Lecture 21

Review $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ $\quad \lambda_t \neq 0$.

First column hook lengths $\quad h_{i1}^{\lambda} = \lambda_i - i + t$

Def Let $r \geq t$. Then $(\lambda_1 - 1 + r, \lambda_2 - 2 + r, \dots, \lambda_r - r + r)$ is called a <u>sequence of β-numbers</u> for λ.

Rmk Easy to recover λ from a sequence of β #'s.

Ex $\quad \lambda = (6,2,2,1)$ hook lengths $(r=4)$ $\quad (9,4,3,1)$

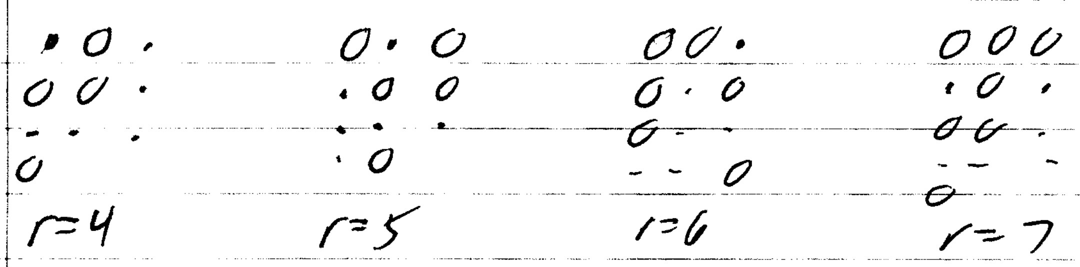
$r=5 \quad (10,5,4,2,0)$

$r=6 \quad (11,6,5,3,1,0)$

$r=7 \quad (12,7,6,4,2,1,0)$

Abacus Given λ as above, and $r \geq t$, represent λ on an abacus with r beads by putting a bead at each spot in seq of β #'s

Ex $\quad \lambda = (6,2,2,1)$ $\quad p = 3$



Rmk Recover 1^{st} col hook lengths by counting 1^{st} gap as "0" then count from there

Prop Removing a rim p-hook from λ corr to replacing a β-number with $\beta_i - p$, i.e. sliding a bead up one spot.

Adding (Removing) an addable (removeable) node corresponds to sliding a bead one spot right (left) on the abacus

COR The p-core $\tilde{\lambda}$ is well defined

<u>p-content</u>

Def Fix p. The ~~[illegible]~~ <u>residue</u> of a node (i,j) is $j-i \bmod p$. The <u>p-content</u> of λ is $(c_0, c_1, \dots, c_{p-1})$ where $c_i = \#$ nodes in λ with residue i.

Ex $\lambda = (54221)$ $p=3$

0	1	2	0	1
2	0	1	2	
1	2			
0	1			
2				

p-content $(4, 5, 5)$

Thm Let $\lambda, \mu \vdash d$. Then $\tilde{\lambda} = \tilde{\mu}$ iff λ, μ have same p-content.

Proof A rim p-hook clearly contains one node of each p-residue. Thus $\tilde{\lambda} = \tilde{\mu} \Rightarrow \lambda, \mu$ same p-content.

Conversely, suppose we have abacus w/ same # of beads on each runner, corr to empty diagram. Build diagram of λ by adding one node at a time. Check node of res i corr to $\overset{i-1}{0} \to \overset{i}{\bullet}$. Thus λ, μ have same # of beads on each runner, so $\tilde{\lambda} = \tilde{\mu}$.

Def Given $\lambda \vdash d$, the <u>weight</u> w is the # of rim p-hooks removed to get $\tilde{\lambda}$. Thus $\tilde{\lambda} \vdash d - pw$.

(Murphy '83)

Thm Let R be an integral domain. Then $Z(R\Sigma_d)$ is the ring of completely symmetric polynomials in the operators $L_2, L_3, \ldots, L_n$.

Rmk Exercise to show these commute w/ all $(a, a+1)$, harder is to show we have right dimension.

COR (After some Work)
<u>Nakayama's Conjecture</u> S^λ and S^μ are in same block of $R\Sigma_d$ if & only if $\tilde{\lambda} = \tilde{\mu}$.

"Proof" Recall S^λ has G-Z basis, we know how the L_i act.

Murphy constructs explicitly the block idempotents from the JM elements.

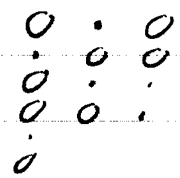
- First proof 1947 Brauer/Robinson
- We already proved λ is a p-core $\leftrightarrow$ no rim p-hooks $\leftrightarrow$ max power $p \mid \dim S^\lambda$ $\leftrightarrow S^\lambda$ in own block

Thm Let $\lambda \vdash d$ have weight w and p-core $\tilde{\lambda} \vdash d-pw$. The # of partitions of d with p-core $\tilde{\lambda}$ depends only on w and is:

$$b(w) = \sum_{(w_0, \ldots, w_{p-1})} p(w_0) \cdots p(w_{p-1})$$

sum over all tuples of nonnegative integers so $\sum w_i = w$, where $p(u) = \#$ partitions of u

Proof Given μ with $\tilde{\mu} = \tilde{\lambda}$, the $p(w_i)$ tells us how beads on runner i move.

