Lecture 23

Review B a block with p-core $b = (b_1, b_2, \ldots, b_r)$ and weight w.

Put ~~B~~ b on an abacus with $r + pw$ beads. Let Γ_i be the # of beads on runner i.

Suppose $\Gamma_i = \Gamma_{i-1} + K$, $K \geq w$, for some $i \geq 2$.
Let $\overline{B}$ be block of Σ_{n-k} having p-core obtained from swapping runners $i-1, i$ on abacus of b.

Thm B and $\overline{B}$ are Morita Equivalent.

Notation $\overline{b}$ is p-core corr. to $\overline{B}$
$e = e_B$ central idempotent corr to B
$\overline{e} = e_{\overline{B}}$ central idempotent corr to $\overline{B}$.

Restriction / Induction

Recall In char 0,
$$\mathrm{Res}_{\Sigma_{n-1}} S^\lambda \cong \bigoplus_{\substack{\text{remov} \\ \text{nodes } A}} S^{\lambda_A}$$

$$\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} S^\lambda = \bigoplus_{\substack{\text{addable} \\ \text{nodes } B}} S^{\lambda^B}$$

Thm In char p, $\mathrm{Res}_{\Sigma_{n-1}} S^\lambda$ has a filtration with factors $\cong$ to the various S^{λ_A}. Similarly for $\mathrm{Ind}\, S^\lambda$.

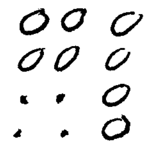
Suppose $H<G$, b a block of kH, U a kG-module

$\mathrm{Res}_b\, U$ denotes the summand of U in block b

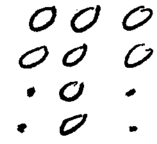
Similarly $\mathrm{Ind}_H^B V$ denotes summand in block B

Moreover, given a block B and a B module U, U is naturally a kG-module by letting other blocks act as 0.

<u>Example</u> $p=3$ $w=2$ $b=(4,2)$



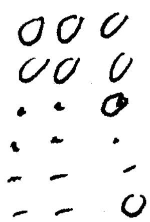
b



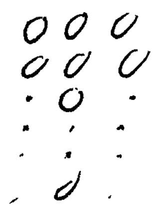
$\bar{b} =$  $=(3,1)$

<u>Prop</u> Swapping columns $i-1, i$ of abacus gives a bijection between partitions $\lambda \in B$ and $\bar{\lambda} \in \bar{B}$.

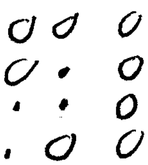
<u>Proof</u> The map $\lambda \to \bar{\lambda}$ is clearly 1-1. But 2 blocks of weight w have same # of $\mathbb{C}\Sigma_d$ simples, so onto.

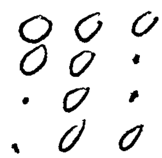


$\lambda = (10,2)$

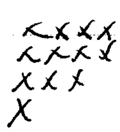
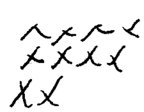


$\bar{\lambda} = (9,1)$





etc...

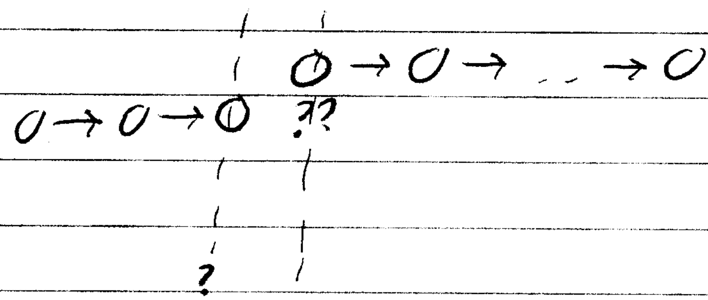
$\lambda =$  $= (4431)$ $\bar{\lambda} =$  $= (442)$

<u>Tedious Lemma</u> The map $\Phi: \lambda \to \bar{\lambda}$ preserves lexicographic (dictionary) order on partitions

<u>Slightly Less Tedious Lemma</u> The map Φ preserves p-regular/p-singular, i.e. λ is p-reg $\Leftrightarrow$ $\bar{\lambda}$ is p-reg

<u>Proof</u> We know the # of p-regular partitions in a block is also an invariant of w, so ETS λ p-singular $\Rightarrow$ $\bar{\lambda}$ p-sing.

