

Lecture 13

- Review
- We classified irreducible $\mathbb{C}\Sigma_n$ modules $\forall n \geq 1$ at once, denoted them $\{S^\lambda \mid \lambda \vdash n\}$
 - $\dim S^\lambda = \#$ standard Young tableaux of shape λ
 - SYT of shape $\lambda \leftrightarrow$ paths $\emptyset \rightarrow \dots \rightarrow \lambda$ in Young lattice
 \leftrightarrow GZ basis vectors
 - $T \leftrightarrow$ residue sequence = weights of corresponding GZ basis vectors.
 - Branching graph is Young graph
 - Hook formula $\dim S^\lambda = n! / \prod \text{hook lengths}$
 - $S^\lambda \otimes S^{\mu} \cong S^{\nu}$
 - Explicit formulas for matrix of S_i in terms of GZ basis, shows \mathbb{Q} is a splitting field.

Problem From beginning this approach needed us to work over \mathbb{C} , we used Wedderburn to define A_n , etc.

"Classical" approach to representation theory, over any field

Def Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s) \vdash n$. The Young subgroup $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \dots \times \Sigma_{\lambda_s}$

Def The permutation module $M^\lambda \cong \text{Ind}_{\Sigma_\lambda}^{\Sigma_n} k$, $\dim M^\lambda = \frac{n!}{\prod \lambda_i!}$

Tabloids Let t be a λ -tableau. Define C_t, R_t, t_1, t_2 and tabloids.

Rank The set of λ -tabloids give a basis of M^λ .

Specht Modules

Def For t a tableau, $K_t = \sum_{\sigma \in C_t} (\text{sgn } \sigma) \sigma$, $P_t = \sum_{\sigma \in C_t} \sigma$. The

polytabloid is $e_t = K_t \{t\} = \sum_{\sigma \in C_t} \text{sgn } \sigma \{ \sigma t \}$.

Ex $t = \begin{array}{|c|c|} \hline 12 \\ \hline 34 \\ \hline \end{array}$ $e_t = \frac{12}{34} - \frac{13}{24} - \frac{23}{14} + \frac{34}{12}$

Exc $\pi e_t = e_{\pi t}$

Def The Specht module $S^\lambda \subseteq M^\lambda$ is the submodule spanned by all the e_t as t runs over the set of λ -tableaux.

Remarks

1. e_t depends on t not just $\{t\}$
2. S^λ is cyclic, gen by any e_t
3. S^λ defined over any field, not clear that dimension is independent.

Interlude on \otimes and \perp

Situation M a kG -module with a nondegenerate, symmetric, G -invariant bilinear form \langle, \rangle .

Example M^λ , set $\langle \{t_1\}, \{t_2\} \rangle = \delta_{\{t_1\}, \{t_2\}}$

Def For $U \subseteq M$, $U^\perp = \{m \in M \mid \langle u, m \rangle = 0 \ \forall u \in U\}$

Exc U a submodule $\Rightarrow U^\perp$ is a submodule

Lemma 1 Let $0 \subseteq U \subseteq V \subseteq M$. Then $V/U \cong (U^\perp/V^\perp)^\perp$

Proof Define $\theta: V \rightarrow (U^\perp/V^\perp)^\perp$ by $\theta(v) = \psi_v$ where $\psi_v(x + V^\perp) = \langle v, x \rangle$. Check:

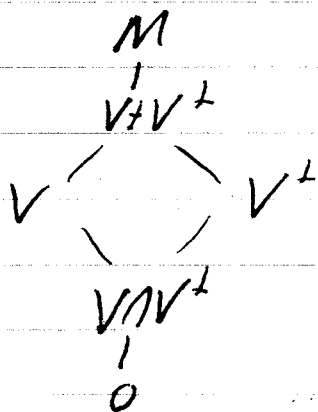
- $\text{Ker } \theta = U$
- ψ_v is well-defined, as to (use dim)
- ψ_v is a G -module map

Cor 1 $V \cong (M/V^\perp)^\perp$
 PE Set $U = 0$.

Lemma 2 $(U+V)^\perp = U^\perp \cap V^\perp$
 PE By Def

Thm $V/V \cap V^\perp$ is self dual.

Proof



$$\begin{aligned}
 V/V \cap V^\perp &\cong (V+V^\perp)/V^\perp \stackrel{\text{Thm}}{\cong} \\
 &\cong (V/(V+V^\perp)^\perp)^\perp \stackrel{\text{Lemma 1}}{\cong} \\
 &= (V/V^\perp \cap (V^\perp)^\perp)^\perp \stackrel{\text{Lemma 2}}{\cong} \\
 &= (V/V^\perp \cap V)^\perp
 \end{aligned}$$

since $(V^\perp)^\perp = V$ (obviously $V \subseteq (V^\perp)^\perp$, use dim //

Def Given a basis $\langle e_i \rangle$, the Gram matrix of \langle, \rangle wrt $\{e_i\}$ is

$$G_{ij} = \langle e_i, e_j \rangle$$

Thm $\dim(V/V \cap V^\perp) = \text{rank of Gram matrix}$

Proof Define $\mathcal{G}: V \rightarrow V^*$ by $\mathcal{G}(v) = \langle v, - \rangle$. Choosing a basis of V and dual basis of V^* , check that matrix of \mathcal{G} is the Gram matrix. Thus

$$\text{Rank} = \dim V - \dim \ker \mathcal{G} = \dim V - \dim(V \cap V^\perp) //$$

Back to \mathbb{Z}_n ... Lemma to a λ -tbl, $u \in M$, $\ker u$ is a mult of e_0 .

Thm (James' Submodule Thm)

Suppose $U \subseteq M$. Then either $S^\lambda \subseteq U$ or $U \subseteq (S^\lambda)^\perp$.

Proof Suppose $\ker u = 0 \forall u, \forall u \in U$. Then $\langle e_0, u \rangle = \langle \lambda e_0, \ker u \rangle = 0 \forall u, u$
 so $U \subseteq (S^\lambda)^\perp$.

Otherwise $\exists u \in U, \lambda e_0 \in \ker u \neq 0 \Rightarrow e_0 \in U \Rightarrow S^\lambda \subseteq U$

COR Either $S^\lambda \cap (S^\lambda)^\perp$ is the unique max submodule of S^λ or is all of S^λ .

Proof Any submodule of S^λ is either all of S^λ or also in $(S^\lambda)^\perp$

Thm $S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$ is either zero or absolutely irreducible.

If it is nonzero then $S^\lambda \cap (S^\lambda)^\perp = \text{rad } S^\lambda$ and $S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$ is self-dual!

Now use lots of combinatorics to prove over \mathbb{C}

1. $\text{Hom}_{\mathbb{C}\Sigma_n}(S^\lambda, M^\mu) \neq 0 \Rightarrow \lambda \triangleright \mu$

2. $\text{Hom}_{\mathbb{C}\Sigma_n}(S^\lambda, M^\lambda) \cong \mathbb{C}$

3. $\langle \cdot, \cdot \rangle$ is inner product, $S^\lambda \perp S^{\lambda'} \neq 0$.

COR $\{S^\lambda\}$ are a complete set of irred $\mathbb{C}\Sigma_n$ -modules.

Aside Inside $\mathbb{C}\Sigma_n$ Let t be a λ -tableau

$$M^\lambda \cong \mathbb{C}\Sigma_n P_t$$

$$S^\lambda \cong \mathbb{C}\Sigma_n P_t K_t$$