Ex  $w = 7$

$(2,1,1), (2,1), \phi$

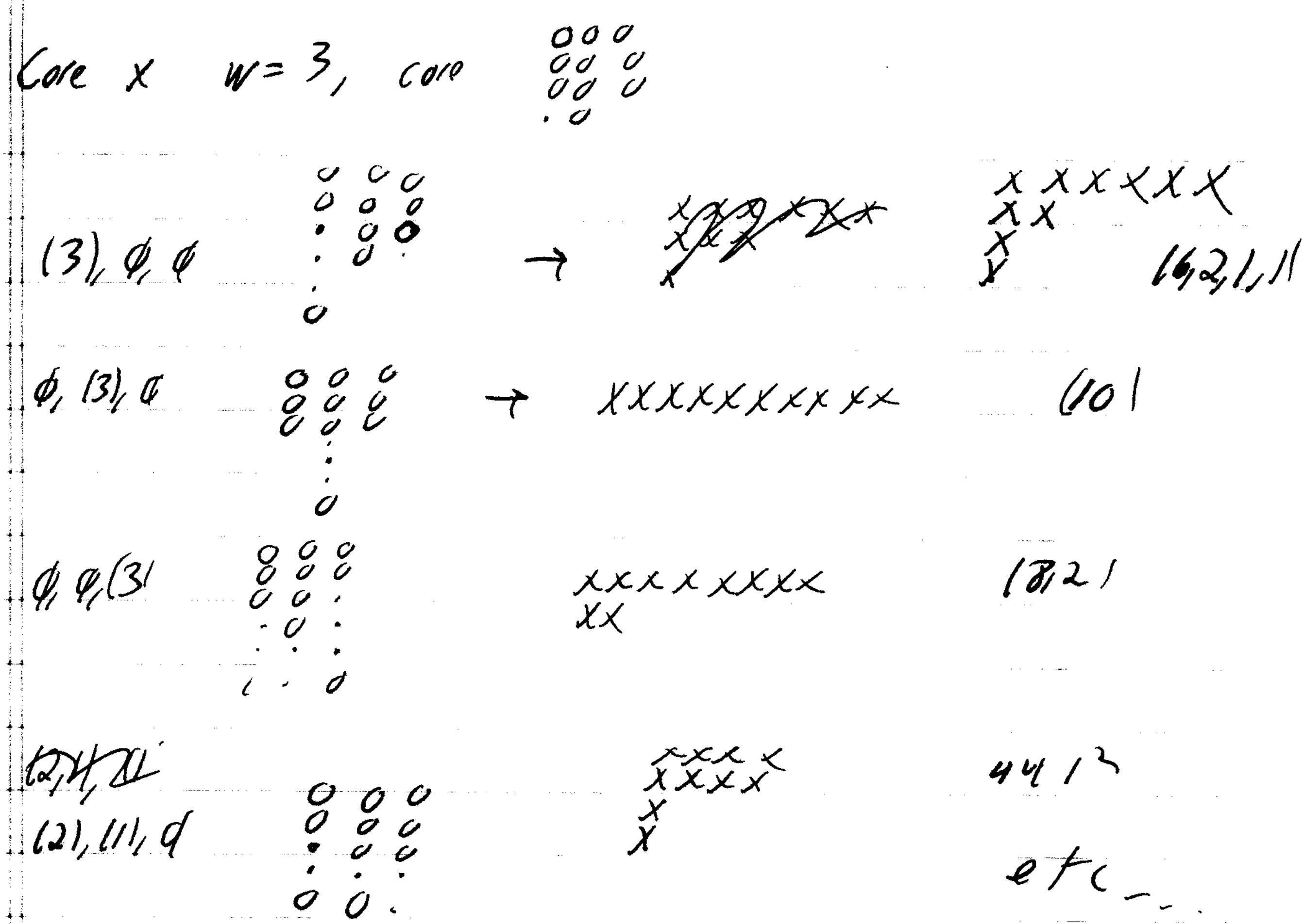
called the p-quotient. //

Rmk λ is uniquely determined by its p-core & p-quotient.

Ex $p=3$ $d=10$ p-cores $\begin{array}{l} x \end{array}$, $\begin{array}{lll} x & x & x \\ x \end{array}$, $\begin{array}{ll} x & x \\ x \\ x \end{array}$,

(Rmk no 3-cores of 7,

Thm (Granville, Ono '96) For every integer d and $t \geq 4$, $\exists$ a partition of d which is a t-core.



<u>Thm</u> The # of p-regular $\lambda \vdash d$ with core $\tau \vdash d-pw$ also depends only on w, and is

$$b'(w) = \sum p(w_1)\cdots p(w_{p-1})$$

$(w_1, \dots, w_{p-1})$ as above.

<u>Thm.</u> Let B be a block of $k\Sigma_d$ with weight w. A defect group of B is $\cong$ to a ~~subgr~~ Sylow p subgroup of Σ_{pw}.

<u>Ex</u> 1. $p=2$ no blocks of defect 2.

2. $w<p$ then defect group is elementary abelian.