# Lecture 16

<u>Review</u> <u>Def</u> $H \le G$. A $kG$-module $U$ is <u>relatively $H$ projective</u> if $U | V^G$ for some $H$-module $V$. In this case we always have $U | (U_H)^G$.

<u>Thm</u> Given $U$ indec $\exists$ a $p$-subgroup $Q$ and an indec. $Q$-module $S$ such that

1. $U | S^G$, $S | U_Q$
2. If $U$ is rel $H$-projective then $gQg^{-1} \le H$ for some $g \in G$
3. If $S'$ is indec $Q$-module and $U | S'^G$ then $S' \cong g(S)$ some $g \in N_G(Q)$

<u>Rmk</u> For $g \in N_G(Q)$ then $q \to gqg^{-1} \in \mathrm{Aut}\, Q$, this is an example of twisting a module by a group automorphism

<u>Def</u> $Q$ is a <u>vertex</u> of $U$, $S$ is a <u>source</u>

## <u>Properties of Vertices & Sources</u>

<u>Lemma</u> Let $U$ be an indec $kG$-module w/ vertex $Q$, $Q \le H$. Then $\exists$ an indecomposable $kH$ module $V$ satisfying any 2 of:

1. $V | U_H$  2. $U | V^G$  3. $V$ has vertex $Q$

<u>Rmk</u> Eventually can get all 3 at once

Proof $1 \Rightarrow 2$

Since $U$ is rel $H$ proj, $U | (U_H)^G$ so choose summand $V | U_H$ with $U | V^G$ //

$2 \Rightarrow 3$ Let $S$ be a source, so $U | S^G = (S^H)^G$. Choose $V | S^H$ so $U | V^G$.
Need $V$ to have vertex $Q$.

Since $V | S^H$, $V$ is rel $Q$ projective. Suppose a vertex $R \lneq Q$. Then
$\exists$ $W$ an $R$-module so $V | W^H \Rightarrow U | W^G \Rightarrow$ vertex of $U \leq W$ $\divideontimes$.
Thus $R = Q$

$G$
$|$
$H$
$|$
$Q$

$1 \Rightarrow 3$

As above, our source $S$ can be chosen so $S | U_Q$. Write

$U_H = \bigoplus$ ___ and choose a summand $V$ with $\boxed{S | V_Q}$
so $\boxed{V | U_H}$

Claim $V$ has vertex $Q$

Pf $V | U_H$ so $V | (S^G)_H$ so $V | \left(s(S)_{H \cap sQs^{-1}}\right)^H$ some $s$

Thus $V$ has vertex $R \leq H \cap sQs^{-1}$. ETS $R$ & $Q$ conjugate in $H$.

Now $V | W_R^{\uparrow H}$ and $S | V_Q$ so $S | (W^H)_Q$. Thus

$S$ is relatively $Q \cap hRh^{-1}$ for some $h \in H$. But $S$ has vertex $Q$
so $Q \cap hRh^{-1} = Q \Rightarrow Q \leq hRh^{-1}$

But $R \leq sQs^{-1}$. Thus $|R| = |Q|$ and $Q = hRh^{-1}$ //

# Trivial Intersections and the Stable Category

**Assume** Sylow subgroup $P$ is trivial intersection, i.e. $P \cap gPg^{-1}$ is always $P$ or $1$. (ex: $|P| = p$)

Let $L = N_G(P)$

**Thm** $\exists$ a 1-1 correspondence between $\cong$ classes of nonprojective indecomposable $kG$ & $kL$ modules, such that if $U \in \text{mod } kG$ corresponds to $V \in \text{mod } kL$ then

$$U_L \cong V \oplus \text{PROJ}$$
$$V^G \cong U \oplus \text{PROJ}.$$

**Proof** Apply Mackey to an indec $V$.

$$(V^G)_L \cong \bigoplus_{s \in L \backslash G / L} \left(s(V)_{L \cap sLs^{-1}}\right)^L. \quad \text{Note } P \trianglelefteq L$$

If $s \notin L$ then $P$ & $sPs^{-1}$ are unique Sylow's of $L$ and $sLs^{-1}$ so $P \cap sPs^{-1}$ is Sylow of $L \cap sLs^{-1}$ so $|L \cap sLs^{-1}|$ is coprime to $p$.

Thus $\quad (V^G)_L \cong V \oplus \text{Projective}$

Write $\quad V^G \cong U_1 \oplus U_2 \oplus \cdots \oplus U_n$. Since Sylow $\leq L$, then WLOG $U_1$ is not projective, $U_2 \to U_n$ are projective. So

$$\boxed{V^G \cong U \oplus \text{Proj} \quad \text{and} \quad U|_L \cong V \oplus \text{Proj}}$$

So we have a bijection, does every $U$ arise

For $U \in kG$-mod, $U$ is rel $L$-projective. So $U \mid V^G$ for some nonprojective $kL$ module $V$. //
(not proj)

COR Let $U_1, U_2$ nonproj indec $kG$-modules $V_1, V_2$ corr $kL$ modules.

There exists non-split $0 \to U_1 \to U \to U_2 \to 0$ if & only if
There exists nonsplit $0 \to V_1 \to V \to V_2 \to 0$.

Pf Tedious and not enlightening.

## Stable Maps

Def Let $U_1, U_2$ be $kG$-modules and $f: U_1 \to U_2$ a module homomorphism.
Say $f$ factors through a projective if $\exists$ a projective module $P$ such that

+ maps $\psi, p$

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ U_1 \arrow[ur, "\psi"] \arrow[rr, "f"] & & U_2 \end{tikzcd}$$

Check The set of such $f$ is a subspace of $\mathrm{Hom}_{kG}(U_1, U_2)$.

Def $\overline{\mathrm{Hom}}_{kG}(U_1, U_2) = \mathrm{Hom}_{kG}(U_1, U_2) / \text{subspace factoring through a proj.}$

Thm Suppose $U_1, U_2$ are nonproj indec $kG$-modules corr to $V_1, V_2$ $kL$-modules.
Then
$$\overline{\mathrm{Hom}}_{kG}(U_1, U_2) \cong \overline{\mathrm{Hom}}_{kL}(V_1, V_2)$$

<u>Proof</u> ETS $\overline{Hom}_{kG}(V_1^G, U_2) \cong \overline{Hom}_{kL}(V_1, U_{2L})$.

But $Hom_{kG}(V^G, U) \cong Hom_{kL}(V, U_L)$ so ETS this $\cong$ preserves property of factoring through a projective.

So suppose $\nu \in Hom_{kL}(V, U_L)$ and $\hat{\nu} \in Hom_{kG}(V^G, U)$

$$V^G \cong V \oplus gV \oplus \cdots$$

Then $\hat{\nu}$ extends $\nu$. So if $\hat{\nu}$ factors through proj $P$ then $\nu$ factors through $P_L$, also proj.

Suppose

$$\begin{tikzcd} & Q \arrow[dr, "\beta"] & \\ V \arrow[ur, "\alpha"] \arrow[rr, "\nu"] & & U_L \end{tikzcd}$$

$Q$ proj $kL$ module.

Check

$$\begin{tikzcd} & Q^G \arrow[dr, "\beta^G"] & \\ V^G \arrow[ur, "\alpha^G"] & & U \end{tikzcd}$$

//

<u>Example</u> $G = SL(2,p)$ $P = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right\}$ $L = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| ac = 1\right\}$

- <u>Define</u> stable module category stmod $kG$.
  - Schanuel's Lemma
  - stable equivalence $kG$ mod & $kL$ mod in TI case