So we have at least $p-B$ #'s in a row, i.e.



If $? \neq i-1, i$ clearly swap keeps $\bar{\lambda}$ p-singular

If $? = i-1$ then ?? is a bead since $w \leq k$, so still ok. //

DA

Branching Behavior

Let $S^\lambda \in B$. Let $\Lambda^{(1)}, \dots, \Lambda^{(p)}$ be column weights in λ, so $w = \sum \lambda_{(i)}$

Lemma If abacus for λ has bead in spot $(j, i-1)$ then also in (j, i).

Pf. Define v by $\Lambda^{(i-1)} = w - v$ so $\Lambda^{(i)} \leq v$.
Beads in col $i-1$ lie in rows $1 \to \Gamma_{i-1} + w - v$
But $\Lambda^{(i)} \leq v$ means 1^{st} $\Gamma_i - v$ rows of col i are filled

$$\Gamma_i - v = \Gamma_{i-1} + k - v \geq \Gamma_{i-1} + w - v \qquad //$$

Lemma Let $S^\lambda \in B$. Then

$$S^\lambda \downarrow_{\bar{B}} \sim k!\, S^{\bar{\lambda}}$$

$$S^{\bar{\lambda}} \uparrow^{B} \sim k!\, S^{\lambda}$$

Proof. We are removing k nodes, net effect has to be to shift k beads runner i to $i-1$, there are $k!$ ways. //

Recall $S^\lambda \propto \underset{\text{if preg}}{D^\lambda} + D^\mu$'s μ smaller.

Thm Let $D^\lambda \in B$ (so λ is p-reg)
1. $D^\lambda \downarrow_{\bar{B}} \sim k!\, D^{\bar{\lambda}}$
2. $D^{\bar{\lambda}} \uparrow^{B} \sim k!\, D^\lambda$
3. B and $\bar{B}$ have same dec-matrix
4. Same Cartan matrix

Proof Choose $\lambda_1 > \lambda_2 > \ldots > \lambda_4$ p-reg in B. $\bar{\lambda}_1 > \bar{\lambda}_2 > \ldots > \bar{\lambda}_4$ in $\bar{B}$.
True for D^{λ_1}! Now induct. //

Prop $S^{\lambda} \cdot \bar{e} \cong S^{\bar{\lambda}} \otimes_F k\Sigma_k$ as a $\Sigma_{n-k} \times \Sigma_k$ module.
$D^{\lambda} \cdot \bar{e} \cong D^{\bar{\lambda}} \otimes_F k\Sigma_k$ " "

Pf Look at actual bases.

The Morita Equivalence

$$\begin{tikzcd}
\mathrm{mod}_B \arrow[r, shift left, "G_1"] & \mathrm{mod}\, k\Sigma_n \arrow[l, shift left, "G_1'"] \arrow[r, shift left, "G_2"] & \mathrm{mod}(k\Sigma_{n-k} \times k\Sigma_k) \arrow[l, shift left, "G_2'"] \arrow[d, shift right, "G_3"'] \\
& \mathrm{mod}\,\bar{B} \arrow[r, shift right, "G_4'"'] & \mathrm{mod}\, k\Sigma_{n-k} \arrow[l, shift right, "G_4"'] \arrow[u, shift right, "G_3'"']
\end{tikzcd}$$

- G_1, G_1', G_4, G_4' as discussed early.
- G_2, G_2' restriction and induction.

For G_3 $\Sigma_{n-k} \times \Sigma_k$ modules $=$ Σ_k^{op}–Σ_{n-k} bimodule

$$G_3(M) = k \otimes_{k\Sigma_k^{op}} M \qquad \text{right } k\Sigma_{n-k} \text{ module}$$

$$G_3'(N) = \mathrm{Hom}_k({}_{\Sigma_k^{op}}k, N_{\Sigma_{n-k}})$$