Lecture 17

Recall $P \in Syl_p(G)$. Suppose ${}^gP \cap P = gPg^{-1} \cap P = \begin{cases} e & \text{or} \\ P \end{cases}$ $\forall g \in G$.

Let $L = N_G(P)$. Then

Thm Induction & Restriction induce a 1-1 corr. btw indec, nonproj kG and kL modules.

$$U_L \cong V \oplus Proj, \quad V^G \cong U \oplus Proj.$$

Moreover induction & restriction induce a stable equivalence.

Green Correspondence - Generalizes above

Notation Fix a p-subgroup Q and choose $N_G(Q) \leq H \leq G$.

$$\mathcal{X} = \{ X \leq G \mid X \leq Q \cap gQg^{-1} \text{ for some } g \in G, g \notin H \}$$

$$\mathcal{Y} = \{ Y \leq G \mid Y \leq H \cap gQg^{-1} \text{ for some } g \in G, g \notin H \}$$

Note that $\mathcal{X} \leq \mathcal{Y}$ and $Q \notin \mathcal{Y}$.
Note also each subgroup in $\mathcal{X}$ is proper in Q.

Lemma Assume: M is an indec kH-module that is rel. Q-projective.

1. $(M^G)_H \cong M \oplus M'$ where each summand in M' is proj. relative to a subgroup in $\mathcal{Y}$.

2. Write $M^G \cong V \oplus V'$, V indec and $M \mid V_H$. Then every summand of V' is projective relative to a subgroup in $\mathcal{X}$.

Proof 1. Choose U indec kQ module with $\boxed{U_Q^{\uparrow H} \cong M \oplus M_0}$, $\boxed{(U_Q^{G})_H \cong (M^G)_H \oplus (M_0^G)_H}$

By Mackey $\boxed{(U_Q^{\uparrow G})_H \cong U_Q^{\uparrow H} \oplus U'}$ where U' is rel $\mathcal{Y}$ projective.

Combine $\square$'s, $(M^G)_H \oplus (M_0^G)_H \cong M \oplus M_0 \oplus U'$.

Cancel M, M_0 to get $(M^G)_H \cong M \oplus M'$ with $M' \mid U'$ as desired. //

2. Let $M^G \cong V \oplus V'$ with V indec, $M \mid V_H$.

Choose V_1 a summand of V', it is rel Q-projective since V is.
Choose $Q_1 \leq Q$ a vertex and a source S_1. So

$\boxed{S_1 \mid V_1 \downarrow^G_{Q_1}}$. Choose $\boxed{M_1 \mid (V_1)_H}$ so $\boxed{S_1 \mid M_1 \downarrow^H_{Q}}$

~~Now M_1 is a rel Q-proj H-module~~
Now $M_1 \mid (V_1)_H \mid (M^G)_H$ so M_1 is rel $\mathcal{Y}$-proj by 1,
i.e. has vertex $H \cap gQg^{-1}$

Now $S_1 \mid M_1 \downarrow^H_Q$ so $S_1 \mid ((??)^{H}_{H \cap gQg^{-1}})_Q \Rightarrow S_1$ is rel $\mathcal{X}$ proj.

Thus V_1 is rel $\mathcal{X}$ proj. //

Thm (Green Correspondence)

Suppose $Q \leq G$ is a p-subgroup and $Q \trianglelefteq N_G(Q) \leq H \leq G$. Then $\exists$ a 1-1 correspondence between indecomposable kH modules with vertex Q and indecomposable kH modules with vertex Q, given as follows:

1. V indec $k\mathbf{G}$ w/ vertex Q then V_H has unique summand $f(V)$ w/ vertex Q. Remaining summands have vertex in $\mathcal{Y}$.

2. M indec kH module w/ vertex Q then M^G has a unique summand $g(M)$ with vertex Q, and other summands have vertices in $\mathcal{X}$.

3. $f(g(M)) \cong M$, $g(f(V)) \cong V$

4. Correspondence preserves being trivial source.

Remark In the TI case $\mathcal{X} = \{e\}$ by definition, so $M^G \cong g(M) \oplus \text{Proj}$.

Suppose $\mathcal{Y}$ is not just $\{e\}$. So we have a TI Sylow P and

$e \neq gPg^{-1} \cap H$, $g \notin H$.

Then $gPg^{-1} \cap H$ is a ~~Sy~~ P-subgroup of H, hence conjugate into P. Thus $\exists\, x \in H$ with

$$x(gPg^{-1} \cap H)x^{-1} \leq P$$
$$xgP(xg)^{-1} \cap H \leq P \Rightarrow xgP(xg)^{-1} \cap P \neq \{e\}$$
$$\Rightarrow xg \in N_G(P) \leq H$$
$$\Rightarrow g \in H \quad ⨳.$$

Thus $\mathcal{Y} = \{e\}$ and $V_H \cong f(V) \oplus \text{Proj}$

Proof of Green Corespondence

1. Given V indec kG-mod w/ vertex Q, source S so $V \mid S\uparrow_Q^G$. Let $S\uparrow^H \cong M \oplus M'$ where M is indec and $V \mid M^G$.

By Lemma part 1, $(M\uparrow^G)_H \cong M \oplus$ Rel $\mathcal{Y}$ proj   Want $V_H \cong M \oplus \mathcal{Y}$ proj

Now $V \mid (V_H)^G$ so V_H has a summand with vertex Q. But $V \mid M^G$ and $Q \notin \mathcal{Y}$ so $M \mid V_H$ and M has vertex Q, other summands rel $\mathcal{Y}$ proj.

Let $f(V) = M$.

2. Suppose M is indec kH module with vertex Q. Always $M \mid (M^G)_H$ so choose V indec, $V \mid M^G$ so $M \mid V_H$.

By Lemma part 2, $M^G \cong V \oplus$ rel $\mathcal{X}$ proj, so choose $g(M) = V$.

3. Start with $V \in kG$-mod, with source S.

G V

M' is rel $\mathcal{Y}$ proj

Thus $V \mid M^G$, $V \nmid M'^G$.

H $S\uparrow^H \cong M \oplus M'$

If start with M, $V \mid M^G$

Choose same V!

Q S

Source is constant.

Green Correspondence Remarks

1. It is ubiquitous in representation theory

2. There are results comparing $\mathrm{Hom}_{kG}(V_1, V_2)$ and $\mathrm{Hom}_{kH}(f(V_1), f(V_2))$
 but not directly, i.e. mod out by maps factoring
 through relatively H-projective modules

3. <u>Thm</u> Suppose U indec kG module w/ vertex Q and M the
 corresponding kH-module

 1. For a kG module W, $U | W$ iff $M | W_L$.

 2. ~~If W is~~ If W is indec and $M | W_L$ then $W \cong U$.

 i.e. 2 is a sort of converse, if you "cover" the Green corr
 then you are the Green corr.