Lecture 18

Recall A an algebra, write $A = A_1 \oplus \cdots \oplus A_s$ a direct sum of 2-sided ideals, each of which is indecomposable as an algebra.
The A_i are called block algebras of A.

Properties

1. $1 = e_1 + \cdots + e_s$ where $\{e_i\}$ are primitive central idempotents.
   - e_i is an identity in A_i
   - $A_i = Ae_i = e_iA$, $A_iA_j = 0$ $i \neq j$.

2. Exercise: If $A = B \oplus C$ check $B = B \cap A_1 \oplus \cdots \oplus B \cap A_s$
   so the decomposition into blocks is unique

Prop Let M be an ~~indecomposable~~ A-module. Then
$$M \cong e_1M \oplus e_2M \oplus \cdots \oplus e_sM.$$
In particular if M is indecomposable then $e_iM \neq 0$ for a unique i. Say M is in the block A_i.

Rmk 1. Any A-module M has a unique decomposition $M = M_1 \oplus \cdots \oplus M_s$ where M_i, if $\neq 0$, lies in the block A_i.

2. Suppose M lies in the block A_i, this is iff <u>e_i acts as identity on M</u>. Thus every submodule and quotient module of M lies in same block. Moreover $\mathrm{Hom}_A(M,N) = 0$ if M, N in diff. blocks.

3. Think of a block as a basket that indec modules are put in

Remarks

1. If S, T are simple and $\exists$ nonsplit $\frac{S}{T}$ or $\frac{T}{S}$ then S, T are in the same block. Moreover if S, T are comp factors of any indec, then they are in same block

2. If ${}_AA = P_1 \oplus P_2 \oplus .. \oplus P_t$ then a block A_i is just the direct sum of all the P's contained in that block.

Thm Let S, T be simple A-modules TFAE:

1. S & T are in same block
2. $\exists$ a chain of simple modules $S = S_1, S_2, .., S_m = T$ such that S_i, S_{i+1} are comp factors in the same PIM.
3. $\exists$ a chain $S = T_1, T_2, .., T_n = T$ of simple modules so for each T_i, T_{i+1} there is a nonsplit extension of one by the other.

Proof Recall that given $\frac{T_i}{T_{i+1}}$ that $\exists$ $P(T_i) \to \frac{T_i}{T_{i+1}} \to 0$. Thus $3 \to 2$. Also $2 \to 1$ is clear from remark 1.

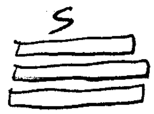
<u>$1 \to 2$</u> Suppose S, T are in same block B. Write

$$B = P_1 \oplus .. \oplus P_s \oplus Q_1 \oplus .. \oplus Q_t$$

where all the comp factors in P's are related to S by 2 and all in the Q are not. Thus $\{P_i\}$ have no composition factors in common with $\{Q_i\}$ so all maps in $End(A)$ preserve this decomp, i.e. $\oplus P_i$ is an ideal $\divideontimes$

Thus all the simples in the block are related as in 2

2→3 Let P be a PIM, it is enough to show any 2 comp factors in P are related as in 3 so enough to show all are related to top.

$P =$ 

radical layers.

ETS given a module w/ 2 radical layers, every module in bottom is related to some in top

WLOG $\dfrac{S_1 S_2 \cdots S_m}{T} = U$ $\quad \operatorname{rad} U = T$, is there an $\dfrac{S_i}{T}$?

By correspondence Thm, for each i $\exists$ a submodule $W_i \subseteq U$ with

$$U/W_i \cong \bigoplus_{k \neq i} S_k \quad \text{and} \quad U = \sum W_i .$$

Thus $\operatorname{rad} U = \operatorname{rad} A\, U = \sum \operatorname{rad} A\, W_i$, so some W_i has nontrivial radical, i.e. some $W_i = \dfrac{S_i}{T}$ //

Defect Groups

For each block B we will associate a p-subgroup $D \leq G$ controlling in many respects the rep theory of the block.

Examples

1. $\mathbb{C}G$ each block has one simple module
2. $k\Sigma_3$ $p=3$ $\quad k\Sigma_3 \cong \begin{smallmatrix} k \\ sgn \\ k \end{smallmatrix} \oplus \begin{smallmatrix} sgn \\ k \\ sgn \end{smallmatrix}$ only one block
3. Simple module is projective $\longleftrightarrow$ ~~[illegible]~~ Block is a matrix algebra
4. $SL_2(p)$ 3 blocks (p 48–50)

Notation $\delta : G \to G \times G$ $g \to (g,g)$ diagonal homom.

