

Lecture 19

- Review
- A block of kG means $kG \cong B \oplus \dots$ as algebras and B is indecomposable.
 - Any indec. kG -module is in a unique block
 - $S \sim T$ if $\exists \frac{S}{I} \circ \frac{T}{J}$ generates equiv relation of being in same block among simple modules.
 - Think of kG as a left $G \times G$ -module via $(g, g_2) \circ g = g, g g_2^{-1}$, the blocks are just indecomposable summands.

Thm As a $k[G \times G]$ module B has a vertex of the form $S(D)$ for some p -subgroup D .

Props D is unique up to conjugacy and is called the defect group of the block. If $|D| = p^d$ then d is the defect of the block.

Def The block containing the trivial module k is called the principal block.

Thm Let B be a block of kG with defect group D . Then any indecomposable kG -module in B has vertex contained in D .

Props Also true that D actually occurs as the vertex of some module in B . Thus B is semisimple iff it has defect zero.

Proof

Recall from Omnibus Lemma that $U \otimes V^G \cong (U_H \otimes V)^G$

Cor U is rel Q -projective \Rightarrow so is any $U \otimes M$.

Now suppose U is in a block B , we must show U is rel D -projective for D the defect group

Consider B as a KG -module via conjugation $g \circ B = g B g^{-1}$.

Note $B \cong \text{Res}_{S(G)}^{G \times G} B$ using obvious $\cong G \cong S(G)$

Claim B is relatively D -projective.

Pf $B \cong \text{Res}_{S(G)}^{G \times G} B$ but by def B is rel $S(D)$ proj so

$$B \mid \text{Res}_{S(G)}^{G \times G} \text{Ind}_{S(D)}^{G \times G} *$$

which, by Mackey is a \oplus of modules induced from

$$S(G) \cap (g, g) S(D) (g, g)^{-1} \quad (*)$$

But if $(g, g)(d, d)(g_1^{-1}, g_1^{-1}) \in S(G)$ then $g_1 d g_1^{-1} = g d g^{-1}$

Thus $(g, g)(d, d)(g_1^{-1}, g_1^{-1}) = (g, g_1)(d, d)(g_1^{-1}, g_1^{-1}) \in S(g) S(D) S(g_1^{-1})$

Thus all subgroups in $(*)$ are conjugate in $S(G)$ to subgroups of $S(D) //$

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Since B is relatively D -projective then so is $B \otimes U$. Thus the following is sufficient

Claim $U \mid B \otimes U$

PF Define $\pi: B \otimes U \rightarrow U$ by $\pi(B \otimes u) = Bu$

• Check balanced map

$$\pi(g \circ (B \otimes u)) = \pi(gBg^{-1} \otimes gu) = gBg^{-1}gu = gBu = g\pi(B \otimes u)$$

Since U is in the block, $\pi(e_B \otimes u) = e_B u = u$ so π is onto.

Define $i: U \rightarrow B \otimes U$ by $i(u) = e_B \otimes u$, clearly $\pi \circ i = \text{id}_U$

$$i(gu) = e_B \otimes gu$$

$$g(i(u)) = g(e_B \otimes u) = g e_B g^{-1} \otimes gu = e_B \otimes gu \text{ since } e_B \text{ is central!}$$

Thus $B \otimes U \cong U \oplus \dots$ //

Thm Suppose B is a block w/ defect group D . Then

1. Let $D \leq P \in \text{Syl}_p(G)$. Then $\exists c \in C_G(D)$ with $D = P \cap c P c^{-1}$
2. D contains every normal p -subgroup of G
3. D is largest normal p -subgroup of $N_G(D)$

Example

Claim The principal block B_0 has Sylow subgroups of G as vertex.

Proof ETS the vertex of the trivial module is $P \in \text{Syl}(G)$, we know the source is trivial. Suppose vertex is $Q < P$. Then

$$k_P \mid \text{Ind}_Q^P k_Q. \quad \text{But}$$

$$\text{Hom}_Q(k_Q, \text{Ind}_Q^P k_Q) \cong \text{Hom}_Q(k_Q, k_Q) \cong k. \quad \text{The } \text{Ind}_Q^P k_Q \text{ is indec. //}$$

The principal block is in many ways the most complicated and most important.

Brauer Correspondence

Suppose $H \leq G$ and B is a block of G and b a block of H .

Def: IF $\text{Res}_{H \times H}^{G \times G} B \cong b \oplus \dots$ for a unique B , write $B = b^G$.

Say B corresponds to b . Given b , say b^G is defined if such a B exists.

Lemma Let b as above w/ defect D .

1. If b^G is defined, then $D \leq \text{some DEF}_{GP \text{ of } b^G}$
2. If $H \leq K \leq G$ and $b^K, (b^K)^G$ and b^G all def then $b^G = (b^K)^G$
3. IF $C_G(D) \leq H$ then b^G is defined (For example $H = C_G(D)$ or $N_G(D)$)

Brauer's 1st Main Thm

Suppose $D \leq N_G(D) \leq H \leq G$, D a p -subgroup. Then

$b \rightarrow b^G$ gives a 1-1 correspondence between blocks of kG with defect D and blocks of kH with defect G .

Say $B \leftrightarrow b$ are Brauer correspondents.

Rmk 1. Respects Green corr