute to degree i . We do not use any of the basis elements containing Bocksteins, so the construction is the same for $p = 2$ and p odd. In either case, each s_j must satisfy $s_r \geq 1$ and $s_j > ps_{j+1}$ for $1 \leq j < r$. Let s_1, s_2, \dots, s_r be defined as follows. First choose s_r so that

$$1 \leq s_r \leq x.$$

Now we need $s_{r-1} > ps_r$, so choose s_{r-1} such that

$$px + 1 \leq s_{r-1} \leq px + x.$$

Proceed in this way for $s_{r-2}, s_{r-3}, \dots, s_2$:

$$\begin{aligned} p^2x + px + 1 &\leq s_{r-2} \leq p^2x + px + x, \\ p^3x + p^2x + px + 1 &\leq s_{r-3} \leq p^3x + p^2x + px + x, \\ &\vdots \\ p^{r-2}x + p^{r-3}x + \dots + px + 1 &\leq s_2 \leq p^{r-2}x + p^{r-3}x + \dots + px + x \end{aligned}$$

Observe that there were x choices for each of s_r, s_{r-1}, \dots, s_2 . Finally, choose

$$\begin{aligned} s_1 := &(p^{r-1}x + p^{r-2}x + \dots + px + x) + (x - s_r) + (px + x - s_{r-1}) \\ &+ (p^2x + px + x - s_{r-2}) + \dots + (p^{r-2}x + p^{r-3}x + \dots + px + x - s_2). \end{aligned} \tag{3.5}$$

Observe that $s_1 > ps_2$ so $\{s_1, s_2, \dots, s_r\}$ is an allowable sequence. From (3.5), it is clear that

$$s_1 + s_2 + \dots + s_r = (p^{r-1} + 2p^{r-2} + 3p^{r-3} + \dots + (r-1)p + (r-1))x. \tag{3.6}$$

Since there were x choices for each of $s_2, s_3, s_4, \dots, s_r$ and then s_1 is determined, so we have x^{r-1} distinct basis elements. From (3.6) and the remark after Theorem 3.2, for p odd these basis elements all occur in degree i and for $p = 2$ they occur in degree $i/2$. In either case, i is just a linear function of x and the dimension of \mathbf{H}_i is at least x^{r-1} , giving the result. \square

By combining Theorems 3.1 and 3.4, the complexity of $\text{Lie}(n)$ can be computed for any n . This proves the Erdmann–Lim–Tan Conjecture [4].

COROLLARY 3.5. *For all $n \in \mathbb{N}$, $c_{\Sigma_n}(\text{Lie}(n)) = t$, where $p^t \mid n$ and $p^{t+1} \nmid n$.*

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