Lecture 22

Review

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \vdash d$, $\lambda_t \neq 0$. For any $r \geq t$ we can represent λ on an abacus with r beads.

Thm Let $\lambda \vdash d$. Then λ has a well-defined p-core $\tilde{\lambda} \vdash d - pw$, where w is the p-weight of λ. For $\lambda, \mu \vdash d$, S^λ and S^μ are in the same block iff $\tilde{\lambda} = \tilde{\mu}$.

Recall

1. λ is uniquely determined by its p-core and p-quotient, a sequence of partitions $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(p-1)}$ of total weight w.
2. The # of irred $\mathbb{C}\Sigma_d$ modules in a block is a function only of w. Similarly for $k\Sigma_d$
3. Often use $wp + t$ beads to represent λ.
4. The Defect group of a block of weight w is $\cong$ to a Sylow p-subgroup of Σ_{pw}.
5. Adding/Removing nodes & rim p-hooks easy to represent on abacus

Def Two categories $\mathcal{C}$ & $\mathcal{D}$ are <u>equivalent</u> if $\exists$ functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{L}: \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{L}$ is naturally $\cong$ to $id_{\mathcal{D}}$ and similarly $\mathcal{L} \circ \mathcal{F} \simeq 1_{\mathcal{C}}$.

Equivalently: $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is an equivalence if for any objects $C_1, C_2 \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(C_1), \mathcal{F}(C_2))$ is a bijection and every object in $\mathcal{D}$ is $\cong$ to an object of form $\mathcal{F}(C_1)$.

Def Algebras A_1, A_2 are <u>Morita Equivalent</u> if mod A_1 and mod A_2 are equivalent.

<u>Ex</u> R and $M_n(R)$ $\quad$ R a ring w 1.

<u>Donovan Conjecture</u> Fix a p-group D. $\exists$ only finitely many block algebras, up to Morita equivalence, with defect group $\cong$ to D.

<u>Thm</u> (Scopes '91) Fix $w \geq 0$. For $d \geq 1$ There are only finitely many $k\Sigma_d$ blocks of weight w up to Morita equivalence.

Moreover..

- # of non equiv blocks is $\leq \prod_{i=1}^{p} ((i-1)(w-1)+1)$
- Every Morita type occurs for some Σ_d,

$$d \leq \frac{p^2(p-1)^2(w-1)^2}{4+wp}$$

Cartan Matrix

Recall: Decomposition matrix D has

$$\begin{array}{cc} & \text{Irr } kG \text{ modules} \\ \begin{array}{c}\text{Irr} \\ \mathcal{O}G \\ \text{modules}\end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array}$$

Def Cartan matrix C has rows and columns indexed by irreducible kG-modules (equiv indec proj modules) and C_{ij} =

$$C_{ij} = \mathrm{mult}(P_i, D_j) = \mathrm{Hom}(P_j, P_i)$$
$$= \mathrm{Hom}(P_i, P_j) = C_{ji}$$

Ex Σ_3 $p = 3$

$$\begin{array}{cc} & \begin{array}{cc} D^{3} & D^{21} \end{array} \\ \begin{array}{c} S^{3} \\ S^{21} \\ S^{13} \end{array} & \begin{pmatrix} 1 & \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

PIMs
$$\begin{array}{cc} k & \mathrm{sgn} \\ \mathrm{sgn} & k \\ k, & \mathrm{sgn} \end{array}$$

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Thm $C = D^{tr} D$.

Rmk Just as decomposition matrices of a block makes sense, so do Cartan matrices of a block.

Rmk Cartan matrix in invariant under Morita Equivalence (up to ordering)

Scopes' Idea

Fix a block B with p-core $(b_1, b_2, \ldots, b_r)$ and weight w.
Take an $r+pw$ elt Bset and repr. core on abacus.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be # of beads on each runner.

Ex. $p=3$, core $= (9,7,5,3,1^2)$

$$\begin{array}{ccccccccc} \times&\times&\times&\times&\times&\times&\times&\times&\times \\ \times&\times&\times&\times&\times&\times&\times&& \\ \times&\times&\times&\times&\times&&&& \\ \times&\times&\times&&&&&& \\ \times&&&&&&&& \\ \times&&&&&&&& \end{array}$$

weight $= 2$
$r+pw = 12$

h.l. $1, 2, 5, 8, 11, 14$

$$\begin{array}{ccc} \circ&\circ&\circ \\ \circ&\circ&\circ \\ \cdot&\circ&\circ \\ \cdot&\cdot&\circ \\ \cdot&\cdot&\circ \\ \cdot&\cdot&\circ \\ \cdot&\cdot&\circ \end{array}$$

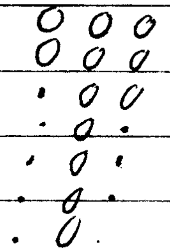
$\Gamma_1 = 2 \quad \Gamma_2 = 3 \quad \Gamma_3 = 7$

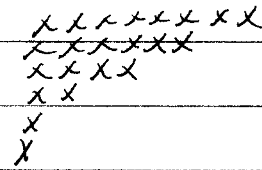
Thm (Scopes)
Suppose for some $i \geq 2$, $\Gamma_i = \Gamma_{i-1} + k$ with $k \geq w$.
Let $\overline{B}$ be block of Σ_{n-k} with core $\overline{b}$, where $\overline{b}$ has abacus obtained by swapping runners $i-1$ and i.

Then B and $\overline{B}$ are Morita equivalent.

Ex B block of Σ_{32} w/ core $(9,7,5,3,1^2)$

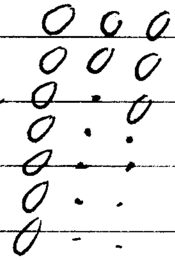
$\bar{B}$

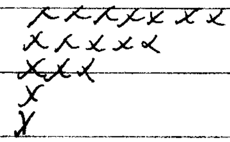




block of Σ_{28} w/ core $(8,6,4,2,1^2)$

Repeat!



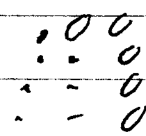


block of Σ_{23} core $(7,5,3,1^2)$

Proof of Thm Assume Morita equivalences as above. Suppose B, B' blocks of weight w of Σ_N, Σ_M w/ $N > M$. Suppose $\exists$ $B_0 = B', B_1, \dots, B_L = B$ so each pair Morita equiv as above. Each family has a unique block ancestor of all in block.

Take such a block. Write out its 1^{st} column hook lengths. Since it's a p-core, 1^{st} runner is empty.

EX. h.l. 1, 2, 5, 8, 11



Keep going

Let θ_i be # of beads on each runner.

$\theta_1 = 0$, $\theta_i \le \theta_{i-1} + w - 1$ ..

Moreover Largest 1^{st} col hook length is at most $p(p-1)(w-1)$