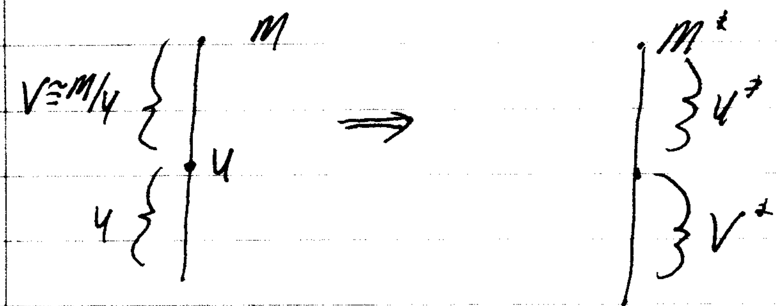
Lecture 8

Review

- U a left kG-module, U^* is also via $(g\psi)(u) = \psi(g^{-1}u)$
- $0 \to U \to M \to V \to 0$ gives $0 \to V^* \to M^* \to U^* \to 0$ so



- We proved $kG \cong kG^*$, and as a corollary that any projective kG-module is injective, and vice-versa. In particular PIM's have simple socle.

Symmetric Algebras

Def: Let A be a f.d. algebra$/k$. Say A is Frobenius if $\exists$ a linear map $\lambda: A \to k$ such that $\ker \lambda$ contains no nonzero left or right ideals

Def If further $\lambda(ab) = \lambda(ba)$ $\forall a,b \in A$, say A is a symmetric algebra.

Rmk Define a bilinear form by $(a,b) = \lambda(ab)$. Notice $(ab,c) = (a,bc)$. Having a nondegenerate bilinear form of this form is equiv. to Frobenius. If form is symmetric then so is A.

Def A is <u>self-injective</u> if ${}_AA$ is an injective module (equivalently if projective $\longleftrightarrow$ injective in A-mod)

Thm A Frobenius $\Rightarrow$ A self-injective.
Proof Mimic our proof that $kG \cong (kG)^*$

Example kG is a symmetric algebra, define $\lambda(\sum_{g\in G} a_g g) = a_e$

Lemma Let $e = e^2 \in A$ and M any A-module. (Rmk: Ae is an A-submodule of A) Then:
$$eM \cong \mathrm{Hom}_A(Ae, M) \quad \text{as vector spaces.}$$

Proof $LHS \to RHS \quad em \longrightarrow (ae \to aem)$
$RHS \to LHS \quad \alpha \longrightarrow \alpha(e) = e\alpha(e) \in eM$
Check //

Rmk Note $A \cong \mathrm{Hom}_A(A,A) = \mathrm{End}_A(A)$ as vector spaces but opposite algebras.

Thm Suppose P is a PIM for a symmetric algebra A. Then $P/\mathrm{rad}\,P \cong \mathrm{soc}\,P$.

Proof Let $A \cong P \oplus R$ and $1 = e + e'$. Check $e^2 = e$ and $P = Ae$.

Thus $\mathrm{soc}(P) = \mathrm{soc}(P)e$ is a left ideal in A. Let λ be as in the def. of symmetric, so $\mathrm{soc}(P) \not\subseteq \ker\lambda$. Thus $\exists x \in \mathrm{Soc}(P)$ with $\lambda(xe) \neq 0$ so $\lambda(ex) \neq 0$ so $e\,\mathrm{Soc}(P) \neq 0$ so $\mathrm{Hom}_A(P, \mathrm{Soc}(P)) \neq 0$. Thus $P/\mathrm{rad}\,P \leq \mathrm{soc}(P)$ but we know $\mathrm{soc}(P)$ is simple.

Tensor Products

Easiest Case: U, V vector spaces bases $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_m\}$,
$U \otimes_k V$ is vector space w/ basis $\{u_i \otimes v_j\}$

Def $U \otimes V$ is a kG-module via $g(u \otimes v) = gu \otimes gv$.

WARNING: Not true that $a(u \otimes v) = au \otimes av$ $\forall a \in kG$.
In general cannot take $\otimes$ of two R-modules and get an R-module.

Rmk 1. Easy to see $\otimes_k$ is commutative, associative.
2. For arbitrary A-modules, no $\otimes$, need a Hopf-algebra structure.

Thm Let V be faithful kG-module, and P an indec. projective.
Then $\exists$ some n so $V \otimes V \otimes \cdots \otimes V \cong P \oplus Q$.

Proof Some nontrivial Galois Theory.

Recall $\mathrm{Hom}_k(U, V)$ is a kG-module via $(g\varphi)(u) = g\varphi(g^{-1}u)$.

Observation: $\varphi \in \mathrm{Hom}_{kG}(U, V) \iff \varphi(gu) = g\varphi(u)$ $\forall g, u$
$\iff g^{-1}\varphi(gu) = \varphi(u)$ $\forall g, u$
$\iff g\varphi = \varphi$

Thus $\mathrm{Hom}_{kG}(U, V)$ is the set of fixed points of G in $\mathrm{Hom}_k(U, V)$.

Relations between $\otimes$ and duals

Thm

1. $\mathrm{Hom}_k(U,V) \cong U^* \otimes V$ as kG-modules

2. $\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(U, V^* \otimes W)$ as kG-modules

COR of 2 $\mathrm{Hom}_{kG}(U \otimes V, W) \cong \mathrm{Hom}_{kG}(U, V^* \otimes W)$ as vector spaces

Proof As vector spaces the $\cong$ in 1 is easy

$$U^* \otimes V \xrightarrow{\tau} \mathrm{Hom}_k(U,V)$$
$$\varphi \otimes v \xrightarrow{\tau} (u \to \varphi(u) \cdot v) \quad \text{natural } \cong \text{ of vector spaces}$$

Now $g(\varphi \otimes v) = g\varphi \otimes gv \to (u \to (g\varphi)(u) \cdot gv)$

$$(g\varphi)(u) = g\varphi(g^{-1}u)$$

$$g \cdot \tau(\varphi \otimes v)(u) = g\tau(\varphi \otimes v)(g^{-1}u)$$
$$= g \cdot \varphi(g^{-1}u) \cdot gv$$

Proof of 2

$$\mathrm{Hom}_k(U \otimes V, W) \cong (U \otimes V)^* \otimes W \quad \text{by 1}$$
$$\cong U^* \otimes V^* \otimes W$$
$$\cong \mathrm{Hom}_k(U, V^* \otimes W) \quad \text{by 1}$$

<u>Key Thm</u> Let V be an arbitrary kG-module and P projective. Then $V \otimes P$ is projective.

<u>Proof</u> Since $V \otimes P \mid V \otimes F$, ETS $V \otimes kG$ is free.

<u>idea</u> Let $V = \langle v_1, \ldots, v_n \rangle$. Let F_i be submodule gen by $v_i \otimes 1$ so $F_i \cong kG$.

Check $V \otimes kG \cong F_1 \oplus \ldots \oplus F_n$

<u>In Alperin</u>

1. V_p is p-dimensional $SL(2,p)$ module.
   - Sylow is $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$, V_p uniserial so projective for Sylow so proj.

2. $V_2 \otimes V_p$ is projective $\cong P_{p-1}$

Keep going to get all projective covers!