Lecture 5

Review Last Class: $G = SL_2(p)$, $V = V_2$ is 2-dimensional natural module.

Let $V_i = S^{i-1}(V)$ i-1st symmetric power, i-dimensional!
$V_1 \cong k$ trivial module

Thm Let char $k = p$. Then $\{V_1, V_2, \ldots, V_p\}$ gives a complete set of nonisomorphic simple G-modules

Clifford Theorem Let $H \trianglelefteq G$ and S a simple kG-module. Then $\mathrm{Res}^G_H S = S_H$ is semisimple.

Lemma Let V be a kH submodule of S_H. Then so is $gV = \{gv \mid g \in G, v \in V\}$.

Proof $hgv = gg^{-1}hgv = g\tilde{h}v \in gV$. //

Proof of Thm
Let T be a simple kH submodule of S_H. Then

$$\sum_{g \in G} gT \text{ is clearly a } G\text{-submodule,}$$

and hence all of S. Thus S_H is a sum of simple H-modules, and hence is semisimple as an H-module. //

Rmk 1 Much more can be said, "Clifford Theory"
Rmk 2 $G = \Sigma_n$, $H = \Sigma_{n-1} \leq G$, S a simple Σ_n-module. Then $\mathrm{Res}_{\Sigma_{n-1}} S$ is poorly understood and very from from being semisimple.

Recall U is indecomposable if $U \not\cong M \oplus N$.

Recall A ring is local if it has a unique maximal (left) ideal.

Fact A k-algebra A is local if and only if $A/\mathrm{rad}\,A \cong k$,
i.e. A has only one simple module which is 1-dim

Prop An A-module U is indecomposable iff $\mathrm{End}_A(U)$ is local.

Proof First we get an alternate description of a local algebra.

Lemma A is local iff every nonzero element is nilpotent or invertible.
Pf Suppose every nonzero elt is inv or nilp. Then true of $A/\mathrm{rad}\,A$ also, so $A/\mathrm{rad}\,A \cong k$, else $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is neither.

Conversely suppose A is local, so $A/\mathrm{rad}\,A \cong k$
If $a \in \mathrm{rad}\,A$ then a is nilpotent.
Else $a = \lambda \cdot 1 + r$, $\lambda \neq 0$ and $r \in \mathrm{rad}\,A$, so r is nilpotent.

Then $\frac{1}{\lambda}\left(1 - \frac{r}{\lambda} + \frac{r^2}{\lambda^2} - \frac{r^3}{\lambda^3} + \dots\right)$ is an inverse for a.

$$\left(\text{Recall } \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \dots\right) \quad //$$

Proof of Prop If $U = U_1 \oplus U_2$ then projections are clearly not inv or nilpotent, so $\mathrm{End}\,U$ is not local.

Conversely suppose $\mathrm{End}\,U$ is not local and choose $\varphi \in \mathrm{End}\,U$ neither invertible nor nilpotent.

Def Let U_λ be gen λ-e-space of ρ, so $U_\lambda = \{u \mid (\rho - \lambda I)^n u = 0 \text{ some } n\}$

Check U_λ is a submodule!
ρ invertible $\leftrightarrow U_0 = 0$
ρ nilpotent $\leftrightarrow U = U_0$ Thus $U \cong U_0 \oplus U_\lambda$'s //

Krull-Schmidt Thm Suppose $M \cong U_1 \oplus \cdots \oplus U_s$
$\cong V_1 \oplus \cdots \oplus V_t$ and $\{U_i\}, \{V_j\}$
are indecomposable. Then $s = t$ and $\exists\, \sigma \in \Sigma_t$ so $U_i \cong V_{\sigma(i)}$.

Cor (Cancellation) If $M \oplus U \cong M \oplus V$ then $U \cong V$.

Rmk We are still assuming everything is fin dim.

Warning Classifying indecomposable kG-modules is usually "impossible".

One Example $G = \langle g \mid g^n = e \rangle$, U an indecomposable G-module
By arg above U must be a Jordan block $J_r(\lambda)$ for
some n^{th} root of unity. If $n = p^a e$ with $p \nmid e$,
then λ is an e^{th} root of unity.

Conclude Must have a Jordan block size $r \leq p^a$
and choice of e eigenvalues So
Exactly n $\cong$ classes!

Exercise These modules are all uniserial, with
same simple module occurring.

Representation Type

<u>Def</u> An algebra A has <u>finite representation type</u> if there are only finitely many indec A-modules, up to $\cong$.
Otherwise say A has infinite type.

<u>Infinite type:</u>

Tame: Roughly finitely many 1-parameter families of $\cong$ classes in each dimension

Wild: Module category contains that of $k\langle X, Y\rangle$ No hope of classifying

<u>Thm</u> (Drozd, Crawley-Boevey) These 3 are disjoint, every f.d. algebra is one of them and mutually exclusive.

<u>Rmk</u>

<u>Thm</u> Theory of f.d. $k\langle X, Y\rangle$ modules is undecidable.

<u>Thm</u> (Bondarenko, Drozd '1977) char $k = p$, G finite.

i) kG has finite type iff G has cyclic Sylow p subgroups

ii) kG has tame type iff $p = 2$ and Sylow two subgroups are $\cong V, D_{2n}, SD_{2n}$ or gen. quaternion

iii) Else, kG is wild

Rmk Reps of Klein 4-group in Benson Sect 4.3 $\approx$ 4 pages
Reps of Dihedrals $\approx$ 8 pages in Benson

Special Case Alperin considers $C_p \times C_p$, which is wild for p odd!

Let $G = \langle x, y \mid x^p = y^p = e,\ xy = yx \rangle$
Let $V_n = \langle v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m \rangle$

$$X: \begin{array}{l} v_i \to w_i \\ w_i \to 0 \end{array} \qquad Y: \begin{array}{l} v_i \to w_{i+1} \\ v_n \to 0 \\ w_j \to 0 \end{array} \qquad 1 \le i \le n-1$$

$$\begin{tikzcd} v_1 \arrow[d, "x"] \arrow[dr, "y"] & v_2 \arrow[d, "x"] \arrow[dr] & \cdots & v_n \arrow[d, "x"] \\ w_1 & w_2 & & w_n \end{tikzcd}$$

So $x^2 = y^2 = xy = yx = 0$

Thus $(I+x)^p = (I+y)^p = I$ so kG acts by $xv = (I+x)v$
$yv = (I+y)v$

$$x \to \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \qquad y \to \begin{pmatrix} I & 0 \\ \begin{smallmatrix} 0 & & \\ 1 & \ddots & \\ 0 & \ddots 1 & 0 \end{smallmatrix} & I \end{pmatrix}$$

Exercise $\mathrm{End}(V_n)$ is all linear maps commuting w/ x & y which is

$$\begin{pmatrix} A & 0 \\ C & A \end{pmatrix} \qquad A = \begin{pmatrix} \lambda_1 & & & \\ \lambda_2 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \lambda_n & \cdots & \lambda_2 & \lambda_1 \end{pmatrix}$$

Check rad is codim 1.

Thus V_n is indecomposable.