Rmk kG is a left $G \times G$ module via $(g_1, g_2) \circ g = g_1 g g_2^{-1}$.

* Thought of as a left $G \times G$ module, the indecomposable summands of kG are the blocks!

Thm Let B be a block. As a $k[G \times G]$ module B has a vertex of the form $\delta(D)$ for some p-subgroup D.

Def D is called a defect group of the block. If $|D| = p^d$, d is the defect of the block.

Rmk δH and δK are conjugate in $G \times G \Leftrightarrow H, K$ are conjugate.
Thus D is well-defined up to conjugacy.

Proof
Key idea is $kG \cong \mathrm{Ind}_{\delta G}^{G \times G} k$, $\underset{\text{reps}}{\text{cosets}}$ of δG are of form $(g, e) \cdot \delta G$.

Thus kG is relatively δG projective $G \times G$ module.
Thus any summand is as well. //

Thm Let B be a block of kG with defect group D. Then any indec. kG-module in B has vertex contained in D.

Rmk Also true that D actually occurs as vertex of some module in B. Thus B is semisimple iff it has defect zero.

<u>Proof</u> Recall from the omnibus lemma that $U \otimes V^G \cong (U_H \otimes V)^G$.

<u>Cor</u> U is rel Q-proj $\Rightarrow$ so is any $U \otimes M$.

Suppose U is in block B. We consider B as a kG-module via conjugation:

$$g \cdot \beta = g\beta g^{-1} \quad \forall \beta \in B$$

<u>Note</u> $B \cong \mathrm{Res}^{G\times G}_{\delta G} B$ if we use obvious $\cong G \cong \delta G$.

<u>Define</u> $\pi : B \otimes U \to U$ by $\pi(\beta \otimes u) = \beta u$.

- balanced
- $g(\beta \otimes u) = g\beta g^{-1} \otimes gu \xrightarrow{\pi} g\beta g^{-1} gu = g\beta u = g\pi(\beta\otimes u)$

Thus π is a G-module map

<u>Define</u> $i : U \to B \otimes U$ $\quad u \to e_B \otimes u$ $\quad e_B$ is the block idempotent — acting as identity on modules in B

$g(i(u)) = g(e_B \otimes u) = ge_B g^{-1} \otimes gu = e_B \otimes gu$ since e_B central.

$i(gu) = e_B \otimes gu$

Finally check that $\pi(i(u)) = \pi(e_B \otimes u) = e_B u = u$.

Thus $B \otimes U \cong U \oplus$ — so ETS $B \otimes U$ is rel D-projective

so ETS B is rel D-projective

Claim B is rel D projective

Pf Equivalently $\mathrm{Res}^{G\times G}_{\delta G} B$ is rel δD proj.

Now by def the $G\times G$ module B is rel δD proj. Thus

B is induced from some $k[\delta D]$ module.

$\mathrm{Res}^{G\times G}_{\delta G}\,\mathrm{Ind}^{G\times G}_{\delta D} -$ by Mackey is a $\oplus$ of modules induced from

$$\delta(G) \cap (g_1,g_2)\,\delta D\,(g_1,g_2)^{-1}$$

Suppose $(g_1,g_2)(d,d)(g_1,g_2)^{-1} \in \delta G$. Then
$$g_1 d g_1^{-1} = g_2 d g_2^{-1}$$

so
$$(g_1,g_2)(d,d)(g_1,g_2^{-1}) = (g_1,g_1)(d,d)(g_1,g_1)^{-1}$$
$$\in \delta(g_1)\,\delta(D)\,\delta(g_1)^{-1}$$

So all these are conjugate in $\delta(G)$ to subgroups of $\delta(D)$ //

Thm Suppose B is a block of defect group D

1. Let $D \le P \in \mathrm{Syl}_p(G)$. Then $\exists\, c \in C_G(D)$ with $D = P \cap cPc^{-1}$

2. D contains every normal p-subgroup of G.

3. $D =$ largest normal p-subgroup of $N_G(D)$.

Rmk $1 \to 2, 3$