Lecture 9

Review

1. Suppose U, V are kG-modules. Then so is $U \otimes V = U \otimes_k V$ with action $g(u \otimes v) = gu \otimes gv$, extended linearly to kG.

2. $\mathrm{Hom}_k(U, V) \cong U^* \otimes_k V$

   $\mathrm{Hom}_{kG}(U \otimes V, W) \cong \mathrm{Hom}_{kG}(U, V^* \otimes W)$

3. P free (projective) $\Rightarrow$ $U \otimes_k P$ is free (projective)

4. $\otimes_k$ is exact, given $0 \to U \to M \to V \to 0$ and $N \in kG$-mod we get $0 \to U \otimes N \to M \otimes N \to V \otimes N \to 0$.

Induction & Restriction

Def $H \leq G$, $U \in kG$-mod then $U_H = U\downarrow_H = \mathrm{Res}^G_H U$ is <u>restriction</u> of U to H.

Problem Suppose $H \leq G$, V an H-module. Produce a corresponding G-module. Note: Very rarely is V the restriction of some kG-module.

Setup: Let $B \subset A$ be a subalgebra, and $V \in B$-mod.

Def The <u>induced module</u> $\mathrm{Ind}_B^A V = V^A \mathrel{\hat{=}} {}_A A_B \otimes_B V$.

Rmk For all we know this could be zero!

Def The coinduced module $CoInd_B^A V \cong Hom_B({}_B A_A, V)$
with action:
$$(a\psi)(a') = \psi(a'a)$$

General Properties

1. $Ind_B^A - = A_B \otimes_B -$ is right exact
2. Coind is left exact
3. $Ind_B^A -$ is left adjoint to restriction, i.e. $M \in A\text{-mod}, N \in B\text{mod}$

$$Hom_A(Ind_B^A N, M) \cong Hom_B(N, Res_B M)$$

4. Coind is right adjoint to restriction, i.e.

$$Hom_A(M, CoInd_B^A N) \cong Hom_B(Res_B M, N)$$

Special Case: $H \leq G$, $B \cong kH \leq A \cong kG$

Recall 1. $B_B \otimes_B U \cong U$

2. $kG\downarrow_H \cong kH \oplus \cdots \oplus kH$

COR Let $H \leq G$ be finite. Then Ind_H^G is an exact functor and is $\cong$ to CoInd.

Suppose $G/H = \{e, g_2, g_3, \dots, g_n\}$ coset reps

$kG \otimes_{kH} U$ spanned by $g \otimes_H U$
spanned by $g_i h \otimes_H u = g_i \otimes u$

Thm $\mathrm{Ind}_H^G U \cong \bigoplus_{s \in G/H} s \otimes U$, $\dim U^G = [G:H] \dim U$

Rmk 1. Action is $gs = \tilde{s} h$ $\quad s\cup$ $\quad g(s \otimes u) = gs \otimes u$
$= \tilde{s} h \otimes u$
$= \tilde{s} \otimes hu \in \tilde{s} \otimes U$

2. One copy of U for each coset, action of g permutes around copies
3. Thus ~~dim U^G =~~ $(U^G)_H \cong U \oplus \underline{\cdots}$

Example $\mathrm{Ind}_H^G R$ is just the permutation representation on the cosets of H.

Example
Let U be the $k\Sigma_3$-module $(12) \to \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ $(23) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$(123) \to \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\Sigma_4 = \Sigma_3 \cup (14)\Sigma_3 \cup (24)\Sigma_3 \cup (34)\Sigma_3$

$(12)e = e(12)$
$(12)(14) = (142) = (24)(12)$
$(12)(24) = (124) = (14)(12)$
$(12)(34) = (34)(12)$

$$(12) \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$(1234)\,e = (14)(123)$

$(1234)(14) = (234) = (24)(23)$

$(1234)(24) = (12)(34) = (34)(12)$

$(1234)(34) = (123) = e\cdot(123)$

$$(1234) \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Omnibus Lemma V_1, V_2, V RH modules, U a RG-module.

1 If V is free (projective) then V^G is free (projective).

2 $(V_1 \oplus V_2)^G \cong V_1^G \oplus V_2^G$

3 $(V^*)^G \cong (V^G)^*$

4 If $L \le H \le G$, W an L-module, then $(W^H)^G \cong W^G$

5 $U \otimes V^G \cong (U_H \otimes V)^G$

Proof

2. $RG \otimes_H (V_1 \oplus V_2) \cong (RG \otimes_H V_1) \oplus (RG \otimes_H V_2)$

1) $RG \otimes_H RH \cong \oplus\, RG$, now use 2.

3. $RG \otimes_{RH} V^* \cong (RG^* \otimes_H V)^* \cong (RG \otimes_H V)^*$

4. ~~RH~~ $RG \otimes_H RH \otimes_L$ ~~RL~~ $W \cong RG \otimes_L W$

5. $U \otimes RG \otimes_H V \cong$ ~~U⊗V⊗~~ $RG \otimes_H (U \otimes V)$

A universal property of induced modules:

$\mathrm{Ind}_H^G V \cong e\otimes V \oplus g\otimes V \oplus \cdots$

Suppose $\psi: e\otimes V \to U$ is a kH-homomorphism, $U \in \mathrm{mod}\, kG$.

Define $\bar{\psi}(g\otimes v) = g\psi(e\otimes v)$, check $\bar{\psi}$ is a kG-module map $V^G \to U$.

Def (Alperin) $H \le G$. A kG-module U is <u>relatively H-free</u> if $\exists$ a kH submodule X of U such that any kH homomorphism $\psi: X \to M$ extends uniquely to a kG-hom $U \to M$.

COR V^G is relatively H free

Rmk Free $\longleftrightarrow$ Relatively $\{e\}$ free

Thm Suppose U is a kG-module, generated by a kH submodule X and $\dim U = |G:H| \cdot \dim X$. Then $U \cong X^G$.

Rmk Alperin uses this "test" in many proofs.

<u>**Mackey Thm**</u> Let $L, H \le G$, $V \in kH$ mod.

$$(V^G)_L \cong \bigoplus_{s \in L\backslash G/H} \left((s\otimes V)_{L\cap sHs^{-1}} \right)^L.$$